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NONWANDERING SETS OF MAPS ON THE CIRCLE

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ABSTRACT. Let f be a continuous map of the circle S^1 into itself. And let $R(f),\Lambda(f),\Gamma(f)$, and $\Omega(f)$ denote the set of recurrent points, ω-limit points, γ-limit points, and nonwandering points of f, respectively. In this paper, we show that each point of $\Omega(f)\setminus\overline{R(f)}$ is one-side isolated, and prove that
(1) $\Omega(f)\setminus\Gamma(f)$ is countable and
(2) $\Lambda(f)\setminus\Gamma(f)$ and $\overline{R(f)}\setminus\Gamma(f)$ are either empty or countably infinite.

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1. Introduction. Let I be the unit interval, S^1 the circle, and X a topological space. And let $C^0(X,X)$ denote the set of continuous maps from X into itself. For any $f\in C^0(X,X)$, let $P(f),R(f),\Lambda(f),\Gamma(f)$, and $\Omega(f)$ denote the set of periodic points, recurrent points, ω-limit points, γ-limit points and nonwandering points of f, respectively.

For any $f\in C^0(I,I)$, in 1980, Z. Nitecki [6] has proved that each point of $\Omega(f)\setminus\overline{P(f)}$ is isolated in $\Omega(f)$ if f is piecewise monotone and is not flat on any subinterval of I. In 1984, J. C. Xiong [7] has proved that each point of $\Omega(f)\setminus\overline{P(f)}$ is one-side isolated in $\Omega(f)$, for a continuous self map of interval I. And, in 1988, J. C. Xiong [9] also showed that $\Omega(f)\setminus\Gamma(f)$ is countable and that $\Lambda(f)\setminus\Gamma(f)$ and $\overline{P(f)}\setminus\Gamma(f)$ are either empty or countably infinite.

In this paper, we obtain the following similar results for maps of the circle:

THEOREM 1.1. *Let $f\in C^0(S^1,S^1)$. Then each point of $\Omega(f)\setminus\overline{R(f)}$ is one-side isolated in $\Omega(f)$.*

THEOREM 1.2. *Let $f\in C^0(S^1,S^1)$. Then*
(1) *$\Omega(f)\setminus\Gamma(f)$ is countable.*
(2) *$\Lambda(f)\setminus\Gamma(f)$ and $\overline{R(f)}\setminus\Gamma(f)$ are either empty or countably infinite.*

2. Preliminaries and definitions. Let X be a compact metric space and $f\in C^0(X,X)$. For any positive integer n, we define f^n inductively by $f^1=f$ and $f^{n+1}=f\circ f^n$. Let f^0 denote the identity map of X. The *forward orbit* $\mathrm{Orb}(x)$ of $x\in X$ is the set $\{f^k(x)\mid k=0,1,2,\dots\}$. Usually, the forward orbit of x is simply called the *orbit* of x.

A point $x\in X$ is called a *periodic point* of f if, for some positive integer n, $f^n(x)=x$. The period of x is the least such integer n. We denote the set of periodic points of f by $P(f)$. A point $x\in X$ is called a *recurrent point* of f if there exists a sequence $\{n_i\}$ of positive integers with $n_i\longrightarrow\infty$ such that $f^{n_i}(x)\longrightarrow x$. We denote the set of recurrent points of f by $R(f)$. A point $x \in X$ is called a *nonwandering point* of f if, for every neighborhood U of x, there exists a positive integer m such that $f^m(U) \cap U \neq \phi$. We denote the set of nonwandering points of f by $\Omega(f)$.

A point $y \in X$ is called an *ω-limit point* of x if there exists a sequence $\{n_i\}$ of positive integers with $n_i \longrightarrow \infty$ such that $f^{n_i}(x) \longrightarrow y$. We denote the set of ω-limit points of x by $\omega(x)$. Define $\Lambda(f) = \bigcup_{x \in X} \omega(x)$. A point $y \in X$ is called an *α-limit point* of x if there exists a sequence $\{n_i\}$ of positive integers with $n_i \longrightarrow \infty$ and a sequence $\{y_i\}$ of points such that $f^{n_i}(y_i) = x$ and $y_i \longrightarrow y$. The symbol $\alpha(x)$ denotes the set of α-limit points of x. A point $y \in X$ is called a *γ-limit point* of x if $y \in \omega(x) \cap \alpha(x)$. The symbol $\gamma(x)$ denotes the set of γ-limit points of x and $\Gamma(f) = \bigcup_{x \in X} \gamma(x)$.

Let R be the set of reals and Z be the set of integers. Formally, we think of the circle S^1 as R/Z and use $\pi\colon R \longrightarrow R/Z$ to denote the canonical projection. In fact, the map $\pi\colon R \longrightarrow S^1$ is an example of a covering map since it wraps R around S^1 without doubling back (i.e., without critical points). To study the dynamics of the circle map, it is helpful to use a *lifting*. Let f be a continuous map on the circle. We say that a continuous map F from R into itself is a lifting of f if $f \circ \pi = \pi \circ F$. We use the following notations throughout this paper.

Let $a, b \in S^1$ with $a \neq b$, and let $A \in \pi^{-1}(a), B \in \pi^{-1}(b)$ with $|A - B| < 1$ and $A < B$. Then we write $\pi((A,B)), \pi([A,B]), \pi([A,B))$ and $\pi((A,B])$ to denote the open, closed, and half-open arcs from a counterclockwise to b, respectively, and we denote it by $(a,b), [a,b], [a,b)$, and $(a,b]$. For $x, y \in [a,b]$ with $a \neq b$, let $X \in \pi^{-1}(x), Y \in \pi^{-1}(y)$ with $X, Y \in [A,B]$, then we define for $x, y \in [a,b], x > y$ if and only if $X > Y$. Let C be a subset of a closed arc $[a,b]$, then we define $\sup C = \pi(\sup(\pi^{-1}(C) \cap [A,B]))$ and $\inf C = \pi(\inf(\pi^{-1}(C) \cap [A,B]))$.

In particular, for $a, b, c \in S^1, a < b < c$ means that b lies in the open arc (a,c), that is, $b \in (a,c)$.

Let X be I or S^1 and $Y \subset X$. Let $x \in Y$. A point $x \in X$ is said to be *left-sided isolated* (resp., *right-sided isolated)* in Y if, for some $\epsilon > 0$, $(x - \epsilon, x) \cap Y = \phi$ (resp., $(x, x + \epsilon) \cap Y = \phi$). A point x is said to be *one-side isolated* in Y if x is either left-side or right-side isolated in Y, and a point x which is both a right-sided and a left-sided isolated in Y is said to be *isolated* in Y.

Let $x \in S^1$ and $f \in C^0(S^1, S^1)$ be given. Then we use the symbols $\omega_+(x)$ (resp., $\omega_-(x)$) to denote the set of all points $y \in S^1$ such that there exists a sequence $\{n_i\}$ of positive integers with $n_i \longrightarrow \infty$ such that $f^{n_i}(x) \longrightarrow y$ and $y < \cdots < f^{n_i}(x) < \cdots < f^{n_2}(x) < f^{n_1}(x)$ (*resp.* $f^{n_1}(x) < f^{n_2}(x) < \cdots < f^{n_i}(x) < \cdots < y$). It is clear that if $x \notin P(f)$, then $\omega(x) = \omega_+(x) \cup \omega_-(x)$. Define $\Lambda_+(f) = \bigcup_{x \in S^1} \omega_+(x)$ and $\Lambda_-(f) = \bigcup_{x \in S^1} \omega_-(x)$.

Also, we use the symbols $\alpha_+(x)$ (*resp.* $\alpha_-(x)$) to denote the set of all points $y \in S^1$ such that there exists a sequence $\{n_i\}$ of positive integers with $n_i \longrightarrow \infty$ and a sequence $\{x_i\}$ of points such that $x_i \longrightarrow y, f^{n_i}(x_i) = x$ for every $i > 0$ and $y < \cdots < x_i < \cdots < x_2 < x_1$ (*resp.* $x_1 < x_2 < \cdots < x_i < \cdots < y$). It is clear that if $x \notin P(f)$, then $\alpha(x) = \alpha_+(x) \cup \alpha_-(x)$.

Define $\gamma_+(x) = \omega_+(x) \cap \alpha_+(x)$ and $\gamma_-(x) = \omega_-(x) \cap \alpha_-(x)$. Also, we define $\Gamma_+(f) = \bigcup_{x \in S^1} \gamma_+(x)$ and $\Gamma_-(f) = \bigcup_{x \in S^1} \gamma_-(x)$.

Let Y be an arc in S^1 and let $\overline{Y}$ denote the closure of Y as usual. A point $y \in S^1$ is called a *right-sided* (resp., *left-sided*) *accumulation point* of Y if, for any $z \in S^1$, $(y,z) \cap Y \neq \phi$ (*resp.* $(z,y) \cap Y \neq \phi$).

The right-side closure $\overline{Y}_+$ (*resp.* left-side closure $\overline{Y}_-$) is the union of Y and the set of right-sided (*resp.* left-sided) accumulation points of Y. A point which is both a right-sided and a left-sided accumulation point of Y is called a *two-sided accumulation point* of Y.

3. Main results. The following lemmas are founded in [3].

Lemma 3.1. *Let $f \in C^0(S^1,S^1)$ and $x \in \Omega(f)$. Then we have $x \in \alpha(x)$.*

Lemma 3.2. *Let $f \in C^0(S^1,S^1)$ and $I = [a,b]$ be an* arc *for some $a,b \in S^1$ with $a \neq b$, and let $I \cap P(f) = \phi$.*

(a) *Suppose that there exists $x \in I$ such that $f(x) \in I$ and $x < f(x)$. Then*
  (i) *if $y \in I, x < y$, and $f(y) \notin [y,b]$, then $[x,y]$ f-covers $[f(x),b]$, and*
  (ii) *if $y \in I, y < x$, and $f(y) \notin [y,b]$, then $[y,x]$ f-covers $[f(x),b]$.*

(b) *Suppose that there exists $x \in I$ such that $f(x) \in I$ and $x > f(x)$. Then*
  (i) *if $y \in I, x < y$, and $f(y) \notin [a,y]$, then $[x,y]$ f-covers $[a,f(x)]$, and*
  (ii) *if $y \in I, y < x$, and $f(y) \notin [a,y]$, then $[y,x]$ f-covers $[a,f(x)]$.*

Lemma 3.3. *Let $f \in C^0(S^1,S^1)$. Then we have*

$$P(f) \subset R(f) \subset \Gamma(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f) \subset CR(f). \tag{1}$$

The following lemma is due to [5]

Lemma 3.4. *Let $f \in C^0(S^1,S^1)$, and let $K \subset S^1$ with $f(K) \subset K$. If $x \in \Omega(f) \setminus K$, then $f^n(x) \notin K^\circ$ for any $n \geq 1$.*

The idea of the proof of the following lemma is due to [7].

Lemma 3.5. *Let $f \in C^0(S^1,S^1)$, and let $K \subset S^1$ have only finitely many connected components and $f(K) = K$. Then we have $\overline{K} \setminus K \subset P(f)$.*

Proof. By continuity of f, we have $f(\overline{K}) \subset \overline{f(K)}$. And by the compactness of $\overline{K}, f(\overline{K}) \subset S^1$ is closed. Thus, $\overline{f(K)} \subset \overline{f(\overline{K})} = f(\overline{K})$. Therefore, $f(\overline{K}) = \overline{f(K)} = \overline{K}$. Hence, for each $x \in \overline{K} \setminus K$, there exists $x' \in \overline{K} \setminus K$ such that $f(x') = x$, i.e., $f(\overline{K} \setminus K) = \overline{K} \setminus K$. By the finiteness of $\overline{K} \setminus K$, $\overline{K} \setminus K \subset P(f)$. □

Proposition 3.6. *Let $f \in C^0(S^1,S^1)$. Suppose that $x \in \Omega(f) \setminus \overline{R(f)}$.*

(1) *If $x \in \alpha_+(x)$, then there exists $z \in S^1$ such that $f^i(z,x) \cap (z,x) = \phi$ for all $i \geq 1$.*

(2) *If $x \in \alpha_-(x)$, then there exists $u \in S^1$ such that $f^i(x,u) \cap (x,u) = \phi$ for all $i \geq 1$.*

Proof. We only need to prove part (1). There exists $a,b \in S^1$ such that $x \in (a,b)$ and $(a,b) \cap \mathrm{Orb}(x) = \phi$. Let $V = (a,x)$ and let $W = \cup_{i=0}^{\infty} f^i(V)$. Then $x \in \overline{W}$. Since $x \in \alpha_+(x)$, there exist a positive integer m and a point $y \in (x,b)$ such that $f^m(y) = x$. By Lemma 3.2,

$$[x,y]\ f^m\text{-covers } [a,x]. \tag{2}$$

We claim that $x \notin W$. To show this, suppose that $x \in W$. Then there exist a positive integer j and a point $x_0 \in (a,x)$ such that $f^j(x_0) = x$. By Lemma 3.2,

$$[x_0,x]\, f^j\text{-covers } [x,b]. \tag{3}$$

In particular, $[x_0,x]\, f^j$-covers $[x,y]$.

By (2),

$$[x_0,x]\, f^j\text{-covers } [x_0,y]. \tag{4}$$

Thus,

$$[x_0,x]\, f^{j+m}\text{-covers itself}, \tag{5}$$

and, hence, f^{j+m} has a periodic point in (a,b), a contradiction. Hence, we have $x \in \overline{W} \setminus W$.

Assume that the proposition is false, i.e., for each $z \in (a,x)$, there is some $i \geq 1$ such that $(z,x) \cap f^i(z,x) \neq \phi$. Note that $V \subset f(W)$. Because, for each $y' \in V$, there is some $i \geq 1$ such that $(y',x) \cap f^i(y',x) \neq \phi$. There exists $x_0 \in (y',x)$ such that $f^i(x_0) \in (y',x)$. By Lemma 3.2, either

$$[x_0,x]\, f^i\text{-covers } [f^i(x_0),b] \quad \text{or} \quad [x_0,x]\, f^i\text{-covers } [a,f^i(x_0)]. \tag{6}$$

Particularly, either

$$[x_0,x]\, f^i\text{-covers } [x,b] \quad \text{or} \quad [x_0,x]\, f^j\text{-covers } [a,f^i(x_0)]. \tag{7}$$

If

$$[x_0,x]\, f^i\text{-covers } [x,b], \tag{8}$$

then

$$[x_0,x]\, f^j\text{-covers } [x,y]. \tag{9}$$

By (2),

$$[x,y]\, f^m\text{-covers } [x_0,x]. \tag{10}$$

Hence,

$$[x_0,x]\, f^{i+m}\text{-covers itself}. \tag{11}$$

Thus, f^{j+m} has a periodic point in (a,b). This is a contradiction. Therefore,

$$[x_0,x]\, f^j\text{-covers } [a,f^i(x_0)]. \tag{12}$$

Thus, $y' \in f^i(x_0,x) \subset f^i(V) \subset f(W)$ since $y' \in (a,f^i(x_0))$. Thus, for each $i = 1,2,3,\ldots,l-1$, $f^i(V) \cap f^{l+i}(V) \neq \phi$, and $f^{l+i}(V) \cap f^{2l+i}(V) \neq \phi$, $\ldots$. Therefore, $U_i = \cup_{m=0}^{\infty} f^{ml+i}(V)$ is connected and $W = \cup_{i=0}^{l-1} U_i$ has only finitely many connected components. Now, by Lemma 3.5, $x \in \overline{W} \setminus W \subset P(f)$. This is in contradiction with the assumption of this proposition. □

The following theorem follows immediately from the proposition.

Theorem 3.7. *Let $f \in C^0(S^1,S^1)$. Then each point of $\Omega(f)\setminus\overline{R(f)}$ is one-side isolated in $\Omega(f)$.*

Corollary 3.8. *Let $f \in C^0(S^1,S^1)$. Then $\Omega(f)\setminus\overline{R(f)}$ is countable which is nowhere dense in S^1.*

The following proposition is found in [1].

Proposition 3.9. *Let $f \in C^0(S^1,S^1)$. Then we have*
(1) $\overline{R(f)}_+ \setminus R(f) \subset \Lambda(f)_+$.
(2) $\overline{R(f)}_- \setminus R(f) \subset \Lambda(f)_-$.

Proposition 3.10. *Let $f \in C^0(S^1,S^1)$. Then we have $\overline{R(f)}_+ \cap \overline{R(f)}_- \setminus R(f) \subset \Gamma(f)$.*

Proof. If $P(f) = \phi$, then we have the desired results since $\overline{R(f)} = \Gamma(f)$ [2]. Suppose that $P(f) \neq \phi$. Let $z \in \overline{R(f)}_+ \cap \overline{R(f)}_- \setminus R(f)$. Then there exist $a,b \in S^1$ with $a < b$ such that $z \in (a,b)$ and $(a,b)\cap \mathrm{Orb}(z) = \phi$. By Proposition 3.9, $z \in \Lambda(f)_+ \cap \Lambda(f)_-$. Then there exist y_1, y_2 such that $a < y_1 < z < y_2 < b$ with $z \in \omega(y_1)\cap\omega(y_2)$. Since $\overline{P(f)} = \overline{R(f)}$ [4], $z \in \overline{P(f)}_+ \cap \overline{P(f)}_- \setminus P(f)$. Then there exists u_i of periodic point of f with $a < y_1 < u_1 < u_2 < \cdots < z$ and $u_i \longrightarrow z$. Let p_i be the period of u_i with respect to f. Then $f^{p_i}(u_i) = u_i$ for all $i \geq 1$. Then either $[u_i,z]\, f^{p_i}$-covers $[a,u_i]$ or $[u_i,z]\, f^{p_i}$-covers $[u_i,b]$.

We may assume that, for infinitely many i, either

$$[u_i,z]\, f^{p_i}\text{-covers } [a,u_i] \quad \text{or} \quad [u_i,z]\, f^{p_i}\text{-covers } [u_i,b]. \tag{13}$$

Then we consider two cases.

Case I. $[u_i,z]\, f^{p_i}$-covers $[a,u_i]$ for infinitely many i. There exists $z_i \in [u_i,z]$ such that $f^{p_i}(z_i) = y_1$. Since $u_i \longrightarrow z$, $z_i \longrightarrow z$. Thus, $z \in \alpha(y_1)$ and, hence, $z \in \omega(y_1)\cap\alpha(y_1) \subset \Gamma(f)$.

Case II. $[u_i,z]\, f^{p_i}$-covers $[u_i,b]$ for infinitely many i. There exists $z_i' \in [u_i,z]$ such that $f^{p_i}(z_i') = y_1$. Since $u_i \longrightarrow z$, $z_i' \longrightarrow z$. Thus, $z \in \alpha(y_2)$ and, hence, $z \in \omega(y_2)\cap\alpha(y_2) \subset \Gamma(f)$. □

The idea of the proof of the following lemma is due to [8].

Lemma 3.11. *Let $f \in C^0(S^1,S^1)$ and $Y \subset S^1$. Then $\overline{Y}\setminus(\overline{Y}+\cap\overline{Y}_-)$ is countable.*

Proof. For each $y \in \overline{Y}_+ \setminus \overline{Y}_-$, there is some $u_y \in S^1$ such that $(u_y,y)\cap Y = \phi$. The family of $\{(u_y,y) \mid y \in \overline{Y}_+\setminus\overline{Y}_-\}$ is countable because it is disjoint. Hence, $\overline{Y}_+\setminus\overline{Y}_-$ is countable. Similarly, $\overline{Y}_-\setminus\overline{Y}_+$ is also countable. Therefore,

$$\overline{Y}\setminus(\overline{Y}_+\cap\overline{Y}_-) = (\overline{Y}_+\setminus\overline{Y}_-)\cup(\overline{Y}_-\setminus\overline{Y}_+) \tag{14}$$

is countable. □

Theorem 3.12. *Let $f \in C^0(S^1,S^1)$. Then*
(1) *$\Omega(f)\setminus\Gamma(f)$ is countable.*
(2) *$\Lambda(f)\setminus\Gamma(f)$ and $\overline{R(f)}\setminus\Gamma(f)$ are either empty or countably infinite.*

Proof. (1) We know that $\overline{R(f)} \setminus (\overline{R(f)_+} \cap \overline{R(f)_-})$ is countable by Lemma 3.11. By Proposition 3.10, $\overline{R(f)} \setminus \Gamma(f)$ is also countable. By Corollary 3.8, $\Omega(f) \setminus \overline{R(f)}$ is countable. Hence, $\Omega(f) \setminus \Gamma(f)$ is countable.

(2) It is easy to prove that $f(\omega(x)) = \omega(x)$ and $f(\overline{R(f)}) = \overline{R(f)}$ for $x \in S^1$. Hence, $f(\Lambda(f)) = \Lambda(f)$. Suppose that $\Lambda(f) \setminus \Gamma(f) \neq \phi$ (resp., $\overline{R(f)} \setminus \Gamma(f) \neq \phi$). Then we take $z_1 \in \Lambda(f) \setminus \Gamma(f)$ (resp., $z_1 \in \overline{R(f)} \setminus \Gamma(f)$). We can take $z_2 \in \Lambda(f) \setminus \Gamma(f)$ (resp., $z_2 \in \overline{R(f)} \setminus \Gamma(f)$) such that $z_1 = f(z_2)$. Continuing this process, we can take $z_i \in \Lambda(f) \setminus \Gamma(f)$ (resp., $z_i \in \overline{R(f)} \setminus \Gamma(f)$) such that $z_i = f(z_{i+1})$ for all $i = 1, 2, \ldots$. Since $z_i \notin (f)$ for all $i \geq 1$, the points $z_1, z_2, \ldots$ are pairwise disjoint. Hence, $\Lambda(f) \setminus \Gamma(f)$ (resp., $\overline{R(f)} \setminus \Gamma(f)$) is infinite and, hence, $\Lambda(f) \setminus \Gamma(f)$ (resp., $\overline{R(f)} \setminus \Gamma(f)$) is countably infinite. □

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