DIFFERENCE SEQUENCE SPACES

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ABSTRACT. In [1]

$$S_r(\Delta) : = \{x = (x_k) : (k^r|\Delta x_k|)_{k=1}^{\infty} \in c_0\}$$

for $r \geq 1$ is studied. In this paper, we generalize this space to $S_r(p, \Delta)$ for a sequence of strictly positive reals. We give a characterization of the matrix classes $(S_r(p,\Delta), \ell_\infty)$ and $(S_r(p,\Delta), \ell_1)$.

KEY WORDS AND PHRASES: Difference sequence spaces, Köthe-Toeplitz duals, matrix transformations.

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1. INTRODUCTION

Let ℓ_∞, c and c_0 be the sets of all bounded, convergent and null sequences of $x = (x_k)_1^\infty$, respectively. Let w denote the set of all complex sequences and ℓ_1 denote the set of all convergent and absolutely convergent series.

Let z be any sequence and Y be any subset of w. Then

$$z^{-1} \cdot Y = \{x \in w : zx = (z_k x_k)_1^\infty \in Y\}.$$

For any subset X of w, the sets

$$X^\alpha = \bigcap_{x \in X} \left(x^{-1} \cdot \ell_1\right) \quad \text{and} \quad X^\beta = \bigcap_{x \in X} \left(x^{-1} \cdot cs\right)$$

are called the α- and β-duals of X.

We define the linear operators $\Delta, \Delta^{-1} : w \to w$ by

$$\Delta x = (\Delta x_k)_1^\infty = (x_k - x_{k+1})_1^\infty,$$

and

$$\Delta^{-1} x = \left(\Delta^{-1} x_k\right)_1^\infty = \left(\sum_{j=1}^{k-1} x_j\right)_1^\infty,$$

such that

$$\Delta^{-1}x_1 = 0.$$

Let

$$S_r(\Delta) : = \{x \in w : (k^r|\Delta x_k|)_{k=1}^{\infty} \in c_0\}, \text{ see [1]}.$$

In this paper we extend the space $S_r(\Delta)$ to $S_r(p,\Delta)$ in the same manner as c_0, c, ℓ_∞ were extended to $c_0(p)$, $c(p)$, $\ell_\infty(p)$, respectively (cf. [2],[3],[4]). We also determine the α- and β-duals of our new sequence space. Let $p = (p_k)_1^\infty$ be an arbitrary sequence of positive reals and $r \geq 1$, then we define

$$S_r(p,\Delta) : = \{x \in w : (k^r \Delta x_k)_1^\infty \in c_0(p)\},$$

where

$$c_0(p) : = \left\{x \in w : \lim_{k\to\infty} |x_k|^{p_k} = 0\right\}.$$

If $p = e = (1,1,1,...)$, then the set $S_r(p,\Delta)$ reduces to the set $S_r(\Delta)$. For $r = 0$, $S_r(p,\Delta)$ is the same as $\Delta c_0(p)$ (cf. [5],[6],[7]).

We will need the following lemmas:

LEMMA 1 (Corollary 1 in [7]). Let $(P_n)_{n=1}^\infty$ be a sequence of nondecreasing positive reals. Then $a \in (P_n)^{-1} \cdot cs$ implies $R = (R_n) \in (P_n)^{-1} \cdot c_0$ where $R_n = \sum_{k=n+1}^{\infty} a_k$ $(n = 1,2,...)$.

LEMMA 2 (Lemma 1(b) in [8]). Let $p = (p_k)_{k=1}^\infty$ be a strictly positive sequence such that $p \in \ell_\infty$. Then $A \in (c_0(p), \ell_1)$ if and only if

$$(*) \qquad B(M) = \sup_{\substack{N\subset\mathbb{N}\\ N \text{ finite}}} \left(\sum_{k=1}^{\infty} \left|\sum_{n\in N} a_{nk}\right| M^{-1/p_k}\right) < \infty$$

for some integer $M \geq 2$.

2. THE α- AND β-DUALS OF $S_r(p,\Delta)$

THEOREM 2.1. Let $p = (p_k)_1^\infty$ be a strictly positive sequence and $r \geq 1$. Then

(a)
$$[S_r(p,\Delta)]^\alpha = \bigcup_{N>1} D_r^{(1)}(p),$$

(b)
$$[S_r(p,\Delta)]^\beta = C_r(p) = \bigcap_{v\in c_0^+} D_r^{(2)}(p) \bigcap \bigcup_{N>1} D_r^{(3)}(p),$$

where

$$D_r^{(1)}(p) : = \left(\Delta_r^{-1} N^{-1/p}\right)^{-1} \cdot \ell_1 = \left\{a \in w : \sum_{k=1}^{\infty} |a_k| \left|\sum_{j=1}^{k-1} \frac{N^{-1/p_j}}{j^r}\right| < \infty\right\}$$

$$D_r^{(2)}(p) : = \left(\Delta_r^{-1} v^{1/p}\right)^{-1} \cdot cs = \left\{a \in w : \sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} \frac{v_j^{1/p_j}}{j^r} \text{ converges}\right\},$$

$$D_r^{(3)}(p) : = \left\{a \in w : R \in \left(\frac{N^{-1/p}}{k^r}\right)^{-1} \cdot \ell_1\right\} = \left\{a \in w : \sum_{k=1}^{\infty} |R_k| \frac{N^{-1/p_k}}{k^r} < \infty\right\},$$

$$\Delta_r x = (k^r \Delta x_k)_{k=1}^\infty, \Delta_r^{-1} x = \left(k^r \Delta^{-1} x_k\right)_{k=1}^\infty,$$

and c_0^+ is the set of all positive sequences in c_0.

PROOF. (a) Let $a \in \bigcup_{N>1} D_r^{(1)}(p)$. Then

$$a \cdot s(1/N_0) \in \ell_1 \quad \text{for some} \quad N_0 \geq 2, \tag{2.1}$$

where

$$s(1/N_0) = \left(s_k\left(\frac{1}{N_0}\right)\right)_{k=1}^{\infty} = \left(\sum_{j=1}^{k-1} \frac{N_0^{-1/p_j}}{j^r}\right)_{k=1}^{\infty}.$$

Since $s\left(\frac{1}{N_0}\right)$ is increasing, (2.1) implies that

$$a \in \ell_1. \tag{2.2}$$

Let $x \in S_r(p, \Delta)$. Then for a given $N_0 \in \mathbb{N}$, there exists an $M = M(N_0) \in \mathbb{N}$ such that $\sup_{k \geq M} |k^r \Delta x_k|^{p_k} < \frac{1}{N_0}$, and hence $|\Delta x_k| \leq \frac{N_0^{-1/p_k}}{k^r}$ for all $k = 1, 2, \ldots$, and consequently by (2.1) we have

$$\sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} |\Delta x_j| \leq \sum_{k=1}^{\infty} |a_k| s_k(1/N_0) < \infty. \tag{2.3}$$

Finally, by (2.2) and (2.3), we get

$$a \in [S_r(p, \Delta)]^{\alpha}.$$

Let $a \notin \bigcup_{N>1} D_r^{(1)}(p)$ Then we can determine a strictly increasing sequence $(k(m))_{m=1}^{\infty}$ of integers such that $k(1) = 1$ and

$$\sum_{k=k(m)}^{k(m+1)-1} |a_k| s_k(1/(m+1)) > 1 \quad (m = 1, 2, \ldots).$$

We define the sequence $x = (x_k)$ by

$$x_k = \sum_{i=1}^{m} \sum_{j=(k(i))}^{\min\{k-1, k(i+1)-1\}} \frac{(i+1)^{-1/p_j}}{j^r}, \qquad (k(m) \leq k \leq k(m+1) - 1; m = 1, 2, \ldots).$$

Then $x \in S_r(p, \Delta)$ and

$$\sum_{k=1}^{\infty} |a_k|\, |x_k| = \sum_{m=1}^{\infty} \sum_{k=k(m)}^{k(m+1)-1} |a_k x_k| > \infty$$

which proves that

$$a \notin [S_r(p, \Delta)]^{\alpha}.$$

Hence, $[S_r(p, \Delta)]^{\alpha} = \bigcup_{n>1} D_r^{(1)}(p)$.

(b) Let $a \in C_r(p)$. Then $a \in cs$, and Abel's summation by parts yields

$$\sum_{k=1}^{n} a_k x_k = -\sum_{k=1}^{n-1} R_k \Delta x_k + R_n \sum_{k=1}^{n-1} \Delta x_k + x_1 \sum_{k=1}^{n} a_k \text{ for all } x, \quad (n = 1, 2, \ldots). \tag{2.4}$$

Further

$$R \in \left(\frac{N_0^{-1/p}}{k^r}\right) \cdot \ell_1 \text{ for some integer } N_0 \geq 2. \tag{2.5}$$

Let $x \in S_r(p, \Delta)$. Then there is a sequence $v \in c_0^+$ such that

$$|\Delta x_k| \leq \frac{v_k^{1/p_k}}{k^r} \ (k = 1, 2, \ldots) \quad \text{and} \quad |\Delta x_k| \leq \frac{N_0^{-1/p_k}}{k^r}$$

for all sufficiently large k. Now, by (2.5)

$$\sum_{k=1}^{\infty} |R_k|\,|\Delta x_k| < \infty.$$

Hence

$$R\Delta x \in \ell_1 \subset cs. \tag{2.6}$$

Finally, by Lemma 1, $a \in \left(\Delta_r^{-1} v^{1/p}\right)^{-1} \cdot cs$ implies that

$$R \in \left(\Delta_r^{-1} v^{1/p}\right)^{-1} \cdot c_0 \tag{2.7}$$

and consequently

$$R_n \sum_{k=1}^{n-1} \Delta x_k \to 0 \ \ (n \to \infty). \tag{2.8}$$

From $a \in cs$, (2.4), (2.6) and (2.8), we conclude that

$$\sum_{k=1}^{\infty} a_k x_k = -\sum_{k=1}^{\infty} R_k \Delta x_k + x_1 \sum_{k=1}^{\infty} a_k \tag{2.9}$$

and $ax \in cs$. Thus $a \in [S_r(p,\Delta)]^\beta$. Now, let $a \in [S_r(p,\Delta)]^\beta$. Then $ax \in cs$ for all $x \in S_r(p,\Delta)$ and $e \in S_r(p,\Delta)$. This implies that $a \in cs$. Let $v \in c_0^+$ be given. Then $x = \Delta_r^{-1} v^{1/p} \in S_r(p,\Delta)$. Hence $a \in \left(\Delta_r^{-1} v^{1/p}\right)^{-1} \cdot cs$, and by Lemma 1, we get (2.7). Therefore (2.8) holds for all $x \in S_r(p,\Delta)$. By (2.4), we get $R\Delta x \in cs$. Since $x \in S_r(p,\Delta)$ if and only if $y = \Delta_r x = (k^r \Delta x_k)_{k=1}^\infty \in c_0(p)$, this implies that

$$\sum_{k=1}^{\infty} |R_k| \frac{N^{-1/p_k}}{k^r} < \infty$$

for some integer $N \geq 2$ (cf. [9], Theorem 6). Hence $[S_r(p,\Delta)]^\beta = C_r(p)$.

3. MATRIX TRANSFORMATIONS

For any infinite complex matrix $A = (a_{nk})_{n,k=1}^\infty$, we write $A_n = (a_{nk})_{k=1}^\infty$ for the sequence in the nth row of A. Let X and Y be two subsets of w. By (X, Y), we denote the class of all matrices A such that the series $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for all $x \in X$ and each $n \in \mathbb{N}$, and the sequence $Ax = (A_n(x))_{n=1}^\infty \in Y$ for all $x \in X$.

THEOREM 3.1. Let $p = (p_k)_1^\infty$ be a strictly positive sequence and $r \geq 1$. Then $A \in (S_r(p,\Delta), \ell_\infty)$ if and only if

(i) $$D_r(v) := \sup_n \left|A_n\left(\Delta_r^{-1} v^{1/p}\right)\right| = \sup_n \left|\sum_{k=1}^{\infty} a_{nk} \sum_{j=1}^{k-1} \frac{v^{1/p_j}}{j^r}\right| < \infty \quad \text{for all} \quad v \in c_0^+,$$

(ii) $$D_r(M) := \sup_n \left(\sum_{k=1}^{\infty} |R_{nk}| \frac{M^{1/p_k}}{k^r}\right) < \infty \quad \text{for some integer} \quad M \geq 2,$$

where $R_{nk} = \sum_{j=k+1}^{\infty} a_{nk}$ for all n and k, and

(iii) $$D_\infty := \sup_n |A_n(e)| = \sup_n \left|\sum_{k=1}^{\infty} a_{nk}\right| < \infty.$$

PROOF. Let the conditions (i), (ii) and (iii) be true and $x \in S_r(p, \Delta)$. By Theorem 2.1(b), conditions (i) and (ii) imply that $A_n \in [S_r(p, \Delta)]^\beta$ for $n = 1, 2, \ldots$. for a given $M \in \mathbb{N}$, there exists a $M' = M'(M) \in \mathbb{N}$ such that $\sup_{k \geq M'} |kr\Delta x_k| \leq \frac{1}{M}$, where $M \geq 2$ is the integer in (ii). By (2.9), we have

$$|A_n(x)| \leq D_r(M) + |x_1| D_\infty \quad (n = 1, 2, \ldots)$$

and hence $Ax \in \ell_\infty$. Conversely, let $A \in (S_r(p, \Delta), \ell_\infty)$. Since $x = \Delta_r^{-1} v^{1/p} \in S_r(p, \Delta)$ for all $v \in c_0^+$, condition (i) follows immediately. Also the necessity of (iii) follows from the fact that $x = e \in S_r(p, \Delta)$. Now, by (i), (iii) and (2.9),

$$A_n(x) = -\sum_{k=1}^{\infty} R_{nk} \Delta x_k + x_1 A_n(e) \quad (n = 1, 2, \ldots).$$

Since $Ax \in \ell_\infty$ and $x_1 Ae \in \ell_\infty$, therefore $(R_n \Delta x)_{n=1}^\infty \in \ell_\infty$. Since $x \in S_r(p, \Delta)$ if and only if $(k^r \Delta x_k)_{k=1}^\infty \in c_0(p)$, and $\left(\sum_{k=1}^{\infty} (R_{nk}/k^r)(k^r \Delta x_k) \right)_{n=1}^\infty \in \ell_\infty$ for all $(k^r \Delta x_k)_{k=1}^\infty \in c_0(p)$, this implies that $D_r(M) < \infty$ for some integer $M \geq 2$, and (ii) holds.

THEOREM 3.2. Let $p = (p_k)_1^\infty$ be a strictly positive sequence such that $p \in \ell_\infty$, and $r \geq 1$. Then $A \in (S_r(p, \Delta), \ell_1)$ if and only if

(i)
$$C_r^{(1)}(v) := \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left| \sum_{n \in N} A_n(\Delta_r^{-1} v^{1/p}) \right| = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left| \sum_{n \in N} \sum_{k=1}^{\infty} a_{nk} \sum_{j=1}^{k-1} \frac{v^{1/p_j}}{j^r} \right| < \infty$$

for all sequences $v \in c_0^+$,

(ii)
$$C_r^{(2)}(M) := \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left(\sum_{k=1}^{\infty} \sum_{n \in N} |R_{nk}| \frac{M^{-1/p_k}}{k^r} \right) < \infty$$

for some integer $M \geq 2$, and

(iii)
$$D_r^{(3)} := \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left| \sum_{n \in N} A_n(e) \right| < \infty.$$

PROOF. Let conditions (i), (ii) and (iii) hold. Then $A_n \in [S_r(p, \Delta)]^\beta$. Let $xS_r(p, \Delta)$. For a given $M \in \mathbb{N}$ there exists a $M' = M'(M) \in \mathbb{N}$ such that $\sup_{k \geq M'} |k^r \Delta x_k|^{p_k} < \frac{1}{M}$. Now, by (2.9) and the inequality in [10], p. 33, we have

$$\sum_{n=1}^{m} |A_n(x)| \leq 4\left(C_r^{(2)}(M) + |x_1| D_r^{(3)}\right) < \infty.$$

Since $m \in N$ is arbitrary, we have $Ax \in \ell_1$. Conversely, let $A \in (S_r(p, \Delta), \ell_1)$. Then

$$\left| \sum_{n \in N} A_n(x) \right| \leq \sum_{k=1}^{\infty} |A_n(x)| < \infty$$

for all $x \in S_r(p, \Delta)$ and for all finite subsets N of $\mathbb{N}$. Therefore the necessity of (iii) and (i) follows immediately, since e and $x = \Delta_r^{-1} v^{1/p} \in S_r(p, \Delta)$ for every sequence $v \in c_0^+$. Further we have

$$\left(\sum_{k=1}^{\infty} \frac{R_{nk}}{k^r} k^r \Delta x_k \right)_{n=1}^{\infty} \in \ell_1 \quad \text{for all} \quad (k^r \Delta x_k)_{k=1}^{\infty} \in c_0(p),$$

and hence (ii) holds by Lemma 2.

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