Internat. J. Math. & Math. Sci.
Vol. 22, No. 1 (1999) 209–211
S 0161-1712⟨99⟩22209-0


# A CHARACTERIZATION OF RANDOM APPROXIMATIONS

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(Received 1 October 1996 and in revised form 25 March 1997)

ABSTRACT. By using Hahn-Banach theorem, a characterization of random approximations is obtained.

Keywords and phrases. Random approximations, characterization, normed space.

1991 Mathematics Subject Classification. 41A50, 41A65, 60H25.

**1. Introduction and preliminaries.** Random approximation theory is a lively and fascinating field of research lying at the intersection of approximation theory and probability theory. It has received much attention for the past two decades after the publication of a survey article by Bharucha-Reid [4] in 1976. For more details, see [1, 2, 3, 5, 6, 7, 8, 9] and references therein. Random approximation theorems are required for the theory of random equations. The aim of this note is to obtain a characterization of random approximation via the Hahn-Banach theorem. Let $(\Omega,\Sigma)$ be a measurable space with $\Sigma$ a sigma algebra of subsets of $\Omega$. Let $X$ be a normed space and $M$ be a nonempty subset of $X$. A map $T:\Omega\times M\longrightarrow X$ is called a random operator if for each fixed $x\in M$, the map $T(\cdot,x):\Omega\longrightarrow X$ is measurable. Let $B_r(x):=\{z\in X:\|z-x\|\le r\}$ and $\delta(M,x):=\inf_{u\in M}\|x-u\|$. In the sequel, cl, int, and $X'$ stand for the closure, interior, and normed dual of $X$.

In our proof, we use the following geometric version of the Hahn-Banach theorem regarding the separation of convex sets: Let $A$ and $B$ be two disjoint convex sets in a normed space $X$. Moreover, assume that $A$ is open. Then, there is an $f\in X'$ and a real number $c$ such that $\operatorname{Re} f(x)>c$ for $x\in A$, and $\operatorname{Re} f(x)\le c$ for $x\in B$.

## 2. The results

**THEOREM.** *Let $M$ be a nonempty convex subset of a complex normed space $X,T:\Omega\times M\longrightarrow X$ be a random operator, and $\xi:\Omega\longrightarrow M$ be a measurable map such that $T(\omega,\xi(\omega))\notin \operatorname{cl}(M)$. Then $\xi$ is a random best approximation for $T$, i.e., $\|\xi(\omega)-T(\omega,\xi(\omega))\|=\delta(M,T(\omega,\xi(\omega)))$ if and only if there exists $f\in X'$ with the following properties:*

(a) $\|f\|=1$,
(b) $f(T(\omega,\xi(\omega))-\xi(\omega))=\|T(\omega,\xi(\omega))-\xi(\omega)\|$, *and*
(c) $\operatorname{Re} f(x-\xi(\omega))\le 0$ *for all* $x\in M$.

**PROOF.** Necessity: Assume that $\|\xi(\omega)-T(\omega,\xi(\omega))\|=\delta(M,T(\omega,\xi(\omega)))$. Then $M$ and $\operatorname{int}(B_r(T(\omega,\xi(\omega))))$, where $r:=\|T(\omega,\xi(\omega))-\xi(\omega)\|$, are disjoint convex sets. By the separation theorem, there is an $f_{\xi(\omega)}\in X'$ and $c\in R$ such that,

$$\operatorname{Re} f_{\xi(\omega)}(x) \le c \quad \text{for all } x \in M \tag{1}$$

and

$$\operatorname{Re} f_{\xi(\omega)}(y) > c \quad \text{for all } y \in \operatorname{int}\Big(B_r\Big(T(\omega,\xi(\omega))\Big)\Big). \tag{2}$$

The continuity of $f_{\xi(\omega)}$ implies that,

$$\operatorname{Re} f_{\xi(\omega)}(y) \ge c \quad \text{for all } y \in B_r\Big(T(\omega,\xi(\omega))\Big). \tag{3}$$

Since $\xi(\omega) \in M \cap B_r(T(\omega,\xi(\omega)))$, $\operatorname{Re} f_{\xi(\omega)}(\xi(\omega)) = c$. Also, since $T(\omega,\xi(\omega)) \in \operatorname{int} B_r(T(\omega,\xi(\omega)))$, it follows that,

$$\beta := \operatorname{Re} f_{\xi(\omega)}\Big(T(\omega,\xi(\omega))\Big) - c = \operatorname{Re} f_{\xi(\omega)}(T(\omega,\xi(\omega)) - \xi(\omega)) > 0. \tag{4}$$

Let $f = \beta^{-1} r f_{\xi(\omega)}$. This implies that

$$\begin{aligned}\operatorname{Re} f(T(\omega,\xi(\omega)) - \xi(\omega)) &= \operatorname{Re} \beta^{-1} r f_{\xi(\omega)}(T(\omega,\xi(\omega)) - \xi(\omega)) \\ &= \beta^{-1} r \operatorname{Re} f_{\xi(\omega)}(T(\omega,\xi(\omega)) - \xi(\omega)) \\ &= r \\ &= \|(T(\omega,\xi(\omega)) - \xi(\omega))\|.\end{aligned} \tag{5}$$

It further implies that $\|f\| \ge 1$.

Suppose that $\|f\| > 1$. Then there would exist an $h \in X$, with $\|h\| < 1$, such that $f(h)$ is real and $f(h) > 1$. For $y = T(\omega,\xi(\omega)) - rh$, we have,

$$\operatorname{Re} f_{\xi(\omega)}(y) = \operatorname{Re}\Big[f_{\xi(\omega)}\Big(T(\omega,\xi(\omega))\Big) - r f_{\xi(\omega)}(h)\Big] = (c+\beta) - \beta f(h) < c. \tag{6}$$

Since $y \in B_r(T(\omega,\xi(\omega)))$, the above inequality contradicts inequality (3). Hence, $\|f\| = 1$. As $\|f\| = 1$, it follows that $|f(T(\omega,\xi(\omega)) - \xi(\omega))| \le \|T(\omega,\xi(\omega)) - \xi(\omega)\|$. This and equality (5) imply that $f(T(\omega,\xi(\omega)) - \xi(\omega)) = \|T(\omega,\xi(\omega)) - \xi(\omega)\|$. Finally, from inequalities (2) and (3), we obtain,

$$\operatorname{Re} f_{\xi(\omega)}(x - \xi(\omega)) = \operatorname{Re} f_{\xi(\omega)}(x) - \operatorname{Re} f_{\xi(\omega)}(\xi(\omega)) \le 0, \tag{7}$$

for $x \in M$. Since $f = \beta^{-1} r f_{\xi(\omega)}$, where $\beta^{-1} r > 0$,

$$\operatorname{Re} f(x - \xi(\omega)) = \operatorname{Re} \beta^{-1} r f_{\xi(\omega)}(x - \xi(\omega)) \le 0. \tag{8}$$

Sufficiency: Let $M$ be a nonempty set in a complex normed space $X$ and let $\xi : \Omega \longrightarrow M$ be a measurable map. Assume that there is an $f \in X'$ satisfying (a), (b), and (c).

For each $x \in M$,

$$\begin{aligned}\operatorname{Re} f(T(\omega,\xi(\omega)) - x) &\le |f(T(\omega,\xi(\omega)) - x)| \\ &\le \|f\| \|T(\omega,\xi(\omega)) - x\| \\ &= \|T(\omega,\xi(\omega)) - x\|.\end{aligned} \tag{9}$$

It further implies that

$$\begin{aligned} \|T(\omega,\xi(\omega))-x\| &\geq \operatorname{Re} f(T(\omega,\xi(\omega))-x) \\ &= \operatorname{Re} f(T(\omega,\xi(\omega))-\xi(\omega))-\operatorname{Re} f(x-\xi(\omega)) \\ &\geq \operatorname{Re} f(T(\omega,\xi(\omega))-\xi(\omega)) \\ &= \|T(\omega,\xi(\omega))-\xi(\omega)\|. \end{aligned} \tag{10}$$

Hence, $\|T(\omega,\xi(\omega))-\xi(\omega)\| = \delta(M,T(\omega,\xi(\omega)))$. □

## References


[1] I. Beg and N. Shahzad, *Random approximations and random fixed point theorems*, J. Appl. Math. Stochastic Anal. **7** (1994), no. 2, 145–150. CMP 94 14. Zbl 811.47069.

[2] ———, *Applications of the proximity map to random fixed point theorems in Hilbert spaces*, J. Math. Anal. Appl. **196** (1995), no. 2, 606–613. MR 96k:47120. Zbl 868.47044.

[3] ———, *On random approximation and coincidence point theorems for multivalued operators*, Nonlinear Anal. **26** (1996), no. 6, 1035–1041. MR 97c:47073. Zbl 863.47029.

[4] A. T. Bharucha-Reid, *Fixed point theorems in probabilistic analysis*, Bull. Amer. Math. Soc. **82** (1976), no. 5, 641–657. MR 54 1390. Zbl 339.60061.

[5] T.-C. Lin, *Random approximations and random fixed point theorems for continuous* 1*-set-contractive random maps*, Proc. Amer. Math. Soc. **123** (1995), no. 4, 1167–1176. MR 95e:47088. Zbl 834.47049.

[6] N. S. Papageorgiou, *Fixed points and best approximations for measurable multifunctions with stochastic domain*, Tamkang J. Math. **23** (1992), no. 3, 197–203. MR 94b:60074. Zbl 773.60057.

[7] V. M. Sehgal and S. P. Singh, *A theorem on best approximations*, Numer. Funct. Anal. Optim. **10** (1989), no. 1-2, 181–184. MR 90b:41054. Zbl 635.41022.

[8] S. P. Singh and B. Watson, *Proximity maps and fixed points*, J. Approx. Theory **39** (1983), no. 1, 72–76. MR 85c:47057. Zbl 518.47040.

[9] K. K. Tan and X. Z. Yuan, *Random fixed-point theorems and approximation in cones*, J. Math. Anal. Appl. **185** (1994), no. 2, 378–390. MR 95d:47085. Zbl 856.47036.


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