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EXISTENCE AND UNIQUENESS THEOREM FOR A SOLUTION OF FUZZY DIFFERENTIAL EQUATIONS

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ABSTRACT. By using the method of successive approximation, we prove the existence and uniqueness of a solution of the fuzzy differential equation $x'(t) = f(t,x(t))$, $x(t_0) = x_0$. We also consider an ϵ-approximate solution of the above fuzzy differential equation.

Keywords and phrases. Fuzzy set-valued mapping, levelwise continuous, fuzzy derivative, fuzzy integral, fuzzy differential equation, fuzzy solution, fuzzy ϵ-approximate solution.

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1. Introduction. The differential equation

$$x'(t) = f(t,x(t)), \qquad x(t_0) = x_0 \tag{1.1}$$

has a solution provided f is continuous and satisfies a Lipschitz condition by C. Corduneanu [2]. The definition given here generalizes that of Aumann [1] for set-valued mappings. Kaleva [3] discussed the properties of differentiable fuzzy set-valued mappings and gave the existence and uniqueness theorem for a solution of the fuzzy differential equation $x'(t) = f(t,x(t))$ when f satisfies the Lipschitz condition. Also, in [4], he dealt with fuzzy differential equations on locally compact spaces. Park [6, 7] showed existence of solutions for fuzzy integral equations and a fixed point theorem for a pair of generalized nonexpansive fuzzy mappings.

In this paper, we prove the existence and uniqueness theorem of a solution to the fuzzy differential equation (1.1), where $f : I \times E^n \to E^n$ is levelwise continuous and satisfies a generalized Lipschitz condition.

Under some hypotheses, we consider an ϵ-approximate solution of the above fuzzy differential equation.

2. Preliminaries. Let $P_K(R^n)$ denote the family of all nonempty compact convex subsets of R^n and define the addition and scalar multiplication in $P_K(R^n)$ as usual. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$d(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}, \tag{2.1}$$

where $\|\cdot\|$ denotes the usual Euclidean norm in R^n. Then it is clear that $(P_K(R^n),d)$ becomes a metric space.

THEOREM 2.1 [8]. *The metric space $(P_K(R^n),d)$ is complete and separable.*

Let $T = [c,d] \subset R$ be a compact interval and denote

$$E^n = \{u : R^n \longrightarrow [0,1] \mid u \text{ satisfies (i)–(iv) below}\}, \tag{2.2}$$

where

(i) u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,
(ii) u is fuzzy convex,
(iii) u is upper semicontinuous,
(iv) $[u]^0 = \text{cl}\{x \in R^n \mid u(x) > 0\}$ is compact.

For $0 < \alpha \le 1$, denote $[u]^\alpha = \{x \in R^n \mid u(x) \ge \alpha\}$, then from (i)–(iv), it follows that the α-level set $[u]^\alpha \in P_K(R^n)$ for all $0 \le \alpha \le 1$.

If $g : R^n \times R^n \to R^n$ is a function, then, according to Zadeh's extension principle, we can extend g to $E^n \times E^n \to E^n$ by the equation

$$g(u,v)(z) = \sup_{z=g(x,y)} \min\{u(x),v(y)\}. \tag{2.3}$$

It is well known that

$$[g(u,v)]^\alpha = g([u]^\alpha,[v]^\alpha) \tag{2.4}$$

for all $u,v \in E^n$, $0 \le \alpha \le 1$ and g is continuous. Especially for addition and scalar multiplication, we have

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [ku]^\alpha = k[u]^\alpha, \tag{2.5}$$

where u, $v \in E^n$, $k \in R$, $0 \le \alpha \le 1$.

Theorem 2.2 [5]. *If $u \in E^n$, then*

(1) $[u]^\alpha \in P_K(R^n)$ *for all* $0 \le \alpha \le 1$,
(2) $[u]^\alpha \subset [u]^{\alpha_1}$ *for all* $0 \le \alpha_1 \le \alpha_2 \le 1$,
(3) *if* $\{\alpha_k\} \subset [0,1]$ *is a nondecreasing sequence converging to* $\alpha > 0$, *then*

$$[u]^\alpha = \bigcap_{k\ge 1} [u]^{\alpha_k}. \tag{2.6}$$

Conversely, if $\{A^\alpha \mid 0 \le \alpha \le 1\}$ is a family of subsets of R^n satisfying (1)–(3), then there exists $u \in E^n$ such that

$$[u]^\alpha = A^\alpha \quad \text{for } 0 < \alpha \le 1 \tag{2.7}$$

and

$$[u]^0 = \overline{\bigcup_{0<\alpha\le 1} A^\alpha} \subset A^0. \tag{2.8}$$

Define $D : E^n \times E^n \to R^+ \cup \{0\}$ by the equation

$$D(u,v) = \sup_{0\le\alpha\le 1} d([u]^\alpha,[v]^\alpha), \tag{2.9}$$

where d is the Hausdorff metric defined in $P_K(R^n)$.

The following definitions and theorems are given in [3].

Definition 2.1. A mapping $F : T \to E^n$ is *strongly measurable* if, for all $\alpha \in [0,1]$, the set-valued mapping $F_\alpha : T \to P_K(R^n)$ defined by

$$F_\alpha(t) = [F(t)]^\alpha \tag{2.10}$$

is Lebesgue measurable, when $P_K(R^n)$ is endowed with the topology generated by the Hausdorff metric d.

Definition 2.2. A mapping $F : T \to E^n$ is called *levelwise continuous* at $t_0 \in T$ if the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is continuous at $t = t_0$ with respect to the Hausdorff metric d for all $\alpha \in [0,1]$.

A mapping $F : T \to E^n$ is called *integrably bounded* if there exists an integrable function h such that $\|x\| \le h(t)$ for all $x \in F_0(t)$.

Definition 2.3. Let $F : T \to E^n$. The integral of F over T, denoted by $\int_T F(t)$ or $\int_c^d F(t)dt$, is defined levelwise by the equation

$$\begin{aligned}\left(\int_T F(t)dt\right)^\alpha &= \int_T F_\alpha(t)dt \\ &= \left\{\int_T f(t)dt \mid f : T \to R^n \text{ is a measurable selection for } F_\alpha\right\}\end{aligned} \tag{2.11}$$

for all $0 < \alpha \le 1$.

A strongly measurable and integrably bounded mapping $F : T \to E^n$ is said to be *integrable* over T if $\int_T F(t)dt \in E^n$.

Theorem 2.3. *If $F : T \to E^n$ is strongly measurable and integrably bounded, then F is integrable.*

It is known that $[\int_T F(t)dt]^0 = \int_T F_0(t)dt$.

Theorem 2.4. *Let $F, G : T \to E^n$ be integrable, and $\lambda \in R$. Then*
(i) $\int_T (F(t)+G(t))dt = \int_T F(t)dt + \int_T G(t)dt$.
(ii) $\int_T \lambda F(t)dt = \lambda \int_T F(t)dt$.
(iii) $D(F,G)$ *is integrable.*
(iv) $D\left(\int_T F(t)dt, \int_T G(t)dt\right) \le \int_T D(F,G)(t)dt$.

Definition 2.4. A mapping $F : T \to E^n$ is called *differentiable* at $t_0 \in T$ if, for any $\alpha \in [0,1]$, the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at point t_0 with $DF_\alpha(t_0)$ and the family $\{DF_\alpha(t_0) \mid \alpha \in [0,1]\}$ define a fuzzy number $F(t_0) \in E^n$.

If $F : T \to E^n$ is differentiable at $t_0 \in T$, then we say that $F'(t_0)$ is the *fuzzy derivative* of $F(t)$ at the point t_0.

Theorem 2.5. *Let $F : T \to E^1$ be differentiable. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$. Then f_α and g_α are differentiable and $[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$.*

Theorem 2.6. *Let $F : T \to E^n$ be differentiable and assume that the derivative F' is integrable over T. Then, for each $s \in T$, we have*

$$F(s) = F(a) + \int_a^s F'(t)dt. \tag{2.12}$$

Definition 2.5. A mapping $f : T \times E^n \to E^n$ is called *levelwise continuous* at point $(t_0, x_0) \in T \times E^n$ provided, for any fixed $\alpha \in [0,1]$ and arbitrary $\epsilon > 0$, there exists a

$\delta(\epsilon,\alpha) > 0$ such that

$$d\Big([f(t,x)]^\alpha,[f(t_0,x_0)]^\alpha\Big) < \epsilon \tag{2.13}$$

whenever $|t-t_0| < \delta(\epsilon,\alpha)$ and $d([x]^\alpha,[x_0]^\alpha) < \delta(\epsilon,\alpha)$ for all $t \in T$, $x \in E^n$.

3. Fuzzy differential equations. Assume that $f : I \times E^n \to E^n$ is levelwise continuous, where the interval $I = \{t : |t-t_0| \le \delta \le a\}$. Consider the fuzzy differential equation (1.1) where $x_0 \in E^n$. We denote $J_0 = I \times B(x_0,b)$, where $a > 0$, $b > 0$, $x_0 \in E^n$,

$$B(x_0,b) = \{x \in E^n \mid D(x,x_0) \le b\}. \tag{3.1}$$

Definition 3.1. A mapping $x : I \to E^n$ is a solution to the problem (1.1) if it is levelwise continuous and satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s,x(s))\,ds \quad \text{for all } t \in I. \tag{3.2}$$

According to the method of successive approximation, let us consider the sequence $\{x_n(t)\}$ such that

$$x_n(t) = x_0 + \int_{t_0}^t f(s,x_{n-1}(s))\,ds, \quad n = 1,2,\ldots, \tag{3.3}$$

where $x_0(t) \equiv x_0$, $t \in I$.

Theorem 3.1. *Assume that*

(i) *a mapping $f : J_0 \to E^n$ is levelwise continuous,*

(ii) *for any pair $(t,x),(t,y) \in J_0$, we have*

$$d\Big([f(t,x)]^\alpha,[f(t,y)]^\alpha\Big) \le Ld([x]^\alpha,[y]^\alpha), \tag{3.4}$$

where $L > 0$ is a given constant and for any $\alpha \in [0,1]$.

Then there exists a unique solution $x = x(t)$ of (1.1) defined on the interval

$$|t-t_0| \le \delta = \min\left\{a, \frac{b}{M}\right\}, \tag{3.5}$$

where $M = D(f(t,x),\hat{o})$, $\hat{o} \in E^n$ such that $\hat{o}(t) = 1$ for $t = 0$ and 0 otherwise and for any $(t,x) \in J_0$.

Moreover, there exists a fuzzy set-valued mapping $x : I \to E^n$ such that $D(x_n(t),x(t)) \to 0$ on $|t-t_0| \le \delta$ as $n \to \infty$.

Proof. Let $t \in I$, from (3.3), it follows that, for $n = 1$,

$$x_1(t) = x_0 + \int_{t_0}^t f(s,x_0)\,ds \tag{3.6}$$

which proves that $x(t)$ is levelwise continuous on $|t-t_0| \le a$ and, hence on $|t-t_0| \le \delta$. Moreover, for any $\alpha \in [0,1]$, we have

$$d([x_1(t)]^\alpha,[x_0]^\alpha) = d\left(\left[\int_{t_0}^t f(s,x_0)\,ds\right]^\alpha,0\right) \le \int_{t_0}^t d([f(s,x_0)]^\alpha,0)\,ds \tag{3.7}$$

and by the definition of D, we get

$$D(x_1(t),x_0) \le M|t-t_0| \le M\delta = b \tag{3.8}$$

if $|t-t_0| \le \delta$, where $M = D(f(t,x),\hat{o})$, $\hat{o} \in E^n$ and for any $(t,x) \in J_0$.

Now, assume that $x_{n-1}(t)$ is levelwise continuous on $|t-t_0| \le \delta$ and that

$$D(x_{n-1}(t),x_0) \le M|t-t_0| \le M\delta = b \tag{3.9}$$

if $|t-t_0| \le \delta$, where $M = D(f(t,x),\hat{o})$, $\hat{o} \in E^n$ and for any $(t,x) \in J_0$.

From (3.3), we deduce that $x_n(t)$ is levelwise continuous on $|t-t_0| \le \delta$ and that

$$D(x_n(t),x_0) \le M|t-t_0| \le M\delta = b \tag{3.10}$$

if $|t-t_0| \le \delta$, where $M = D(f(t,x),\hat{o})$, $\hat{o} \in E^n$ and for any $(t,y) \in J_0$.

Consequently, we conclude that $\{x_n(t)\}$ consists of levelwise continuous mappings on $|t-t_0| \le \delta$ and that

$$(t,x_n(t)) \in J_0, \qquad |t-t_0| \le \delta, \quad n = 1,2,\dots. \tag{3.11}$$

Let us prove that there exists a fuzzy set-valued mapping $x : I \to E^n$ such that $D(x_n(t), x(t)) \to 0$ uniformly on $|t-t_0| \le \delta$ as $n \to \infty$. For $n = 2$, from (3.3),

$$x_2(t) = x_0 + \int_{t_0}^{t} f(s,x_1(s))\,ds. \tag{3.12}$$

From (3.6) and (3.12), we have

$$\begin{aligned} d\Big([x_2(t)]^\alpha,[x_1(t)]^\alpha\Big) &= d\left(\left[\int_{t_0}^{t} f(s,x_1(s))\,ds\right]^\alpha,\left[\int_{t_0}^{t} f(s,x_0)\,ds\right]^\alpha\right) \\ &\le \int_{t_0}^{t} d\Big([f(s,x_1(s))]^\alpha,[f(s,x_0)]^\alpha\Big)ds \end{aligned} \tag{3.13}$$

for any $\alpha \in [0,1]$.

According to the condition (3.4), we obtain

$$d([x_2(t)]^\alpha,[x_1(t)]^\alpha) \le \int_{t_0}^{t} L d([x_1(s)]^\alpha,[x_0]^\alpha)\,ds \tag{3.14}$$

and by the definition of D, we obtain

$$D(x_2(t),x_1(t)) \le L\int_{t_0}^{t} D(x_1(s),x_0(s))\,ds. \tag{3.15}$$

Now, we can apply the first inequality (3.8) in the right-hand side of (3.15) to get

$$D(x_2(t),x_1(t)) \le ML\frac{|t-t_0|^2}{2!} \le ML\frac{\delta^2}{2!}. \tag{3.16}$$

Starting from (3.8) and (3.16), assume that

$$D(x_n(t),x_{n-1}(t)) \le ML^{n-1}\frac{|t-t_0|^n}{n!} \le ML^{n-1}\frac{\delta^n}{n!} \tag{3.17}$$

and let us prove that such an inequality holds for $D(x_{n+1}(t),x_n(t))$.

Indeed, from (3.3) and condition (3.4), it follows that

$$\begin{aligned} d\Big([x_{n+1}(t)]^\alpha,[x_n(t)]^\alpha\Big) &= d\left(\left[\int_{t_0}^t f(s,x_n(s))ds\right]^\alpha,\left[\int_{t_0}^t f(s,x_{n-1}(s))ds\right]^\alpha\right) \\ &\le \int_{t_0}^t d\Big([f(s,x_n(s))]^\alpha,[f(s,x_{n-1}(s))]^\alpha\Big)ds \\ &\le \int_{t_0}^t Ld\Big([x_n(s)]^\alpha,[x_{n-1}(s)]^\alpha\Big)ds \end{aligned} \tag{3.18}$$

for any $\alpha \in [0,1]$ and from the definition of D, we have

$$D(x_{n+1}(t),x_n(t)) \le L\int_{t_0}^t D(x_n(s),x_{n-1}(s))ds. \tag{3.19}$$

According to (3.17), we get

$$D(x_{n+1}(t),x_n(t)) \le ML^n\int_{t_0}^t \frac{|s-t_0|^n}{n!}ds = ML^n\frac{|t-t_0|^{n+1}}{(n+1)!} \le ML^n\frac{\delta^{n+1}}{(n+1)!}. \tag{3.20}$$

Consequently, inequality (3.17) holds for $n = 1,2,\dots$. We can also write

$$D(x_n(t),x_{n-1}(t)) \le \frac{M}{L}\frac{(L\delta)^n}{n!} \tag{3.21}$$

for $n = 1,2,\dots$, and $|t-t_0| \le \delta$.

Let us mention now that

$$x_n(t) = x_0 + [x_1(t)-x_0] + \cdots + [x_n(t)-x_{n-1}(t)], \tag{3.22}$$

which implies that the sequence $\{x_n(t)\}$ and the series

$$x_0 + \sum_{n=1}^{\infty}[x_n(t)-x_{n-1}(t)] \tag{3.23}$$

have the same convergence properties.

From (3.21), according to the convergence criterion of Weierstrass, it follows that the series having the general term $x_n(t)-x_{n-1}(t)$, so $D(x_n(t),x_{n-1}(t)) \to 0$ uniformly on $|t-t_0| \le \delta$ as $n \to \infty$.

Hence, there exists a fuzzy set-valued mapping $x : I \to E^n$ such that $D(x_n(t),x(t)) \to 0$ uniformly on $|t-t_0| \le \delta$ as $n \to \infty$.

From (3.4), we get

$$d\Big([f(t,x_n(t))]^\alpha,[f(t,x(t))]^\alpha\Big) \le Ld\Big([x_n(t)]^\alpha,[x(t)]^\alpha\Big) \tag{3.24}$$

for any $\alpha \in [0,1]$. By the definition of D,

$$D(f(t,x_n(t)),f(t,x(t))) \le LD(x_n(t),x(t)) \longrightarrow 0 \tag{3.25}$$

uniformly on $|t-t_0| \le \delta$ as $n \to \infty$.

Taking (3.25) into account, from (3.3), we obtain, for $n \to \infty$,

$$x(t) = x_0 + \int_{t_0}^t f(s,x(s))ds. \tag{3.26}$$

Consequently, there is at least one levelwise continuous solution of (1.1).

We want to prove now that this solution is unique, that is, from

$$y(t) = x_0 + \int_{t_0}^{t} f(s, y(s))ds \tag{3.27}$$

on $|t - t_0| \le \delta$, it follows that $D(x(t), y(t)) \equiv 0$. Indeed, from (3.3) and (3.27), we obtain

$$\begin{aligned} d\Big([y(t)]^\alpha, [x_n(t)]^\alpha\Big) &= d\left(\left[\int_{t_0}^{t} f(s, y(s))ds\right]^\alpha, \left[\int_{t_0}^{t} f(s, x_{n-1}(s))ds\right]^\alpha\right) \\ &\le \int_{t_0}^{t} d\Big([f(s, y(s))]^\alpha, [f(s, x_{n-1}(s))]^\alpha\Big)ds \\ &\le \int_{t_0}^{t} Ld\Big([y(s)]^\alpha, [x_{n-1}(s)]^\alpha\Big)ds \end{aligned} \tag{3.28}$$

for any $\alpha \in [0, 1]$, $n = 1, 2, \ldots$.

By the definition of D, we obtain

$$D(y(t), x_n(t)) \le L \int_{t_0}^{t} D(y(s), x_{n-1}(s))ds, \quad n = 1, 2, \ldots. \tag{3.29}$$

But $D(y(t), x_0) \le b$ on $|t - t_0| \le \delta$, $y(t)$ being a solution of (3.27). It follows from (3.29) that

$$D(y(t), x_1(t)) \le bL|t - t_0| \tag{3.30}$$

on $|t - t_0| \le \delta$. Now, assume that

$$D(y(t), x_n(t)) \le bL^n \frac{|t - t_0|^n}{n!} \tag{3.31}$$

on the interval $|t - t_0| \le \delta$. From

$$D(y(t), x_{n+1}(t)) \le L \int_{t_0}^{t} D(y(s), x_n(s))ds \tag{3.32}$$

and (3.31), one obtains

$$D(y(t), x_{n+1}(t)) \le bL^{n+1} \frac{|t - t_0|^{n+1}}{(n+1)!}. \tag{3.33}$$

Consequently, (3.31) holds for any n, which leads to the conclusion

$$D(y(t), x_n(t)) = D(x(t), x_n(t)) \longrightarrow 0 \tag{3.34}$$

on the interval $|t - t_0| \le \delta$ as $n \to \infty$.

This proves the uniqueness of the solution for (1.1). □

Definition 3.2. A mapping $x : L \to E^n$ is an *ϵ-approximate solution* of (1.1) if the following properties hold

(a) $x(t)$ is levelwise continuous on $|t - t_0| \le \delta$,

(b) the derivative $x'(t)$ exists and it is levelwise continuous,

(c) for all t for which $x'(t)$ is defined, we have

$$D\left(x'(t), f(t, x(t))\right) < \epsilon. \tag{3.35}$$

Theorem 3.2. *A mapping $f : J_0 \to E^n$ is levelwise continuous, and let $\epsilon > 0$ be arbitrary. Then there exists at least one ϵ-approximate solution of (1.1), defined on $|t-t_0| \le \delta = \min\{a, b/M\}$, where $M = D(f(t,x),\hat{o})$, $\hat{o} \in E^n$ and for any $(t,x) \in J_0$.*

Proof. In as much as a mapping $f : J_0 \to E^n$ is a levelwise continuous on a compact set J_0, it follows that $f(t,x)$ is uniformly levelwise continuous.

Consequently, for any $\alpha \in [0,1]$, we can find $\delta > 0$ such that $d([f(t,x)]^\alpha, [f(s,y)]^\alpha) < \epsilon$.

Now, we construct the approximate solution for $t \in [t_0, t_0+\delta]$, the construction being completely similar for $t \in [t_0-\delta, t_0]$.

Let us consider a division

$$t_0 < t_1 < \cdots < t_n = t_0 + \delta \tag{3.36}$$

of $[t_0, t_0+\delta]$ such that

$$\max_k (t_k - t_{k-1}) < \lambda = \min\left\{\delta, \frac{\delta}{M}\right\}. \tag{3.37}$$

We define a mapping $x : I \to E^n$ as follows

$$x(t_0) = x_0, \tag{3.38}$$

$$x(t) = x(t_k) + f(t_k, x(t_k))(t - t_k) \tag{3.39}$$

on $t_k < t \le t_{k+1}$, $k = 0, 1, \ldots, n-1$.

It is obvious that a mapping $x : I \to E^n$ satisfies the first two properties from the definition of an ϵ-approximate solution.

Now, we want to prove that the last property is also fulfilled. Indeed, $x'(t) = f(t_k, x(t_k))$ on (t_k, t_{k+1}) and for any $\alpha \in [0,1]$,

$$d\left([x'(t)]^\alpha, [f(t,x(t))]^\alpha\right) = d\left([f(t_k,x(t_k))]^\alpha, [f(t,x(t))]^\alpha\right) < \epsilon \tag{3.40}$$

since $|t - t_k| < \lambda \le \delta$,

$$d([x(t)]^\alpha, [x(t_k)]^\alpha) \le d([f(t_k,x(t_k))]^\alpha, 0)|t-t_k| < M\lambda \le \delta. \tag{3.41}$$

Thus, by the definition of D, we have

$$D(x'(t), f(t,x(t))) < \epsilon \tag{3.42}$$

on $|t-t_0| < \delta$ and $(t,x) \in J_0$.

Theorem 3.2 is completely proved. □

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