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A PROPER SUBCLASS OF MACLANE'S CLASS $\mathscr{A}$

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Abstract. The MacLane's class $\mathscr{A}$ of analytic functions is the class of nonconstant analytic functions in the unit disk that have asymptotic values at a dense subset of the unit circle. In this paper, we define a subclass $\mathscr{R}$ of $\mathscr{A}$ consisting of those functions that have asymptotic values at a dense subset of the unit circle reached along *rectifiable* asymptotic paths. We also show that the class $\mathscr{R}$ is a proper subclass of $\mathscr{A}$ by constructing a function $f \in \mathscr{A}$ that admits no asymptotic paths of finite length.

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1. Preliminaries. In all what follows, f is a nonconstant analytic function on the unit disk.

Definition 1 [2]**.** We say that a simple curve $\Gamma : z(t)$, $0 \le t < 1$ is a boundary path ending at ζ if $|z(t)| \to 1$ as $t \to 1$ and if $\bar{\Gamma} \cap C = \{\zeta\}$. The number a is called an asymptotic value associated with ζ if there is a boundary path Γ such that $f(z(t)) \to a$ as $t \to 1$ and $\bar{\Gamma} \cap C = \{\zeta\}$. In that case, we call Γ an asymptotic path.

Definition 2. We define the set $A(f)$ to be the set of all points ζ at which f has an asymptotic value. In particular, we denote by A_a the set of all points ζ associated with the asymptotic value a, and by A_∞ the set of all points ζ associated with the asymptotic value ∞. We also define the set A^R to be the set of all points on the unit circle at which f has asymptotic values reached along rectifiable asymptotic paths.

Definition 3 [3]**.** If $A(f)$ is a dense subset of C, we say that $f \in \mathscr{A}$, the MacLane class of analytic functions and we define the set $\mathscr{R}$ to be the subset of $\mathscr{A}$ for which A^R is a dense subset of the unit circle C.

Definition 4 [1]**.** Let $H \subset D$ be a relatively closed subset of D. We say that H is an Arakelyan set or $H \in K(D)$ if, for every $z_0 \in D - H$, there is a boundary path $\Gamma_0 \subset D - H$ which connects z_0 to C, that is, if there is a boundary path $\Gamma_0 : z(t)$, $0 \le t < 1$, such that $z(t) \in D - H$, $z(0) = z_0$ and $d(z(t), C) \to 0$ as $t \to 1$. Here, $d(z(t), C)$ denotes the distance from $z(t)$ to C.

Definition 5. Let $H \subset D$. We say that H is a set of tangential approximations (by analytic functions of H) provided that, for each function g continuous on H and analytic on the interior H^0 of H, and for each positive continuous function $\epsilon(t)$, $0 < t < 1$, there is an analytic function f on D such that, for all $z \in H$,

$$|f(z) - g(z)| < \epsilon(d(z, C)). \tag{1}$$

Note that when $H^0 = \phi$, the function g is only required to be continuous on H.

Lemma 1 [1]. *Let $H \in K(D)$, and $H^0 = \phi$. Then H is a set of tangential approximation. This is Arakelyan's theorem.*

2. Main theorem. $\mathcal{R}$ is a proper subset of $\mathcal{A}$.

Proof. The strategy is to construct a function $f \in \mathcal{A}$ by approximating a function g on an Arakelyan set H, with $H^0 = \phi$, using Lemma 1.

The set H is the union of

(a) a sequence of circles converging to the circumference C, each having small equally spaced gaps in it, and

(b) the boundary paths that snake through the gaps.

The gaps in a circle have a total length that approaches zero quickly as the circles approach the circumference. Also, the gaps on consecutive circles are rotated enough so that the asymptotic paths approaching a point of the circumference that pass through the circles only in the gaps (of most of the circles) have infinite length.

The set H so constructed turns out to be a set of tangential approximation. We define a continuous function g on the set H as follows. On each circle (minus its gaps), the function is constant, and on consecutive circles, it has values $0,1,2,3,0,1,2,3,0,1,2,3,\dots$. Along the asymptotic paths of the set H, the function approaches infinity (uniformly) as the modulus approaches 1. Arakelyan's theorem allows us to extend the function g into an analytic function f with the desired property. The set H and the function g and, consequently, f are constructed in such a way that any asymptotic path along which f converges must be either funneled only through the gaps of the circle (if the limit is other than 0, 1, 2, or 3), or must eventually only hit at most one of four circles (hence, pass through the gaps most of the time). In either cases, the asymptotic path must be of infinite length because it is trying to avoid the circles minus the gaps on which the function keeps alternating between the four finite values. More specifically, we start with a sequence of circles $\{C_n^j\}$ converging to the circumference C each having small equally spaced gaps so that the circle minus the gaps form the first set of arcs $\{\gamma_{n,k}^j\}$, $(n = 2,3,\dots,\ k = 1,2,\dots,n,\ j = 0,1,2,3)$. The arcs $\{\gamma_{n,k}^j\}$ will be positioned so that if any asymptotic path were to avoid a 'good number' of them, that asymptotic path would have to be of infinite length. The second set of arcs $\{\Gamma_{p\backslash q}\}$, $(q = 1,2,3,\dots$ and $p = 1,\dots,q-1)$ consists of boundary paths ending at a dense subset of C. The set H is the union of the sets $\{\gamma_{n,k}^j\}$ and $\{\Gamma_{p\backslash q}\}$. First, we show that $H \in K(D)$ then we define a function g on H with the property that $g \to \infty$ as $|z| \to 1$ along the boundary paths $\Gamma_{p\backslash q}$, while the function g takes four different constant values on subarcs $\{\gamma_{n,k}^j\}$ of four consecutive concentric circles C_n^j, $j = 0,1,2,3$. Finally, we show that g and H satisfy the conditions of Lemma 1 and that, for an appropriate choice of $\epsilon(t) > 0$, the function f in the conclusion of the lemma has the desired property: $f \in \mathcal{A} - \mathcal{R}$. □

Construction of the set H.

$$H = \left(\cup_{j=0}^{3} \cup_{n=1}^{\infty} \cup_{k=1}^{n} \left(\gamma_{n,k}^j\right)\right) \cup \left(\cup_{q=1}^{\infty} \cup_{p=1}^{q-1} \left(\Gamma_{p\backslash q}\right)\right). \tag{2}$$

We only take the values of p and q that are relatively prime ($(p,q)=1$).

(1) Description of the arcs $\gamma_{n,k}^j$, $n=2,3,\dots$; $j=0,1,2,3$; $k=1,2,\dots,n$. We start with a sequence of circles $\{C_n^0\}$, $n=2,3,\dots$, centered at the origin, each of radius $r_n^0=1-1/n$, and we let D_n, $n=2,3,\dots$ be the annulus $D_n=\{r_n^0\le|z|<r_{n+1}^0\}$. Let D_1 be the disc $\{z:|z|<r_2^0\}$. Within each annulus D_n, we define three concentric circles C_n^j, ($j=0,1,2,3$) centered at the origin, of radius $r_n^j=r_n^0+(j/(4n(n+1)))$, where $j=1,2,3$. So, the three circles are equally spaced consecutively between C_n^0 and C_{n+1}^0. Note that here and in all that follows, j is used as an index rather than an exponent in r_n^j. On every circle C_n^j, ($n=2,3,\dots$; $j=0,1,2,3$) we arrange n (equally spaced) arcs $\{\gamma_{n,k}^j\}$, ($k=1,2,\dots,n$) of equal length so that the gaps

$$C_n^j-\cup_j\left(\gamma_{n,k}^j\right) \tag{3}$$

consist of n open arcs $\sigma_{n,k}^j$, each of length $\pi r_n^j/2^n$, such that, for $j=0$ or 2, the arcs $\sigma_{n,k}^j$, $k=1,2,\dots,n$, all have their midpoint at the point $r_n^je^{2k\pi i/n}$, whereas for $j=1$ or 3, the arcs $\sigma_{n,k}^j$, $k=1,2,\dots,n$, have their midpoint at $r_n^je^{(2k+1)\pi i/n}$ (a rotation of angle π/n from the previous case). The arcs $\sigma_{n,k}^j$, which are the gaps on the circles, are rotated enough so that a boundary path funneling through a 'good number' of them would be of infinite length. Note that the length of $\gamma_{n,k}^j$ is

$$\left|\gamma_{n,k}^j\right|=\frac{2\pi r_n^j}{n}-\frac{\pi r_n^j}{n}. \tag{4}$$

In the future, we refer to the arcs $\gamma_{n,k}^j(\sigma_{n,k}^j)$, $j=0,1,2,3$, in D_n as the arcs $\gamma_n(\sigma_n)$ provided there is no ambiguity. The distance from a point of σ_n on C_n^j to a point of σ_{n+1} on C_{n+1}^j is at least

$$\frac{\pi r_n^j}{n}-\frac{\pi r_n^j}{2^n}\quad\text{for } j=0,1,2, \tag{5}$$

because of the arrangement of the arcs γ_n and σ_n. Consider a curve $J_n\subset\overline{D_n}$ such that

$$J_n\cap C_n^j\ne\phi\quad\text{for } j=0,1,2,3. \tag{6}$$

Suppose, in addition, that $J_n\cap\gamma_{n,k}^j\ne\phi$ for at most one value of $j\in\{0,1,2,3\}$. Therefore, J_n crosses a pair of circles C_n^j and C_{n+1}^j for some value of $j\in\{0,1,2,3\}$ at points of some arcs σ_n and σ_{n+1}. By the previous remark, the length of such a path is

$$|J_n|>\pi r_n^j\left(\frac{1}{n}-\frac{1}{2^n}\right). \tag{7}$$

(By doing so, we have made sure that the gaps on consecutive circles are rotated enough so that asymptotic paths approaching a point of the circumference that pass through the circles only in the gaps (of 'most' of the circles) have infinite length.)

(2) Description of the boundary paths $\Gamma_{p/q}$. For $q=1,2,\dots$, and $p=1,2,\dots,q-1$ and $(p,q)=1$, let

$$S=\{e^{2\pi ip/q}\}. \tag{8}$$

Note that S is a dense subset of C. We define a sequence of disjoint boundary paths $\{\Gamma_{p/q}\}$, where p and q are as mentioned above, and such that $\Gamma_{p/q}\cap\{z:|z|=r\}$ consists of exactly one point for all r satisfying $r_q^0 \le r < 1$. ($\Gamma_{p/q}$ is not defined for $r < r_q^0$.) We need to construct the arcs $\Gamma_{p/q} \cap C_n^j$ for $j \in \{0,1,2,3\}$, and $n = q, q+1, \dots$, so that

$$\Gamma_{p/q} \cap C_n^j \ne \Gamma_{p'/q'} \cap C_n^j, \tag{9}$$

if $p/q \ne p'/q'$. In the special case of Γ_1, we make

$$\Gamma_1 \cap C_n^j = \begin{cases} r_n^j e^{\pi i/2^{n+1}} & \text{for } j \in \{0,2\}, \\ r_n^j e^{\pi i/2^{n+1}} e^{\pi i/n} & \text{for } j \in \{1,3\}. \end{cases} \tag{10}$$

For $q > 1$, we make

$$\Gamma_{p/q} \cap C_n^j = \begin{cases} r_n^j e^{(2\pi i \lfloor np/q \rfloor/n)+(p\pi i/2^{n+1}q)} & \text{for } j \in \{0,2\} \\ r_n^j e^{(2\pi i \lfloor p/q \rfloor/n)+(p\pi i/2^{n+1}q)} e^{\pi i/n} & \text{for } j \in \{1,3\}, \end{cases} \tag{11}$$

where $\lfloor\ \rfloor$ is the greatest integer function. The first part of the argument, $2\pi i \lfloor np/q \rfloor/n$, is the center of the 'gap' $\sigma_{n,k}^j$, while the second part of the argument, $p\pi i/2^{n+1}q$, determines the distance from the point $\Gamma_{p/q} \cap C_n^j$ to the midpoint of the arc $\sigma_{n,k}^j$; it ensures that the point of intersection is still within $\sigma_{n,k}^j$. Such paths $\Gamma_{p/q}$ must end at the dense subset S of C since, for any p and q as described above,

$$r_n^j \to 1 \qquad \text{and} \qquad e^{(2\pi i \lfloor np/q \rfloor/n)+(p\pi i/2^{n+1}q)} \longrightarrow e^{2\pi i p/q} \quad \text{as } n \to \infty. \tag{12}$$

Note that two different paths intersect the circle C_n^0 at two different points since if $p/q \ne p'/q'$, then

$$r_n^j e^{(2\pi i \lfloor np/q \rfloor/n)+(p\pi i/2^{n+1}q)} \ne r_n^j e^{(2\pi i \lfloor np'/q' \rfloor/n)+(p'\pi i/2^{n+1}q')}. \tag{13}$$

We define

$$\Gamma_{p/q} : z(t), \quad 0 \le t < 1 \tag{14}$$

to be the polygonal arc that begins at

$$z(0) = r_q^0 e^{(2\pi i p/q)+(p\pi i/2^{q+1}q)}. \tag{15}$$

(Observe that the arc $\Gamma_{p/q}$ starts on the arc $\sigma_{q,p}^0$ whose midpoint is of argument $2\pi i p/q$.) In the annulus $\overline{D_n}$, for $n = q, q+1, \dots$ the arc joins $r_n^0 e^{(2\pi i \lfloor np/q \rfloor/n)+(p\pi i/2^{n+1}q)}$ on C_n^0 successively

to $r_n^1 e^{(2\pi i \lfloor np/q \rfloor/n)+(p\pi i/2^{n+1}q)} e^{\pi i/n}$ on C_n^1,

to $r_n^2 e^{(2\pi i \lfloor np/q \rfloor/n)+(p\pi i/2^{n+1}q)}$ on C_n^2,

to $r_n^3 e^{(2\pi i \lfloor np/q \rfloor/n)+(p\pi i/2^{n+1}q)} e^{\pi i/n}$ on C_n^3, and finally

to $r_{n+1}^1 e^{(2\pi i \lfloor np/q \rfloor/n)+(p\pi i/2^{n+1}q)}$ on C_{n+1}^0.

Observe the rotation provided by the $e^{\pi i/n}$ factor. From the definition of $\Gamma_{p/q}$ and from the fact that $\Gamma_{p/q} \cap C_n^j \ne \Gamma_{p'/q'} \cap C_n^j$ in case $p/q \ne p'/q'$, the paths are disjoint. As an illustration, we take the path $\Gamma_{4/5}$. Its initial point is the point $r_5^0 e^{2\pi i(4/5)+4\pi i/2^6 6}$ on the arc $\sigma_{4,5}^0 \subset C_5^0$. The path $\Gamma_{4/5}$ ends at the point of C of the argument $2\pi i(4/5)$. The second circle that $\Gamma_{4/5}$ crosses at a point on an arc σ_n whose midpoint is of argument

$2\pi i(4/5)$ (more informally, the gap at $2\pi i(4/5)$), is the circle C_{10}^0. The kth such circle is the circle C_{5k}^0. (Note that this is a justification for using the greatest integer function. In fact, the solution for n in the equation $e^{2\pi i\lfloor n4/5\rfloor/n} = e^{2\pi i(4/5)}$ is $n = 5k$.) Now, consider the paths $\Gamma_{1/3}$ and $\Gamma_{1/4}$. They both intersect C_8^0 at different points of the same arc $\sigma_{8,2}^0$. In fact,

$$\Gamma_{1/3} \cap C_8^0 = r_8^0 e^{(2\pi i\lfloor 8(1/3)\rfloor/8)+(\pi i/2^9 3)} = r_8^0 e^{(\pi i/2)+(\pi i/2^9 3)} \tag{16}$$

while

$$\Gamma_{1/4} \cap C_8^0 = r_8^0 e^{(2\pi i\lfloor 8(1/4)\rfloor/8)+(\pi i/2^9 3)} = r_8^0 e^{(\pi i/2)+(\pi i/2^9 4)}. \tag{17}$$

In general, there might be more than two paths $\Gamma_{p/q}$ intersecting a circle C_n^0 at different points of the same arc $\sigma_{n,k}^j$. However, there are finitely many (n^2) such paths since $q \le n$ and $p < q$. Note that the arcs $\{\gamma_n\}$ and $\{\sigma_n\}$ have been arranged so that any such path $\Gamma_{p/q}$ is of infinite length since, $\Gamma_{p/q} \cap D_n$ contains an arc joining a point of $\sigma_{n,k}^j$ to a point of $\sigma_{n,t}^{j+1}$, $j = 0,1,2$, similar to the arcs J_n in a previous remark. Finally, define the set H to be the following disjoint union:

$$H = \left(\cup_{j=0}^3 \cup_{n=1}^\infty \cup_{k=1}^n \left(\gamma_{n,k}^j\right)\right) \cup \left(\cup_{q=1}^\infty \cup_{p=1}^{q-1} \left(\Gamma_{p\backslash q}\right)\right). \tag{18}$$

The set H is relatively closed because every arc $\gamma_{n,k}^j$ is closed and $\overline{\Gamma_{p\backslash q}}$ are closed arcs in $\overline{D_n}$.

Proof of $H \in K(D)$. Let $z_0 \in D - H$. We need to find a path $\Gamma_0 \in D - H$ that connects z_0 to C. Note that $z_0 \notin \gamma_n$ for any n, and $z_0 \notin \Gamma_{p/q}$ for any $\Gamma_{p/q} \subset H$. Choose n so that $z_0 \in D_n$. Since $D - H$ is open in D, we can construct a path Γ_0 in $D - H$ that first joins z_0 to some point $z_0' \in \sigma_n - H$ for some $\sigma_n \subset D$. Observe that there might be more than one path $\Gamma_{p/q}$ crossing the same arc σ_n. Let $\Gamma_h/m \subset H$ be the boundary path in H with the property that $\Gamma_{h/m} \cap \sigma_n$ is the closest to z_0' on σ_n. From z_0', we make the path $\Gamma_0 \subset D - H$ follows $\Gamma_{h/m}$ so closely in $D_n - H$ that Γ_0 intersects no path $\Gamma_{p/q} \subset H$, and so that the distance between Γ_0 and $\Gamma_{h/m}$ in the annulus D_n approaches 0 as $n \to \infty$. Thus, Γ_0 connects z_0 to the boundary without intersecting H, and it ends at the point $e^{2\pi i(h/m)}$ as desired.

Construction of the function g on H.

$$g(z) = \begin{cases} j & \text{for } z \in \gamma_n \subset C_n^j,\ j = 0,1,2,3, \\ \dfrac{1}{1-|z|} & \text{for } z \in \Gamma_{p/q} \text{ for all } p/q,\ q = 1,2,\dots;\ p = 1,2,\dots,q-1. \end{cases} \tag{19}$$

Observe that $g \to \infty$ as $|z| \to 1$ along $\Gamma_{p/q}$. Note that H has no interior and g is continuous on H.

Construction of a function $f \in \mathcal{A} - \mathcal{R}$. Since g and H satisfy the conditions of Lemma 1, there corresponds to every positive continuous function $\epsilon(t) : 0 < t < 1$, some analytic function f on D with the property that

$$|f(z) - g(z)| < \epsilon(d(z,C)) \quad \text{for all } z \in H. \tag{20}$$

Let f denote the function corresponding to $\epsilon_0(t) = 1/9$ for all t.

Since

$$g \longrightarrow \infty \text{ as } |z| \longrightarrow 1 \quad \text{when } z \in \Gamma_{p/q}, \tag{21}$$

then

$$f \longrightarrow \infty \text{ as } |z| \longrightarrow 1 \quad \text{when } z \in \Gamma_{p/q} \tag{22}$$

as well, and since the paths $\Gamma_{p/q}$ end at a dense subset S of C, it follows that ∞ is reached as an asymptotic value at a dense subset of the unit circle, that is A_∞ is a dense subset of C, so that $f \in \mathcal{A}$, the MacLane class.

Since the function g has values that differ by *one* on the arcs γ_n of the different circles C_n^j, $j \in \{0,1,2,3\}$, the function f has values that differ at least by $1-2(1/8) = 3/4$ on the arcs γ_n of the different circles C_n^j, $j \in \{0,1,2,3\}$. Therefore, if n is sufficiently large, no asymptotic path can cross the arcs γ_n on more than one of the four circles C_n^j, $j \in \{0,1,2,3\}$; hence, by a previous remark,

$$|\Gamma_{p/q} \cap D_n| \geq \pi r_n^0 \left(\frac{1}{n} - \frac{1}{2^n}\right) \quad \text{for all } n > q. \tag{23}$$

Finally, since the regions D_n are disjoint and

$$\Gamma = \cup_{n=1}^{\infty} \{\Gamma \cap D_n\}, \tag{24}$$

it follows that, for any asymptotic path Γ,

$$|\Gamma| \geq \sum_n \pi r_n^0 \left(\frac{1}{n} - \frac{1}{2^n}\right) = \infty. \tag{25}$$

In other words, no asymptotic path is rectifiable, and so $f \notin \mathcal{R}$. □

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