Internat. J. Math. & Math. Sci.
Vol. 22, No. 1 (1999) 119–129
S 0161-1712(99)22119-9

FIXED POINT THEOREMS FOR GENERALIZED LIPSCHITZIAN SEMIGROUPS IN BANACH SPACES

BALWANT SINGH THAKUR and JONG SOO JUNG

(Received 12 February 1997 and in revised form 22 July 1997)

ABSTRACT. Fixed point theorems for generalized Lipschitzian semigroups are proved in p-uniformly convex Banach spaces and in uniformly convex Banach spaces. As applications, its corollaries are given in a Hilbert space, in L^p spaces, in Hardy space H^p, and in Sobolev spaces $H^{k,p}$, for $1 < p < \infty$ and $k \geq 0$.

Keywords and phrases. Semitopological semigroup, submean, generalized Lipschitzian semigroup, p-uniformly convex Banach space, uniformly normal structure.

1991 Mathematics Subject Classification. 47H10.

1. Introduction. Let K be a nonempty closed convex subset of a Banach space E. A mapping $T : K \longrightarrow K$ is said to be Lipschitzian mapping if for each $n \geq 1$, there exists a positive real number k_n such that

$$||T^n x - T^n y|| \leq k_n ||x - y|| \tag{1}$$

for all x, y in K. A Lipschitzian mapping is said to be nonexpansive if $k_n = 1$ for all $n \geq 1$, uniformly k-Lipschitzian if $k_n = k$ for all $n \geq 1$, and asymptotically nonexpansive if $\lim_n k_n = 1$, respectively. These mappings were first studied by Geobel and Kirk [6] and Geobel, Kirk, and Thele [8]. Lifshitz [10] showed that in a Hilbert space H, a uniformly k-Lipschitzian mapping T with $k < \sqrt{2}$ has a fixed point. Downing and Ray [3] and Ishihara and Takahashi [9] verified that Lifshitz's theorem is valid for uniformly Lipschitzian semigroup in Hilbert spaces.

Mizoguchi and Takahashi [14] introduced the notion of a submean on an appropriate space and, using a submean, they proved a fixed point theorem for uniformly Lipschitzian semigroup in a Hilbert space. Recently, Tan and Xu [21] generalized the result of Mizoguchi and Takahashi [14] to a Banach space setting and, also, proved a new fixed point theorem for uniformly k-Lipschitzian semigroup in a uniformly convex Banach space.

Now, we consider the following class of mappings, which we call generalized Lipschitzian mapping whose nth iterate T^n satisfies the following condition:

$$\begin{aligned} ||T^n x - T^n y|| \leq a_n ||x - y|| &+ b_n (||x - T^n x|| + ||y - T^n y||) \\ &+ c_n (||x - T^n y|| + ||y - T^n x||) \end{aligned} \tag{2}$$

for each $x, y \in K$ and $n \geq 1$, where a_n, b_n, c_n are the nonnegative constants such that there exists an integer n_0 such that $b_n + c_n < 1$ for all $n \geq n_0$.

This class of generalized Lipschitzian mappings are more general than nonexpansive, asymptotically nonexpansive, Lipschitzian, and uniformly k-Lipschitzian mappings and it can be seen by taking $b_n = c_n = 0$.

In this paper, we prove some fixed point theorems for generalized Lipschitzian semigroups in p-uniformly convex Banach spaces and in uniformly convex Banach spaces. Next, we give its corollaries in a Hilbert space, in L^p spaces, in Hardy space H^p, and in Sobolev spaces $H^{k,p}$, for $1 < p < \infty$ and $k \geq 0$. Our results improve and extend results from [9, 14, 21, 22].

2. Preliminaries. Let $p > 1$ and denote by λ the number in $[0,1]$ and by $w_p(\lambda)$ the function $\lambda \cdot (1-\lambda)^p + \lambda^p \cdot (1-\lambda)$. The functional $\|\cdot\|^p$ is said to be uniformly convex (cf. Zalinescu [24]) on the Banach space E if there exists a positive constant c_p such that, for all $\lambda \in [0,1]$ and $x, y \in E$, the following inequality holds:

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda \|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda) \cdot c_p \cdot \|x-y\|^p. \tag{3}$$

Xu [23] proved that the functional $\|\cdot\|^p$ is uniformly convex on the whole Banach space E if and only if E is p-uniformly convex, i.e., there exists a constant $c_p > 0$ such that the modulus of convexity (see [7]) $\delta_E(\epsilon) \geq c_p \cdot \epsilon^p$ all $0 \leq \epsilon \leq 2$.

Let G be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that, for each $a \in G$, the mapping $t \longrightarrow a \cdot t$ and $t \longrightarrow t \cdot a$ from G onto itself are continuous. A semitopological semigroup G is left reversible if any two closed right ideals of G have nonempty intersection. In this case, $(G, \preceq)$ is a directed system when the binary relation "$\preceq$" on G is defined by $a \preceq b$ if and only if $\{a\} \cup \overline{aG} \supseteq \{b\} \cup \overline{bG}$, where $\overline{D}$ is the closure of set D. Examples of left reversible semigroups include commutative and all left amenable semigroups.

Let $m(G)$ be the Banach space of bounded real valued functions on G with the supremum norm. Suppose X is a subspace of $m(G)$ containing constants. Following Mizoguchi and Takahashi [14], we say that a real valued function μ on X is a submean on X if the following conditions are satisfied:

(i) $\mu(f+g) \leq \mu(f) + \mu(g)$ for all $f, g \in X$;

(ii) $\mu(\alpha f) = \alpha\mu(f)$ for all $f \in X$ and $\alpha \geq 0$;

(iii) if $f, g \in X$ with $f \leq g$, then $\mu(f) \leq \mu(g)$; and

(iv) $\mu(c) = c$ for every constant c.

If μ is a submean on X and $f \in X$, then we denote by either $\mu(f)$ or $\mu_t(f(t))$, according to time and circumstances, the value of μ at f. For $a \in G$ and $f \in m(G)$, we define $(l_a f)(t) = f(at)$ and $(r_a f)(t) = f(ta)$ for all $t \in G$. Let X be a subspace of $m(G)$ containing constants which is l_G-invariant, i.e., $l_a(X) \subseteq X$ for all $a \in G$. Then a submean μ on x is said to be left invariant if $\mu(f) = \mu(l_a f)$ for every $a \in G$ and $f \in X$. A right invariant submean is defined similarly. A submean is called invariant if it is left and right invariant. Let K be a closed convex subset of a Banach space E. Then a collection $\mathcal{S} = \{T_s : s \in G\}$ of mappings of K into itself is said to be a generalized Lipschitzian semigroup on K if the following conditions are satisfied:

(i) $T_{st}x = T_s T_t x$ for all $s, t \in G$ and $x \in K$;

(ii) for each $x \in K$, the mapping $t \longrightarrow T_t x$ from G into K is continuous; and

(iii) for each $s \in G$

$$\|T_s x - T_s y\| \le a_s \|x - y\| + b_s (\|x - T_s x\| + \|y - T_s y\|) + c_s (\|x - T_s y\| + \|y - T_s x\|), \tag{4}$$

for $x, y \in K$, where $a_s, b_s, c_s > 0$ such that there exists a $t_1 \in G$ such that $b_s + c_s < 1$ for all $s \succeq t_1$.

The following lemma is needed to prove the main result:

Lemma 1 [22, Lem. 2.1]. *Let E be a p-uniformly convex Banach space, K a nonempty closed convex subset of E, and $\{x_t : t \in G\}$ a bounded family of elements of E. Also, suppose that for every x in K, the function f on G, defined by*

$$f(t) = \|x_t - x\|^p, \quad t \in G, \tag{5}$$

belongs to X. Set

$$r(x) = \mu_t \|x_t - x\|^p, \quad x \in K \tag{6}$$

and

$$r = \inf\{r(x) : x \in K\}. \tag{7}$$

Then there exists a unique point z in K such that

$$r + c_p \|z - x\|^p \le r(x) \tag{8}$$

for all x in K, where c_p is the constant appearing in (3).

3. Main results.

Now, we prove the first result of this paper.

Theorem 1. *Let K be a nonempty closed convex subset of a p-uniformly convex Banach space E, X an l_G-invariant subspace of $m(G)$ containing constants which has left invariant submean μ, and $\mathcal{S} = \{T_s : s \in G\}$ a generalized Lipschitzian semigroup on K. Suppose that there exists an x_0 in K such that $\{T_s x_0 : x \in G\}$ is bounded and that, for every $u, v \in K$, the function f on G defined by*

$$f(t) = \|T_t u - v\|^p, \quad t \in G, \tag{9}$$

and the function g on G defined by

$$g(t) = 2^{p-1}(\alpha_t^p + \beta_t^p), \quad t \in G \tag{10}$$

belong to X. Then, if $2^{p-1}\{\mu_t(\alpha_t^p + \beta_t^p)\} < 1 + c_p$, where $\alpha_t = (a_t + b_t + c_t)/(1 - b_t - c_t)$, $\beta_t = (2b_t + 2c_t)/(1 - b_t - c_t)$, and c_p is the constant appearing in (3), there exists a $z \in K$ such that $T_s z = z$ for all $s \in G$.

Proof. Since $\{T_s x_0 : s \in G\}$ is bounded, it follows that $\{T_s x : s \in G\}$ is bounded for every $x \in K$. By Lemma 1, we inductively construct a sequence $\{x_n\}_{n=1}^{\infty}$ in K in the following manner:

$$\mu_t \|T_t x_{n-1} - x_n\|^p = \min_{y\in K} \mu_t \|T_t x_{n-1} - y\|^p \tag{11}$$

for $n = 1, 2, \dots$. It follows from Lemma 1 that

$$c_p \|x_n - y\|^p \le \mu_t \|T_t x_{n-1} - y\|^p - \mu_t \|T_t x_{n-1} - x_n\|^p \tag{12}$$

for all $y \in K$ and $n \ge 1$. Since T is generalized Lipschitzian, we get, after a simple calculation,

$$\|T_s x - T_s y\| \le \alpha_s \|x - y\| + \beta_s \|y - T_s y\| \tag{13}$$

for each $x, y \in K$ and $s \in G$, where $\alpha_s = (a_s + b_s + c_s)/(1 - b_s - c_s)$ and $\beta_s = (2b_s + 2c_s)/(1 - b_s - c_s)$. By putting $y = T_s x_n$ into (12), we have

$$\begin{aligned} c_p \|x_n - T_s x_n\|^p &\le \mu_t \|T_t x_{n-1} - T_s x_n\|^p - \mu_t \|T_t x_{n-1} - x_n\|^p \\ &= \mu_t \|T_{st} x_{n-1} - T_s x_n\|^p - \mu_t \|T_t x_{n-1} - x_n\|^p \\ &= \mu_t \|T_s T_t x_{n-1} - T_s x_n\|^p - \mu_t \|T_t x_{n-1} - x_n\|^p \\ &\le \mu_t \Big[\alpha_s \|T_t x_{n-1} - x_n\| + \beta_s \|x_n - T_s x_n\|\Big]^p - \mu_t \|T_t x_{n-1} - x_n\|^p \end{aligned} \tag{14}$$

or

$$(c_p - 2^{p-1}\beta_s^p)\|x_n - T_s x_n\|^p \le (2^{p-1}\alpha_s^p - 1) \cdot \mu_t \|T_t x_{n-1} - x_n\|^p. \tag{15}$$

Therefore, we have

$$\mu_s \|x_n - T_s x_n\|^p \le A \cdot \mu_t \|T_t x_{n-1} - x_n\|^p, \tag{16}$$

where $A = (2^{p-1}\alpha_s^p - 1)/(c_p - 2^{p-1}\beta_s^p) < 1$ by the assumption of the theorem. Since

$$\mu_t \|T_t x_{n-1} - x_n\|^p \le \mu_t \|T_t x_{n-1} - x_{n-1}\|^p \tag{17}$$

by (11), it follows from (13) that

$$\begin{aligned} \mu_t \|T_t x_{n-1} - x_n\|^p &\le A \cdot \mu_t \|T_t x_{n-1} - x_{n-1}\|^p \\ &\le A^n \mu_t \|T_t x_0 - x_0\|^p. \end{aligned} \tag{18}$$

Noticing that

$$\|x_n - x_{n-1}\|^p \le 2^{p-1}\Big(\|x_n - T_t x_{n-1}\|^p + \|T_t x_{n-1} - x_{n-1}\|^p\Big), \tag{19}$$

we get

$$\begin{aligned} \|x_n - x_{n-1}\|^p &\le 2^{p-1}\Big(\mu_t \|x_n - T_t x_{n-1}\|^p + \mu_t \|T_t x_{n-1} - x_{n-1}\|^p\Big) \\ &\le 2^p \mu_t \|T_t x_{n-1} - x_{n-1}\|^p \\ &\le 2^p A^{n-1} \mu_t \|T_t x_0 - x_0\|^p, \end{aligned} \tag{20}$$

which shows that $\{x_n\}$ is a Cauchy sequence and, hence, convergent. Let $z = \lim_{n\to\infty} x_n$. Then, for each $s \in G$, we have

$$\begin{aligned} \|z - T_s z\|^p &\le \Big(\|z - x_n\| + \|x_n - T_s x_n\| + \|T_s x_n - T_s z\|\Big)^p \\ &\le \Big[(1+\alpha_s)\|z - x_n\| + (1+\beta_s)\|x_n - T_s x_n\|\Big]^p \\ &\le 2^{p-1}\Big[(1+\alpha_s)^p\|z - x_n\| + (1+\beta_s)^p \cdot A \cdot \mu_t \|x_n - T_s x_n\|^p\Big] \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{21}$$

Therefore, $T_s z = z$ for all $s \in G$ and the proof is complete. □

Let E be a Banach space, K a nonempty closed convex subset of E, and G an unbounded subset of $[0,\infty)$ such that

$$t + h \in G \quad \text{for all } t, h \in G \tag{22}$$

and

$$t - h \in G \quad \text{for all } t, h \in G \quad \text{with } t > h \tag{23}$$

(e.g., $G = [0,\infty)$ or $G = N$, the set of nonnegative integers). Suppose $\mathcal{S} = \{T_s : s \in G\}$ is a generalized uniformly Lipschitzian semigroup on K, i.e., a family of self-mappings of K satisfying the conditions:

(i) $T_{s+h}x = T_s T_h x$ for all $s, h \in G$ and $x \in K$;

(ii) for each $x \in K$, the mappings $s \longrightarrow T_s x$ from G onto K is continuous when G has the relative topology of $[0,\infty)$; and

(iii)

$$\|T_s x - T_s y\| \le a\|x - y\| + b\Big(\|x - T_s x\| + \|y - T_s y\|\Big) + c\Big(\|x - T_s y\| + \|y - T_s x\|\Big) \tag{24}$$

for all x, y in K and s in G, where a, b, c are nonnegative constants such that $b + c < 1$.

For the rest of this paper, $\lim_t$ and $\overline{\lim}_t$ always stand for $\lim_{t\to\infty, t\in G}$, $\overline{\lim}_{t\to\infty, t\in G}$ respectively.

The normal structure coefficient $N(E)$ of E is defined (cf. [2]) by

$$N(E) = \inf\left\{\frac{\operatorname{diam} K}{r_K(K)} : K \text{ is a bounded convex subset of } E \text{ consisting of more than one point}\right\}, \tag{25}$$

where $\operatorname{diam} K = \sup\{\|x - y\| : x, y \in K\}$ is the diameter of K and $r_K(K) = \inf_{x\in K}\{\sup_{y\in K}\|x - y\|\}$ is the Chebyshev radius of K relative to itself. E is said to have uniformly normal structure if $N(E) > 1$. It is known that a uniformly convex Banach space has the uniformly normal structure and for a Hilbert space H, $N(H) = \sqrt{2}$. Recently, Pichugov [15] (cf. Prus [17]) showed that

$$N(L^p) = \min\{2^{1/p}, 2^{(p-1)/p}\}, \quad 1 < p < \infty. \tag{26}$$

Some estimate for normal structure coefficient in other Banach spaces may be found in [18].

Suppose E is a uniformly convex Banach space. Then it is easily seen that the equation

$$\xi^2 \delta_E^{-1}\left(1-\frac{1}{\xi}\right)\tilde{N}(E) = 1 \tag{27}$$

has a unique solution $\xi > 1$, where $\tilde{N}(E) = N(E)^{-1}$.

Now, recall the definition of an asymptotic center. Let K be a nonempty closed convex subset of a Banach space E and $\{x_t : t \in G\}$ be a bounded family of elements of E. Then the asymptotic radius and asymptotic center of $\{x_t\}_{t\in G}$ with respect to K are the number

$$r_K(\{x_t\}) = \inf_{y\in K} \overline{\lim_t} \|x_t - y\| \tag{28}$$

and the (possibly empty) set

$$A_K(\{x_t\}) = \left\{y \in K : \overline{\lim_t} \|x_t - y\| = r_K(\{x_t\})\right\}, \tag{29}$$

respectively. It is easy to see that if E is reflexive, then $A_K(\{x_t\})$ is nonempty bounded closed and convex and if E is uniformly convex, then $A_K(\{x_t\})$ consists of a single point.

We need the following lemma to prove our next theorem.

Lemma 2 [22, Lem. 3.4]. *Let E be a Banach space with uniformly normal structure. Then for every bounded family $\{x_t\}_{t\in G}$ of elements of E, there exists y in $\overline{\text{co}}(\{x_t : t \in G\})$ such that*

$$\overline{\lim_t} \|x_t - y\| \le \tilde{N}(E) A(\{x_t\}), \tag{30}$$

where $\overline{\text{co}}(D)$ is the closure of the convex hull of $D \subseteq E$ and

$$A(\{x_t\}) = \lim_t \left(\sup\{\|x_i - x_j\| : t \le i, j \in G\}\right) \tag{31}$$

is the asymptotic diameter of $\{x_t\}$.

Now, we are in position to prove our next theorem.

Theorem 2. *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E, and $\mathcal{S} = \{T_s : s \in G\}$ a generalized uniformly Lipschitzian semigroup on K with $(\alpha+\beta) < \xi$, where $\xi > 1$ is the unique solution of (27), $\alpha = (a+b+c)/(1-b-c)$ and $\beta = (2b+2c)/(1-b-c)$. Suppose there is an x_0 in K such that $\{T_s x_0 : s \in G\}$ is bounded. Then there exists z in K such that $T_s z = z$ for all s in G.*

Proof. By induction, we define a sequence $\{x_n\}_0^\infty$ in K in the following manner:

$$x_{n+1} = A_K\left(\{T_t x_n\}_{t\in G}\right) \tag{32}$$

for $n = 0, 1, \ldots$, i.e., x_{n+1} is the unique point in K such that

$$\overline{\lim_t} \|T_t x_n - x_{n+1}\| = \inf_{y\in K} \overline{\lim_t} \|T_t x_n - y\|. \tag{33}$$

Write $r_n = r_K(\{T_t x_n\}_{t\in G})$. Then by Lemma 2, we have

$$\begin{aligned} r_n &= \overline{\lim_t} \|T_t x_n - x_{n-1}\| \\ &\le \tilde{N}(E)\cdot A\Big(\{T_t x_n\}_{t\in G}\Big) \\ &= \tilde{N}(E)\lim_t\Big(\sup\{\|T_i x_n - T_j x_n\| : t \le i,\, j\in G\}\Big) \\ &\le \tilde{N}(E)(\alpha+\beta)\cdot d(x_n), \end{aligned} \tag{34}$$

that is,

$$r_n \le (\alpha+\beta)\cdot\tilde{N}(E)d(x_n), \tag{35}$$

where $d(x_n) = \sup\{\|T_t x_n - x_n\| : t\in G\}$. We may assume that $d(x_n) > 0$ for all $n \ge 0$. Let $n \ge 0$ be fixed and let $\epsilon > 0$ be small enough. First, choose $j \in G$ such that

$$\|T_j x_{n+1} - x_{n+1}\| > d(x_{n+1}) - \epsilon \tag{36}$$

and then choose s_0 in G so large that

$$\|T_s x_n - x_{n+1}\| < r_n + \epsilon \tag{37}$$

and

$$\|T_s x_n - T_j x_{n+1}\| \le \alpha\|T_{s-j}x_n - x_{n+1}\| + \beta\|T_j x_n - x_n\| \le (\alpha+\beta)(r_n+\epsilon) \tag{38}$$

for all $s \ge s_0$. It, then, follows that

$$\Big\|T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\Big\| \le (\alpha+\beta)(r_n+\epsilon)\left(1-\delta_E\left(\frac{d(x_{n+1})-\epsilon}{(\alpha+\beta)(r_n+\epsilon)}\right)\right) \tag{39}$$

for $s \ge s_0$ and, hence,

$$\begin{aligned} r_n &\le \overline{\lim_s}\Big\|T_s x_n - \frac{1}{2}(x_{n+1}+T_j x_{n+1})\Big\| \\ &\le (\alpha+\beta)(r_n+\epsilon)\left(1-\delta_E\left(\frac{d(x_{n+1})-\epsilon}{(\alpha+\beta)(r_n+\epsilon)}\right)\right). \end{aligned} \tag{40}$$

Taking the limit as $\epsilon \longrightarrow 0$, we get

$$r_n \le (\alpha+\beta)\cdot r_n\left(1-\delta_E\left(\frac{d(x_{n+1})}{(\alpha+\beta)r_n}\right)\right) \tag{41}$$

which together with (35) leads to the conclusion

$$d(x_{n+1}) \le (\alpha+\beta)^2\tilde{N}(E)\delta_E^{-1}\left(1-\frac{1}{(\alpha+\beta)}\right)d(x_n). \tag{42}$$

Hence,

$$d(x_n) \le Ad(x_{n-1}) \le A^n d(x_0), \tag{43}$$

where $A = (\alpha+\beta)^2\tilde{N}(E)\delta_E^{-1}(1-(1/(\alpha+\beta))) < 1$ by assumption. Noticing that

$$\begin{aligned} \|x_{n+1}-x_n\| &\le \overline{\lim_t}\|T_tx_n - x_{n+1}\| + \overline{\lim_t}\|T_tx_n - x_n\| \\ &\le r_n + d(x_n) \le 2d(x_n), \end{aligned} \tag{44}$$

we see from (43) that $\{x_n\}$ is a Cauchy sequence and, hence, strongly convergent. Let $z = \lim_n x_n$. Then we have, for each $s \in G$,

$$\begin{aligned} \|z - T_sz\| &\le \|z - x_n\| + \|T_sx_n - x_n\| + \|T_sx_n - T_sz\| \\ &\le (1+\alpha)\|z - x_n\| + (1+\beta)\,d(x_n) \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{45}$$

This completes the proof. □

As a consequence of Theorem 2, we have the following result.

Corollary 1. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $T : K \longrightarrow K$ be a generalized uniformly Lipschitzian mapping with $(\alpha+\beta) < \xi$ (ξ is as in Theorem 2). Then T has a fixed point.*

If we take $b = c = 0$ in Theorem 2, then we have the following result from Theorem 2:

Corollary 2 [22, Thm. 3.5]**.** *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E, and $\mathcal{S} = \{T_s : s \in G\}$ a uniformly k-Lipschitzian semigroup on K with $k < \xi$, where $\xi > 1$ is the unique solution of (27). Suppose there is an x_0 in K such that $\{T_sx_0 : s \in G\}$ is bounded. Then there exists z in K such that $T_sz = z$ for all s in G.*

4. Some applications. Since a Hilbert space H is 2-uniformly convex and the following equality holds:

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{46}$$

for all x, y in H and $\lambda \in [0,1]$.

By Theorem 1 and (46), we immediately obtain the following:

Corollary 3. *Let E be a nonempty closed convex subset of a Hilbert space H, X be an l_G-invariant subspace of $m(G)$ containing constants which has left invariant submean μ, and $\mathcal{S} = \{T_s : s \in G\}$ be a generalized Lipschitzian semigroup on K. Suppose that there exists an x_0 in K such that $\{T_sx_0 : s \in G\}$ is a generalized Lipschitzian semigroup on K. Suppose that there exists an x_0 in K such that $\{T_sx_0 : s \in G\}$ is bounded and that for every u, v in K, then the function f on G defined by*

$$f(t) = \|T_tu - v\|^2, \quad t \in G \tag{47}$$

and the function g on G defined by

$$g(t) = 2(\alpha_t^2 + \beta_t^2), \quad t \in G \tag{48}$$

belong to X. Then, if $\{\mu_t(\alpha_t^2+\beta_t^2)\} < 1$, where $\alpha_t = (a_t+b_t+c_t)/(1-b_t-c_t)$ and $\beta_t = (2b_t+2c_t)/(1-b_t-c_t)$, there exists z in K such that $T_sz = z$ for all s in G.

If $1 < p \le 2$, then we have for all x, y in L^p and $\lambda \in [0,1]$

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)(p-1)\|x-y\|^2 \tag{49}$$

(the inequality (49) is contained in [12, 20]).

Assume that $2 < p < \infty$ and t_p is the unique zero of the function $g(x) = -x^{p-1} + (p-1)x + p - 2$ in the interval $(1,\infty)$. Let

$$c_p = (p-1)(1+t_p)^{2-p} = \frac{1+t_p^{p-1}}{(1+t_p)^{p-1}}. \tag{50}$$

Then we have the following inequality

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)\cdot c_p \cdot \|x-y\|^p \tag{51}$$

for all x, y in L^p and $\lambda \in [0,1]$. (The inequality (51) is essentially due to Lim [11].)

By Theorem 1 and inequality (49) and (51), we immediately obtain the following result.

Corollary 4. *Let K be a closed convex subset of an L^p space, $1 < p < \infty$, X be an l_G-invariant subspace of $m(G)$ containing constants which has a left invariant submean μ, and $\mathcal{S} = \{T_s : s \in G\}$ be a generalized Lipschitzian semigroup on K. Suppose that $\{T_s x_0 : s \in G\}$ is bounded for some $x_0 \in K$ and that for every u, v in K, the functions f and g on G defined as in Theorem 1 belong to X. If $2\mu_s(\alpha_s^2 + \beta_s^2) < p$ when $1 < p \le 2$ and $2^{p-1}\mu_s(\alpha_s^{p-1} + \beta_s^{p-1}) < 1 + c_p$ when $p > 2$, where $\alpha_s = (a_s + b_s + c_s)/(1 - b_s - c_s)$ and $\beta_s = (2b_s + 2c_s)/(1 - b_s - c_s)$, then there exists $z \in K$ such that $T_s z = z$ for all $s \in G$.*

Let H^p, $1 < p < \infty$, denote the Hardy space [5] of all functions x analytic in the unit disk $|z| < 1$ of the complex plane and such that

$$\|x\| = \lim_{r\to 1^-}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|x(re^{i\theta})\right|^p d\theta\right)^{1/p} < \infty. \tag{52}$$

Now, let Ω be an open subset of $\mathbb{R}^n$. Denote by $H^{k,p}(\Omega)$, $k \ge 0$, $1 < p < \infty$, the Sobolev space [1, p. 149] of distribution x such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = a_1 + \cdots + \alpha_n \le k$ equipped with the norm

$$\|x\| = \left(\sum_{|\alpha|\le k}\int_\Omega |D^\alpha x(\omega)|^p d\omega\right)^{1/p}. \tag{53}$$

Let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in \wedge$, be a sequence of positive measure spaces, where index set $\wedge$ is finite or countable. Given a sequence of linear subspaces X_α in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < \infty$ and $q = \max\{2, p\}$ [13], the linear space of all sequences $x = \{x_\alpha \in X_\alpha : \alpha \in \wedge\}$ equipped with the norm

$$\|x\| = \left(\sum_{\alpha\in\wedge}(\|x_\alpha\|_{p,\alpha})^q\right)^{1/q}, \tag{54}$$

where $\|\cdot\|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha,\Sigma_\alpha,\mu_\alpha)$.

Finally, let $L_p = (S_1,\Sigma_1,\mu_1)$ and $L_q = (S_2,\Sigma_2,\mu_2)$, where $1 < p < \infty$, $q = \max\{2,p\}$ and (S_i,Σ_i,μ_i) are positive measure spaces. Denote by $L_q(L_p)$ the Banach spaces [4, III.2.10] of all measurable L_p-value function x on S_2 such that

$$\|x\| = \left(\int_{S_2} (\|x(S)\|_p)^q \mu_2(ds)\right)^{1/q}. \tag{55}$$

These spaces are q-uniformly convex with $q = \max\{2,p\}$ [16, 19] and the norm in these spaces satisfies

$$\|\lambda x + (1-\lambda)y\|^q \le \lambda\|x\|^q + (1-\lambda)\|y\|^q - d\cdot w_q(\lambda)\cdot\|x-y\|^q \tag{56}$$

with a constant

$$d = d_p = \begin{cases} \frac{p-1}{8} & \text{for } 1 < p \le 2 \\ \frac{1}{p\cdot 2^p} & \text{for } 2 < p < \infty. \end{cases} \tag{57}$$

Now, from Theorem 1, we have the following result.

Corollary 5. *Let K be a closed convex subset of the space E, where $E = H^p$, or $E = H^{k,p}(\Omega)$, or $E = L_{q,p}$, or $E = L_q(L_p)$, and $1 < p < \infty$, $q = \max\{2,p\}$, $k \ge 0$, X be an l_G-invariant subspace of $m(G)$ containing constants which has a left invariant submean μ, and $\mathcal{S} = \{T_s : s \in G\}$ be a generalized Lipschitzian semigroup on K. Suppose that $\{T_s x_0 : s \in G\}$ is bounded for some x_0 in K and that for every u,v in K, the functions f and g on G defined as in Theorem 1 belong to X. If $2^{q-1}\mu_s(\alpha_s^q + \beta_s^q) < 1 + d$, where $\alpha_s = (a_s + b_s + c_s)/(1 - b_s - c_s)$ and $\beta_s = (2b_s + 2c_s)/(1 - b_s - c_s)$, then there exists $z \in K$ such that $T_s z = z$ for all $s \in G$.*

Acknowledgement. The second author was supported by the Basic Science Research Institute Program, Ministry of Education, Korea, 1997, Project No. BSRI-97-1405.

References

[1] J. Barros-Neto, *An introduction to the theory of distributions*, Pure and Applied Mathematics, vol. 14, Marcel Dekker, Inc., New York, 1973. MR 57 1113. Zbl 273.46026.

[2] W. L. Bynum, *Normal structure coefficients for Banach spaces*, Pacific J. Math. **86** (1980), no. 2, 427–436. MR 81m:46030. Zbl 442.46018.

[3] D. J. Downing and W. O. Ray, *Uniformly Lipschitzian semigroups in Hilbert space*, Canad. Math. Bull. **25** (1982), no. 2, 210–214. MR 84e:47066. Zbl 475.47045.

[4] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, vol. 7, Interscience Publishers, Ltd., London, 1958. MR 22#8302. Zbl 084.10402.

[5] P. L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics, vol. 38, Academic Press, New York, 1970. MR 42#3552. Zbl 215.20203.

[6] K. Goebel and W. A. Kirk, *A fixed point theorem for transformations whose iterates have uniform Lipschitz constant*, Studia Math. **47** (1973), 135–140. MR 49 1242. Zbl 265.47044.

[7] ———, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, 1990. MR 92c:47070. Zbl 708.47031.

[8] K. Goebel, W. A. Kirk, and R. L. Thele, *Uniformly Lipschitzian families of transformations in Banach spaces*, Canad. J. Math. **26** (1974), 1245–1256. MR 50 10919. Zbl 285.47039.

[9] H. Ishihara and W. Takahashi, *Fixed point theorems for uniformly Lipschitzian semigroups in Hilbert spaces*, J. Math. Anal. Appl. **127** (1987), no. 1, 206–210. MR 88j:47076. Zbl 637.47028.

[10] E. A. Lifshitz, *A fixed point theorem for operators in strongly convex spaces*, Voronež. Gos. Univ. Trudy Mat. Fak. **16** (1975), 23–28 (Russian). MR 57 17401.

[11] T. C. Lim, *On some L^p inequalities in best approximation theory*, J. Math. Anal. Appl. **154** (1991), no. 2, 523–528. MR 92d:41029. Zbl 744.41015.

[12] T. C. Lim, H. K. Xu, and Z. B. Xu, *Some L^p inequalities and their applications to fixed point theory and approximation theory*, Progress in approximation theory (Boston), Academic Press, 1991, pp. 609–624. MR 92j:47112.

[13] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 97, Springer-Verlag, New York, 1979, Function spaces. MR 81c:46001. Zbl 403.46022.

[14] N. Mizoguchi and W. Takahashi, *On the existence of fixed points and ergodic retractions for Lipschitzian semigroups in Hilbert spaces*, Nonlinear Anal. **14** (1990), no. 1, 69–80. MR 91h:47071. Zbl 695.47063.

[15] S. A. Pichugov, *The Jung constant of the space L_p*, Mat. Zametki **43** (1988), no. 5, 604–614 (Russian), Translation:Math. Notes **43**(1988),348-358. MR 90a:46037. Zbl 644.46016.

[16] B. Prus and R. Smarzewski, *Strongly unique best approximations and centers in uniformly convex spaces*, J. Math. Anal. Appl. **121** (1987), no. 1, 10–21. MR 88i:41052. Zbl 617.41046.

[17] S. Prus, *On Bynum's fixed point theorem*, Atti Sem. Mat. Fis. Univ. Modena **38** (1990), no. 2, 535–545. MR 91k:47146. Zbl 724.46020.

[18] ______, *Some estimates for the normal structure coefficient in Banach spaces*, Rend. Circ. Mat. Palermo (2) **40** (1991), no. 1, 128–135. MR 92j:46029. Zbl 757.46029.

[19] R. Smarzewski, *Strongly unique best approximation in Banach spaces. II*, J. Approx. Theory **51** (1987), no. 3, 202–217. MR 88j:41064. Zbl 657.41022.

[20] ______, *On an inequality of Bynum and Drew*, J. Math. Anal. Appl. **150** (1990), no. 1, 146–150. MR 91g:47046. Zbl 716.46023.

[21] K. K. Tan and H. K. Xu, *Fixed point theorems for Lipschitzian semigroups in Banach spaces*, Nonlinear Anal. **20** (1993), no. 4, 395–404. MR 94e:47087. Zbl 781.47044.

[22] H. K. Xu, *Fixed point theorems for uniformly Lipschitzian semigroups in uniformly convex spaces*, J. Math. Anal. Appl. **152** (1990), no. 2, 391–398. MR 91m:47089. Zbl 722.47050.

[23] ______, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), no. 12, 1127–1138. MR 92e:47126. Zbl 757.46033.

[24] C. Zalinescu, *On uniformly convex functions*, J. Math. Anal. Appl. **95** (1983), no. 2, 344–374. MR 85a:26018. Zbl 519.49010.

Thakur: Govt. B. H. S. S. Gariaband, Dist. Raipur, M. P. 493889, India

Jung: Department of Mathematics, Dong-A University, Pusan 604–714, Korea
E-mail address: jungjs@seunghak.donga.ac.kr