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ON θ-GENERALIZED CLOSED SETS

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ABSTRACT. The aim of this paper is to study the class of θ-generalized closed sets, which is properly placed between the classes of generalized closed and θ-closed sets. Furthermore, generalized Λ-sets [16] are extended to θ-generalized Λ-sets and R_0-, $T_{1/2}$- and T_1-spaces are characterized. The relations with other notions directly or indirectly connected with generalized closed sets are investigated. The notion of TGO-connectedness is introduced.

Keywords and phrases. θ-generalized closed, θ-closure, Λ-set, TGO-connected.

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1. Introduction. The first step of generalizing closed sets was done by Levine in 1970 [15]. He defined a set A to be generalized closed if its closure belongs to every open superset of A and introduced the notion of $T_{1/2}$-spaces, which is properly placed between T_0-spaces and T_1-spaces. Dunham [10] proved that a topological space is $T_{1/2}$ if and only if every singleton is open or closed. In [13], Khalimsky, Kopperman, and Meyer proved that the digital line is a typical example of a $T_{1/2}$-space.

Ever since, general topologists extended the study of generalized closed sets on the basis of generalized open sets: regular open, α-open [20], semi-open [14], semi-preopen [1], preopen [19], θ-open [26], δ-open [26], etc.

Extensive research on generalizing closedness was done in recent years as the notions of semi-generalized closed, generalized semi-closed, generalized α-closed, α-generalized closed, generalized semi-preclosed, regular generalized closed, γ-g-closed and (γ,γ')-g-closed sets were investigated [2, 3, 6, 7, 11, 18, 17, 22, 23, 24, 25].

Recently, in [8], Ganster and the first author of this paper defined δ-generalized closed sets and introduced the notion of $T_{3/4}$-spaces, which is properly placed between T_1-spaces and $T_{1/2}$-spaces. They proved that the digital line is $T_{3/4}$.

The aim of this paper is to continue the study of generalized closed sets, this time via the θ-closure operator defined in [26] and characterize $T_{1/2}$-spaces and T_1-spaces in terms of θ-generalized closed sets. Via θ-closure operator, we extend the class of generalized Λ-sets to the class of θ-generalized Λ-sets and study some new characterizations of R_0-spaces and T_1-spaces.

2. Preliminaries concerning generalized closed sets. Throughout this paper, we consider spaces on which no separation axioms are assumed unless explicitly stated. The topology of a given space X is denoted by τ and (X,τ) is replaced by X if there is no chance for confusion. For $A \subseteq X$, the closure and the interior of A in X are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. Sometimes, when there is no chance for confusion, $\overline{A}$ stands for $\mathrm{Cl}(A)$. The θ-interior [26] of a subset A of X is the union of all open sets of X whose closures are contained in A, and is denoted by $\mathrm{Int}_\theta(A)$. The subset A is called *θ-open* [26] if $A = \mathrm{Int}_\theta(A)$. The complement of a θ-open set is called *θ-closed.* Alternatively, a set $A \subset (X,\tau)$ is called θ-closed [26] if $A = \mathrm{Cl}_\theta(A)$, where $\mathrm{Cl}_\theta(A) = \{x \in X : \overline{U} \cap A \neq \emptyset, U \in \tau \text{ and } x \in U\}$. The family of all θ-open sets forms a topology on X and is denoted by τ_θ. We use the name CO-set for sets whose closure is open.

Observation 2.1. (i) *If A is preopen, then* $\mathrm{Cl}_\alpha(A) = \mathrm{Cl}(A) = \mathrm{Cl}_\theta(A)$.
(ii) *Every* CO-*set is preopen.*
(iii) *Every dense subset is a* CO-*set.*
(iv) *Every subset of a space (X,τ) is a* CO-*set if and only if (X,τ) is locally indiscrete.*

Definition 1. A subset A of a space (X,τ) is called
(1) a *generalized closed set* (= *g-closed*) [15] if $A \subseteq U$ and $U \in \tau$ implies that $\overline{A} \subseteq U$,
(2) a *semi-generalized closed set (= sg-closed)* [4] if $A \subseteq U$ and U is semi-open implies that ${}_S\mathrm{Cl}(A) \subseteq U$,
(3) a *generalized α-closed set (=gα-closed)* [17] if $A \subseteq U$ and U is α-open implies that $\mathrm{Cl}_\alpha(A) \subset U$,
(4) a *generalized semi-closed set (= gs-closed)* [2] if $A \subseteq U$ and $U \in \tau$ implies that ${}_\mathrm{s}\mathrm{Cl}(A) \subseteq U$,
(5) an *α-generalized closed set (= α g-closed)* [18] if $A \subseteq U$ and $U \in \tau$ implies that $\mathrm{Cl}_\alpha(A) \subset U$,
(6) a *generalized semi-preclosed set (= gsp-closed)* [7] if $A \subseteq U$ and $U \in \tau$ implies that ${}_{\mathrm{sp}}\mathrm{Cl}(A) \subseteq U$,
(7) a *regular generalized closed set (= r-g-closed)* [23] if $A \subseteq U$ and U is regular open implies that $\bar{A} \subseteq U$.

Definition 2. A topological space (X,τ) is called
(1) *R_0-space* [5] if the closures of every two different points are either disjoint or coincide,
(2) *R_1-space* [5] if every two different points, with distinct closures, have disjoint neighborhoods,
(3) *$T_{1/2}$-space* [15] if every g-closed set is closed, (= every singleton is open or closed [10]),
(4) *kc-space* [27] if every compact set is closed.

Definition 3. Recall that a function $f : (X,\tau) \to (Y,\sigma)$ is called
(1) *g-continuous* [3] if $f^{-1}(V)$ is g-closed in (X,τ) for every closed set V of (Y,σ),
(2) *semi-continuous* [14] if $f^{-1}(V)$ is semi-open in (X,τ) for every open set V of (Y,σ),
(3) *strongly θ-continuous* [21] if, for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(\overline{U}) \subseteq V$.

3. Basic properties of θ-generalized closed sets

Definition 4. A subset A of a topological space (X,τ) is called *θ-generalized closed* (= θ-g-closed) if $\mathrm{Cl}_\theta(A) \subseteq U$, whenever $A \subseteq U$ and U is open in (X,τ).

We denote the family of all θ-generalized closed subsets of a space (X,τ) by $\mathrm{TGC}(X,\tau)$.

The next two results together with the examples following them show that the class of θ-generalized closed sets is properly placed between the classes of g-closed and θ-closed sets.

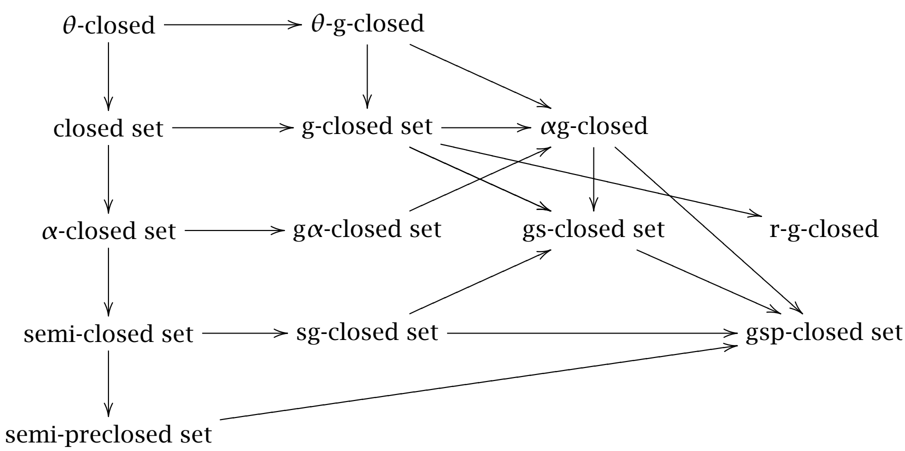
Observation 3.1. *Every θ-closed set is θ-generalized closed.*

Example 3.2. Let $X = \{a,b,c\}$ and let $\tau = \{\emptyset,\{a,b\},X\}$. Set $A = \{a,c\}$. Since the only open superset of A is X, A is clearly θ-generalized closed. But it is easy to see that A is not θ-closed. In fact, it is not even semi-closed since its complement $\{b\}$ has empty interior.

Observation 3.3. *Every θ-generalized closed set is g-closed and hence α g-closed, gs-closed, and r-g-closed.*

Example 3.4. Let $X = \{a,b,c\}$ and let $\tau = \{\emptyset,\{a\},\{a,b\},\{a,c\},X\}$. Set $A = \{c\}$. Clearly, A is closed and hence g-closed. Next, set $U = \{a,c\}$. Note that $X = \mathrm{Cl}_\theta(A) \nsubseteq U \in \tau$. Thus, A is not θ-generalized closed.

The following diagram is an enlargement of a Diagram from [7].



Observation 3.5. *Let (X,τ) be a regular space (not necessarily even T_0). Then a subset A of X is θ-generalized closed if and only if A is generalized closed.*

Lemma 3.6 [12, Thm. 3.1(d), Thm. 3.6(d)]**.** *For a space (X,τ), the following conditions are equivalent*

(1) *X is an R_1-space;*

(2) *for each $x \in X$, $\mathrm{Cl}\{x\} = \mathrm{Cl}_\theta\{x\}$;*

(3) *for each compact set $A \subseteq X$, $\mathrm{Cl}(A) = \mathrm{Cl}_\theta(A)$.*

Proposition 3.7. *If (X,τ) is R_1, then a compact subset K of X is g-closed if and only if K is θ-g-closed.*

Proposition 3.8. *Let A be a preopen subset of a topological space (X,τ). Then the* *following conditions are equivalent*

(1) *A is θ-g-closed;*

(2) *A is g-closed;*

(3) *A is αg-closed.*

Proof. Follows easily from Observation 2.1(i) (note that a preopen g-closed set is a CO-set). □

Lemma 3.9. *If A and B are subsets of a topological space (X,τ), then $\mathrm{Cl}_\theta(A\cup B)=\mathrm{Cl}_\theta(A)\cup\mathrm{Cl}_\theta(B)$ and $\mathrm{Cl}_\theta(A\cap B)\subseteq\mathrm{Cl}_\theta(A)\cap\mathrm{Cl}_\theta(B)$.*

Proposition 3.10. (i) *A finite union of θ-g-closed sets is always a θ-g-closed set.*

(ii) *A countable union of θ-g-closed sets need not be a θ-g-closed set.*

(iii) *A finite intersection of θ-g-closed sets may fail to be a θ-g-closed set.*

Proof. (i) Let $A,B\in\mathrm{TGC}(X)$. Let $U\in\tau$ such that $A\cup B\subseteq U$. By Lemma 3.9, $\mathrm{Cl}_\theta(A\cup B)=\mathrm{Cl}_\theta(A)\cup\mathrm{Cl}_\theta(B)\subseteq U\cup U=U$ since A and B are θ-g-closed. Hence, $A\cup B$ is θ-g-closed.

(ii) Let X be the real line with the usual topology. Since X is regular, by Observation 3.5, every singleton in X is θ-g-closed. Set $A=\bigcup_{i=2}^{\infty}\{1/i\}$. Clearly, A is a countable union of θ-generalized closed sets but A is not θ-generalized closed since $A\subseteq(0,1)$ and $0\in\mathrm{Cl}_\theta(A)$.

(iii) Let $X=\{a,b,c,d,e\}$ and let $\tau=\{\varnothing,\{a,b\},\{c\},\{a,b,c\},X\}$. Set $A=\{a,c,d\}$ and $B=\{b,c,e\}$. Clearly, A and B are θ-generalized closed sets since X is their only open superset. But $C=\{c\}=A\cap B$ is not θ-generalized closed since $C\subseteq\{c\}\in\tau$ and $\mathrm{Cl}_\theta(C)=\{c,d,e\}\nsubseteq\{c\}$. □

Proposition 3.11. *The intersection of a θ-generalized closed set and a θ-closed set is always θ-generalized closed.*

Proof. Let A be θ-generalized closed and let F be θ-closed. Let U be an open set such that $A\cap F\subseteq U$. Set $G=X\backslash F$. Then $A\subseteq U\cup G$. Since G is θ-open, $U\cup G$ is open and since A is θ-generalized closed, $\mathrm{Cl}_\theta(A)\subseteq U\cup G$. Now, by Lemma 3.9, $\mathrm{Cl}_\theta(A\cap F)\subseteq\mathrm{Cl}_\theta(A)\cap\mathrm{Cl}_\theta(F)=\mathrm{Cl}_\theta(A)\cap F\subseteq(U\cup G)\cap F=(U\cap F)\cup(G\cap F)=(U\cap F)\cup\varnothing\subseteq U$. □

Proposition 3.12. *Let $B\subseteq H\subseteq(X,\tau)$ and $(\mathrm{Cl}_\theta)_H(B)$ denote the θ-closure of B in the subspace $(H,\tau\mid H)$. Then*

(i) *$(\mathrm{Cl}_\theta)_H(B)\subseteq\mathrm{Cl}_\theta(B)\cap H$ holds.*

(ii) *If H is open in (X,τ), then $(\mathrm{Cl}_\theta)_H(B)\supset\mathrm{Cl}_\theta(B)\cap H$ holds.*

Theorem 3.13. *Let $B\subseteq H\subseteq(X,\tau)$.*

(i) *If B is θ-g-closed relative to H (i.e., $B\in\mathrm{TGC}(H,\tau\mid H)$), $H\in\mathrm{TGC}(X)$, and $H\in\tau$, then $B\in\mathrm{TGC}(X)$.*

(ii) *If B is θ-g-closed in (X,τ), then B is θ-g-closed relative to H (i.e., $B\in\mathrm{TGC}(H,\tau\mid H)$).*

Proof. (i) Let $B\subseteq U$, where $U\in\tau$. Then $B\subseteq H\cap U$ and, moreover, $(Cl_\theta)_H(B)\subseteq H\cap U$ due to assumption. By Proposition 3.12(ii), $H\cap\mathrm{Cl}_\theta(B)\subseteq H\cap U\subseteq U$. Using the last inclusion, it follows that $H\subseteq H\cup(X\backslash\mathrm{Cl}_\theta(B))=(H\cap\mathrm{Cl}_\theta(B))\cup(X\backslash\mathrm{Cl}_\theta(B))\subseteq U\cup(X\backslash\mathrm{Cl}_\theta(B))$. Since $\mathrm{Cl}_\theta(B)$ is a closed set, $U\cup(X\backslash\mathrm{Cl}_\theta(B))$ is open and thus since $H\in\mathrm{TGC}(X)$, $\mathrm{Cl}_\theta(H)\subseteq U\cup(X\backslash\mathrm{Cl}_\theta(B))$. Now, $\mathrm{Cl}_\theta(B)\subseteq\mathrm{Cl}_\theta(H)\subseteq U\cup(X\backslash\mathrm{Cl}_\theta(B))$. From the last inclusion, it follows that $\mathrm{Cl}_\theta(B) \subseteq U$ or, equivalently, $B \in \mathrm{TGC}(X)$.

(ii) Let V be an open set of $(H, \tau \mid H)$ such that $B \subset V$. Then there exists an open set $G \in \tau$ such that $G \cap H = V$. Since $B \subseteq G \cap H \subseteq G$ and $B \in \mathrm{TGC}(X)$, $\mathrm{Cl}_\theta(B) \subseteq G$. By Proposition 3.12(i), $(\mathrm{Cl}_\theta)_H(B) \subseteq \mathrm{Cl}_\theta(B) \cap H \subseteq G \cap H \subseteq V$. Therefore, B is θ-g-closed relative to H. □

Example 3.14. Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, \{a\}, \{a,b\}, \{a,c,d\}, X\}$. Then $\{\emptyset, X\}$ is the set of all θ-closed sets of (X,τ) and $\mathrm{TGC}(X,\tau) = \{\emptyset, \{b,c\}, \{b,d\}, \{b,c,d\}, \{a,b,d\}, \{a,b,c\}, X\}$. Let $H = \{b,c,d\}$ be a set of X. Then, $\tau \mid H = \{\emptyset, \{b\}, \{c,d\}, H\}$. Note that $\{\emptyset, \{b\}, \{c,d\}, H\}$ is the set of all θ-closed sets of $(H, \tau \mid H)$ and $\mathrm{TGC}(H, \tau \mid H) = \mathcal{P}(H)$. The subset $\{b\}$ of H is θ-g-closed relative to H and H is not open (i.e., $\{b\} \in \mathrm{TGC}(H, \tau \mid H)$, $H \notin \tau$) and $H \in \mathrm{TGC}(X,\tau)$. However, $\{b\} \notin \mathrm{TGC}(X,\tau)$.

Example 3.15. Let (X,τ) be the space in the example above. Set $H = \{a,c,d\}$. Clearly, H is open in (X,τ) and H is not θ-generalized closed in (X,τ). But $B = \{a,c\}$ is θ-generalized closed relative to H. However, B is not θ-generalized closed in (X,τ).

4. Characterizations of $T_{1/2}$-spaces, T_1-spaces and R_0-spaces

Theorem 4.1. *A space (X,τ) is a $T_{1/2}$-space if and only if every θ-generalized closed set is closed.*

Proof.

Necessity. Let $A \subseteq X$ be θ-generalized closed. By Observation 3.3, A is g-closed. Since X is a $T_{1/2}$-space, A is closed.

Sufficiency. Let $x \in X$. If $\{x\}$ is not closed, then $B = X \backslash \{x\}$ is not open and thus the only superset of B is X. Trivially, B is θ-generalized closed. By (2), B is closed or, equivalently, $\{x\}$ is open. Thus, every singleton in X is open or closed. Hence, in the notion of [6, Thm. 6.2(i)], X is a $T_{1/2}$-space. □

Lemma 4.2. *Let $A \subseteq (X,\tau)$ be θ-generalized closed. Then $\mathrm{Cl}_\theta(A) \backslash A$ does not contain a nonempty closed set.*

Theorem 4.3. *A space (X,τ) is a T_1-space if and only if every θ-generalized closed set is θ-closed.*

Proof.

Necessity. Let $A \subseteq X$ be θ-generalized closed and let $x \in \mathrm{Cl}_\theta(A)$. Since X is T_1, $\{x\}$ is closed and thus by Lemma 4.2, $x \notin \mathrm{Cl}_\theta(A) \backslash A$. Since $x \in \mathrm{Cl}_\theta(A)$, then $x \in A$. This shows that $\mathrm{Cl}_\theta(A) \subseteq A$ or, equivalently, that A is θ-closed.

Sufficiency. Let $x \in X$. Assume that $\{x\}$ is not closed. Then $B = X \backslash \{x\}$ is not open and, trivially, B is θ-generalized closed since the only open superset of B is X itself. By (2), B is θ-closed and thus $\{x\}$ is θ-open. Since a singleton is θ-open if and only if it is clopen, $\{x\}$ is clopen. □

The notion of a Λ-set and a generalized Λ-set in a topological space was introduced in [16]. By definition, a subset A of a topological space (X,τ) is called a Λ-*set* [16] if $A = A^\Lambda$, where $A^\Lambda = \cap\{U : U \supset A, U \in \tau\}$. Recall that A is called a generalized Λ-set [16] if $A^\Lambda \subseteq F$, whenever $A \subseteq F$ and F is τ-closed.

Definition 5. (i) For a subset A of (X,τ), we define A_θ^Λ as follows

$$A_\theta^\Lambda = \{x \in X : \mathrm{Cl}_\theta\{x\} \cap A \neq \varnothing\}.$$

In [12], A_θ^Λ is denoted by $\ker_\theta A$.

(ii) A subset A of (X,τ) is called *θ-generalized Λ-set (= θ-g-Λ-set)* if $A_\theta^\Lambda \subseteq F$, whenever $A \subseteq F$ and F is closed in (X,τ).

Observation 4.4. (i) *Every G_δ-set is a Λ-set.*

(ii) [12, Lem. 3.5(a)]. *For any set $A \subseteq X$, $A \subseteq A^\Lambda \subseteq A_\theta^\Lambda \subseteq \mathrm{Cl}_\theta(A)$.*

(iii) *Every θ-closed set is a Λ-set.*

(iv) *Every g-closed Λ-set is closed.*

(v) *Every θ-generalized Λ-set is a generalized Λ-set.*

Remark 4.5. (i) A Λ-set need not be θ-closed. Any singleton of an infinite space with the cofinite topology is a Λ-set (since the space is T_1) but none of the singletons is θ-closed.

(ii) A closed set need not be a Λ-set. In the Sierpinski space ($X = \{a,b\}, \tau = \{\varnothing, \{a\}, X\}$), the set $B = \{b\}$ is closed but B is not a Λ-set. However, in [16, Prop. 3.8], it was shown that in a topological space (X,τ), every subset of X is a generalized Λ-set if and only if every closed set is a Λ-set.

(iii) A generalized Λ-set need not be θ-generalized Λ-set. In an infinite cofinite space X, as mentioned in Remark 4.5, every singleton is a Λ-set and, hence, a generalized Λ-set but none of the singletons is a θ-generalized Λ-set since the θ-closure of every singleton is X.

In [16], it was proved that in T_1-spaces, every set is a Λ-set. Note that the converse is also true.

Proposition 4.6. (i) *A topological space (X,τ) is a T_1-space if and only if every subset of X is a Λ-set.*

(ii) *A topological space (X,τ) is an R_0-space if and only if every singleton of X is a generalized Λ-set.*

Proof. (i) Obvious.

(ii) In [9], Dube showed that a space is R_0 if and only if, for each closed set A, $A = A^\Lambda$. Thus, if X is R_0, then for each singleton $\{x\}$ and each closed set F containing x, we have $\{x\} \subseteq \{x\}^\Lambda \subseteq F^\Lambda = F$. So, $\{x\}$ is a generalized Λ-set. For the reverse assume that $F \subseteq X$ is closed. For each $x \in F$, by assumption, $\{x\}^\Lambda \subseteq F$. Thus, $F^\Lambda = \cup_{x\in F}\{x\}^\Lambda \subseteq F$ according to [16, condition (2.5)]. This shows that $F = F^\Lambda$. □

Observation 4.7. (i) *A subset A of an R_1-space X is generalized Λ-set if and only if A is θ-generalized Λ-set.*

(ii) *In Hausdorff spaces, every subset is a θ-generalized Λ-set.*

(iii) *A topological space X is Hausdorff if and only if X is a kc-space and every closed set of X is a θ-generalized Λ-set.*

5. θ-g-continuous and θ-g-irresolute functions

Definition 6. A function $f : (X,\tau) \to (Y,\sigma)$ is called

(1) *θ-g-continuous* if $f^{-1}(V)$ is θ-g-closed in (X,τ) for every closed set V of (Y,σ),
(2) *θ-g-irresolute* if $f^{-1}(V)$ is θ-g-closed in (X,τ) for every θ-g-closed set V of (Y,σ).

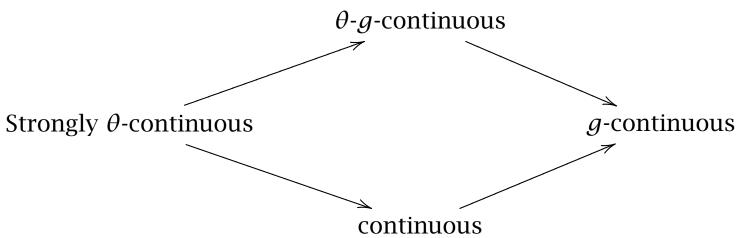
Observation 5.1. *If $f:(X,\tau)\to(Y,\sigma)$ is strongly θ-continuous, then f is θ-g-continuous.*

Example 5.2. Let (X,τ) be the space in Example 3.2. Let $\sigma=\{\emptyset,\{b\},X\}$. Let $f:(X,\tau)\to(X,\sigma)$ be the identity function. Clearly, in the notion of Example 3.2, f is θ-g-continuous but f is not strongly θ-continuous, not even semi-continuous.

Observation 5.3. *Let $f:(X,\tau)\to(Y,\sigma)$ be θ-g-continuous. Then f is g-continuous but not conversely.*

Example 5.4. Let (X,τ) be the space in Example 3.4. Let $\sigma=\{\emptyset,\{a,b\},X\}$. Let $f:(X,\tau)\to(X,\sigma)$ be the identity function. Clearly, f is continuous and hence g-continuous but as shown in Example 3.4, $A=\{c\}\notin \mathrm{TGC}(X,\tau)$ and hence f is not θ-g-continuous.

Example 5.2 and Example 5.4 also show that continuity and θ-g-continuity are independent concepts. Thus, we have the following implications and none of them is reversible.



Example 5.5. Let f be the function in Example 5.2. Let $\nu=\{\emptyset,\{c\},X\}$. Let $g:(X,\sigma)\to(X,\nu)$ be the identity function. It is easily observed that g is also θ-generalized continuous. But the composition function $g\circ f:(X,\tau)\to(X,\nu)$ is not θ-generalized continuous since $\{a,b\}\notin \mathrm{TGC}(X,\tau)$.

Theorem 5.6. *If $f:(X,\tau)\to(Y,\sigma)$ is bijective, open and θ-generalized continuous, then f is θ-g-irresolute.*

Proof. Let $V\in \mathrm{TGC}(Y)$ and let $f^{-1}(V)\subseteq O$, where $O\in\tau$. Clearly, $V\subseteq f(O)$. Since $f(O)\in\sigma$ and since $V\in\mathrm{TGC}(Y)$, $\mathrm{Cl}_\theta(V)\subseteq f(O)$ and thus $f^{-1}(\mathrm{Cl}_\theta(V))\subset O$. Since f is θ-generalized continuous and since $\mathrm{Cl}_\theta(V)$ is closed in Y, $\mathrm{Cl}_\theta(f^{-1}(\mathrm{Cl}_\theta(V)))\subseteq O$ and hence $\mathrm{Cl}_\theta(f^{-1}(V))\subseteq O$. Therefore, $f^{-1}(V)\in\mathrm{TGC}(X)$. Hence, f is θ-g-irresolute. □

Definition 7. A function $f:(X,\tau)\to(Y,\sigma)$ is called *θ-generalized closed* if, for every closed set F of (X,τ), $f(F)$ is θ-g-closed in (Y,σ).

Theorem 5.7. (i) *Let $f:(X,\tau)\to(Y,\sigma)$ be continuous and θ-generalized closed. Then, for a θ-g-closed set A of X, $f(A)$ is θ-g-closed in Y.*

(ii) *Let $f : (X,\tau) \to (Y,\sigma)$ be strongly θ-continuous and closed. Then, f is θ-g-irresolute.*

Proof. (i) Left to the reader.

(ii) Let B be a θ-g-closed set of (Y,σ) and let $U \in \tau$ such that $f^{-1}(B) \subseteq U$. Put $H = \mathrm{Cl}_\theta(f^{-1}(B)) \cap (X \backslash U)$. A map $f : (X,\tau) \to (Y,\sigma)$ is strongly θ-continuous if and only if $f : (X,\tau) \to (Y,\sigma)$ is (γ,id)-continuous in the sense of Ogata [22, Def. 4.12], where $\gamma : \tau \to \mathcal{P}(X)$ is the closure operation and $\mathrm{id} : \sigma \to \mathcal{P}(Y)$ is the identity operation. Using [22, Prop. 4.13(i)] and the fact that $\mathrm{Cl}_\gamma(E) = \mathrm{Cl}_\theta(E)$ and $\mathrm{Cl}_{\mathrm{id}}(E) = \mathrm{Cl}(E)$ for the closure operation γ, the identity operation id and the subset E, we get $f(H) \subseteq f(\mathrm{Cl}_\theta(f^{-1}(B))) \cap f(X \backslash B) \subseteq \mathrm{Cl}(f(f^{-1}(B))) \cap (X \backslash B) \subseteq \mathrm{Cl}(B) \backslash B \subset \mathrm{Cl}_\theta(B) \backslash B$. By Lemma 4.2, $f(H) = \varnothing$ since $f(H)$ is closed. We have $H = \varnothing$ and hence $\mathrm{Cl}_\theta(f^{-1}(B)) \subseteq U$. Therefore, $f^{-1}(B) \in \mathrm{TGC}(X,\tau)$. □

Corollary 5.8. (i) *Under the same assumptions of Theorem 5.6, if (X,τ) is $T_{1/2}$, then (Y,σ) is $T_{1/2}$.*

(ii) *Under the same assumptions of Theorem 5.7(ii), if (X,τ) is $T_{1/2}$ and $f : (X,\tau) \to (Y,\sigma)$ is surjective, then (Y,σ) is $T_{1/2}$.*

Proposition 5.9. *Let $f : (X,\tau) \to (Y,\sigma)$ be a θ-generalized continuous function and let H be a θ-closed subset of X. Then the restriction $f \mid H : (H,\tau \mid H) \to (Y,\sigma)$ is θ-generalized continuous.*

Proof. Let F be a closed subset of (Y,σ). By Proposition 3.11, $H_1 = f^{-1}(F) \cap H$ is θ-generalized closed in (X,τ). Then, by Theorem 3.13(ii), H_1 is θ-g-closed in $(H,\tau \mid H)$. Since $(f \mid H)^{-1}(F) = H_1$, $f \mid H$ is θ-g-continuous. □

Next, we offer the following "Pasting Lemma" for θ-g-continuous functions.

Proposition 5.10. *Let (X,τ) be a topological space such that $X = A \cup B$, where both $A,B \in \mathrm{TGC}(X)$ and $A,B \in \tau$. Let $f : (A,\tau \mid A) \to (Y,\sigma)$ and $g : (B,\tau \mid B) \to (Y,\sigma)$ be θ-generalized continuous functions such that $f(x) = g(x)$ for every $x \in A \cap B$. Then the combination $\alpha : (X,\tau) \to (Y,\sigma)$ is θ-generalized continuous, where $\alpha(x) = f(x)$ for any $x \in A$ and $\alpha(y) = g(y)$ for any $y \in B$.*

Definition 8. A subset A of (X,τ) is called *θ-generalized open* (= θ-g-open) if its complement $X \backslash A$ is θ-generalized closed in (X,τ).

Theorem 5.11. (i) *A subset A of (X,τ) is θ-g-open if and only if $F \subseteq \mathrm{Int}_\theta(A)$, whenever $F \subset A$ and F is closed in (X,τ).*

(ii) *If A is θ-g-open in (X,τ) and B is θ-g-open in (Y,σ), then $A \times B$ is θ-g-open in the product space $(X \times Y, \tau \times \sigma)$.*

Proof. (i) Obvious.

(ii) Let F be a closed subset of $(X \times Y, \tau \times \sigma)$ such that $F \subseteq A \times B$. For each $(x,y) \in F$, $\mathrm{Cl}(\{x\}) \times \mathrm{Cl}(\{y\}) \subseteq \mathrm{Cl}(F) = F \subseteq A \times B$. Then the two closed sets $\mathrm{Cl}(\{x\})$ and $\mathrm{Cl}(\{y\})$ are contained in A and B, respectively. By assumption, $\mathrm{Cl}(\{x\}) \subseteq \mathrm{Int}_\theta(A)$ and $\mathrm{Cl}(\{y\}) \subseteq \mathrm{Int}_\theta(B)$ hold. This implies that, for each $(x,y) \in F$, $(x,y) \in \mathrm{Int}_\theta(A) \times \mathrm{Int}_\theta(B) \subseteq \mathrm{Int}_\theta(A \times B)$ and hence $F \subset \mathrm{Int}_\theta(A \times B)$. By (i) it is clear that $A \times B$ is θ-g-open. □

Proposition 5.12. *The projection $p : (X\times Y,\tau\times\sigma) \to (X,\tau)$ is a θ-g-irresolute map.*

Proof. By definition and Theorem 5.11(ii), for a θ-generalized closed set F of (X,τ), $p^{-1}(x\backslash F) = (X\backslash F)\times Y$ is θ-g-open in $(X\times Y,\tau\times\sigma)$. Therefore, $P^{-1}(F) = F\times Y = X\times Y\backslash(p^{-1}(X\backslash F))$ is θ-generalized closed. □

6. TGO-connected spaces.

In 1991, Balachandran et al. [3] introduced a stronger form of connectedness called GO-connectedness. A set is called *g-open* [15] if its complement is g-closed.

Definition 9. (cf. [15]). A topological space X is called TGO-*connected* (respectively, GO-*connected* [15]) if X cannot be written as a disjoint union of two nonempty θ-g-open (respectively, g-open) sets. A subset of X is called TGO-connected if it is connected as a subspace.

Clearly, every TGO-connected space is connected. The space in [3, Ex. 11] shows that there are connected spaces which are not TGO-connected. Since every θ-generalized closed set is g-closed, every GO-connected space is TGO-connected. Thus, we have the following implications and none of them is reversible.

$$\text{GO-connected} \Longrightarrow \text{TGO-connected} \Longrightarrow \text{Connected}$$

Example 6.1. Let $X = \{a,b,c,d\}$ and let $\tau = \{\varnothing,\{a\},\{a,b\},\{a,c,d\},X\}$. Since $\{c\}$ is both g-closed and g-open, X is not GO-connected. Note that $\text{TGC}(X) = \{\varnothing,\{b,c\}, \{b,d\},\{a,b,c\},\{a,b,d\},\{b,c,d\},X\}$. Hence, X is TGO-connected.

Observation 6.2. (i) [3, Prop. 10]. *For a topological space (X,τ), the following conditions are equivalent.*
- (1) *X is* TGO-*connected;*
- (2) *the only subsets of X, which are both θ-g-open and θ-g-closed, are $\varnothing$ and X;*
- (3) *each θ-generalized continuous function of X into a discrete space Y, with at least two points, is constant.*

(ii) [3, Prop. 12]. *If (X,τ) is a $T_{1/2}$-space, then the following conditions are equivalent*
- (1) *X is* GO-*connected;*
- (2) *X is* TGO-*connected;*
- (3) *X is connected.*

(iii) *A regular space X is* GO-*connected if and only if X is* TGO-*connected.*

(iv) *Let $f : (X,\tau) \to (Y,\sigma)$ be a surjection. Then*
- (a) *If f is θ-generalized continuous and X is* TGO-*connected, then Y is connected.*
- (b) *If f is θ-g-irresolute and X is* TGO-*connected, then Y is* TGO-*connected.*

Corollary 6.3. *If the product space $(X\times Y,\tau\times\sigma)$ is* TGO-*connected, then its factor space (X,τ) is* TGO-*connected.*

Theorem 6.4. *Let $f : (X,\tau) \to (Y,\sigma)$ be θ-g-continuous. Then the image of every θ-closed,* TGO-*connected subset of (X,τ) is connected in (Y,σ).*

Proof. Let H be a θ-closed and TGO-connected set in (X,τ). Then, by Proposition 5.9, the restriction of f to H, $f \mid H : (H,\tau \mid H) \to (Y,\sigma)$, is θ-g-continuous. For f, a function $r_H(f) : (H,\tau \mid H) \to (f(H),\sigma \mid f(H))$ is well defined by $(r_H(f))(x) = f(x)$ for any $x \in H$. Since $f \mid H = j \circ r_H(f)$, where $j : (f(H),\tau \mid f(H)) \to (Y,\sigma)$ is an inclusion. Then it is clear that $r_H(f)$ is θ-g-continuous. In fact, for an open set V of $(f(H),\sigma \mid f(H))$, take an open set $G \in \tau$ such that $G \cap f(H) = V$. Then $r_H(f)^{-1}(V) = (f \mid H)^{-1}(G)$ is θ-g-open. Now, by Observation 6.2(iv), $(f(H),\sigma \mid f(H))$ is connected and hence $f(H)$ is a connected subset of (Y,σ). □

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