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ON WEAK SOLUTION OF A HYPERBOLIC DIFFERENTIAL INCLUSION WITH NONMONOTONE DISCONTINUOUS NONLINEAR TERM

GUO XINGMING

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ABSTRACT. In this paper, a hyperbolic differential inclusion with nonmonotone discontinuous and nonlinear term, which the generalized velocity acts as its variable, is studied and the existence and decay of its weak solution are obtained.

Keywords and phrases. Hyperbolic differential inclusion, nonmonotone discontinuous nonlinear multi-valued system, existence of solution, decay of solution.

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1. Introduction. In the present paper, we investigate the initial boundary value problem of the following degenerate multi-valued hyperbolic differential inclusion:

$$\begin{gathered}\ddot{u}(t)+B(u)(t)+\varphi(\dot{u})(t)\ni f(t), \quad \text{a.e. } t\in[0,T],\\ u(x,t)=0, \quad \text{a.e. } (x,t)\in\sum=\partial\Omega\times[0,T],\\ u(0)=u_0, \qquad \dot{u}(0)=u_1,\end{gathered} \tag{1.1}$$

where B is a linear and symmetric operator; φ is a discontinuous, nonmonotone, and nonlinear set-valued mapping.

Physical motivations for studying equation (1.1) come partly from problems of continuum mechanics, where nonmonotone, nonlinear, discontinuous, and multi-valued constitutive laws and boundary constraints lead to the above variational inequalities (differential inclusions). For example, when elastobody is constrainted by boundary friction, (1.1) denotes its control equation; if we study viscoelastical body and the unilateral problem of plate, (1.1) is also their control equation, etc. [10, 8, 5].

When φ is a nonmonotone multi-valued mapping, generally, for such nonmonotone and discontinuous multi-valued systems, usual monotonicity methods are not valid [1, 6]. When φ degenerates into a class of single-valued mappings and satisfies appropriate conditions, inequation (1.1) become an equation. Equation (1.1) and some of its evolution equations with which it is associated have been investigated and applied intensively [7, 3, 2, 9, 11].

In this paper, we investigate the existence and decay of the weak solutions of the hyperbolic in equation (1.1), with φ and B satisfying adequate conditions under zero boundary conditions.

2. Preliminaries. Let Ω be a bounded open set of R^n with regular boundary Γ. Let T denote a positive real number, $Q=\Omega\times[0,T]$. Suppose that $b\in L^{\infty}_{\text{loc}}(R)$. For every

$\rho > 0$, set

$$\underline{b}_p(\xi) = \operatorname*{ess\,inf}_{|\xi_1-\xi|<p} b(\xi_1), \qquad \overline{b}_p(\xi) = \operatorname*{ess\,sup}_{|\xi_1-\xi|<p} b(\xi_1), \tag{2.1}$$

and

$$\underline{b}(\xi) = \lim_{p\to 0^+} \underline{b}_p(\xi), \qquad \overline{b}(\xi) = \lim_{p\to 0^+} \overline{b}_p(\xi), \qquad \varphi(\xi) = [\underline{b}(\xi), \overline{b}(\xi)]. \tag{2.2}$$

Let $J(\xi) = \int_0^\xi b(t)\,dt$. Then $\partial^c J(\xi) \subseteq \varphi(\xi)$, where $\partial^c J(\xi)$ denotes the Clarke-subdifferential of J.

Remark. If $b(\xi_\pm)$ exists for every $\xi \in R$, then $\varphi(\xi) = \partial^c J(\xi)$. Furthermore, if J is convex, $\varphi(\xi)$ is maximal monotone. If b is continuous at ξ, then $\varphi(\xi)$ is single-valued at ξ ([3]).

Let $V = H_0^1(\Omega)$, $\langle \cdot,\cdot \rangle$ denote the dual pair between $V = H_0^1(\Omega)$ and $V' = H^{-1}(\Omega)$, and $(\cdot,\cdot)$ the inner product of $L^2(\Omega)$ which is compatible with the dual pair. Let $|x|_X$ denote the norm of an element x of a Banach space X.

Consider the following initial boundary value problem of a hyperbolic variational inequation (inclusion):

$$\begin{gathered} \ddot{u}(t) + Bu(t) + g(t) = f(t), \quad \text{a.e. } t \in [0,T], \\ g(x,t) \in \varphi(\dot{u}(x,t)), \quad \text{a.e. } (x,t) \in Q_T = \Omega \times [0,T], \\ u(x,t) = 0, \quad \text{a.e. } (x,t) \in \sum = \partial\Omega \times [0,T], \\ u(0) = u_0, \qquad \dot{u}(0) = u_1, \end{gathered} \tag{2.3}$$

where f, u_0, and u_1 are given.

3. Existence of solution

Theorem 1. *Assume that $f \in L^2(0,T;L^2(\Omega)), u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$. If*

(1) $\exists c > 0, |b(\xi)| \le c(1+|\xi|)$, *a.e.* $\xi \in R$,

(2) $B : H_0^1(\Omega) \to H^{-1}(\Omega)$ *is linear, continuous, symmetric, and semicoercive, i.e.,* $\exists c_1 > 0$, $c_2 > 0$, $c_3 \ge 0$,

$$\begin{gathered} |Bv|_{H^{-1}(\Omega)} \le c_1 |v|_{H_0^1(\Omega)}, \\ \langle Bu, v\rangle = \langle Bv, u\rangle \quad \forall u,v \in H_0^1(\Omega), \\ \langle Bv, v\rangle + c_3|v|^2_{L^2(\Omega)} \ge c_2 |v|^2_{H_0^1(\Omega)} \quad \forall v \in H_0^1(\Omega), \end{gathered} \tag{3.1}$$

then there exists a function u, defined in $\Omega \times [0,T]$, such that

$$\begin{aligned} & u \in L^\infty(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)), \\ & \dot{u} \in L^\infty(0,T;L^2(\Omega)) \cap C([0,T];H^{-1}(\Omega)), \\ & \ddot{u} \in L^2(0,T;H^{-1}(\Omega)), \end{aligned} \tag{3.2}$$

and

$$\begin{gathered} \ddot{u}(t) + Bu(t) + g(t) = f(t) \quad \text{in } L^2(0,T;H^{-1}(\Omega)), \\ g(t) \in \varphi(\dot{u}(x,t)), \quad \text{a.e. } (x,t) \in \Omega \times [0,T], \\ u(0) = u_0, \qquad \dot{u}(0) = u_1. \end{gathered} \tag{3.3}$$

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be a subset of $V = H_0^1(\Omega)$ satisfying $\overline{\operatorname{span}\{e_n\}} = V, (e_i, e_j) = \delta_{ij}$. Moreover, let $x_n = \sum_1^n \omega_i^1 e_i \to u_0$ strongly in $V, y_n = \sum_1^n \omega_i^2 e_i \to u_1$ strongly in $L^2(\Omega)$.

Consider the following regularized equation of inequation (1.1)

$$\ddot{\xi}^n = N^n + h, \qquad \xi^n|_{t=0} = \omega^{1n}, \qquad \dot{\xi}^n|_{t=0} = \omega^{2n}, \tag{3.4}$$

where

$$\xi^n = \{\xi_i^n\}_{1\times n}, \qquad \omega^{1n} = \{\omega_i^1\}_{1\times n}, \qquad \omega^{2n} = \{\omega_i^2\}_{1\times n}, \qquad h = \{\langle f, e_i\rangle\}_{1\times n},$$
$$N^n = \{N_i^n\}_{1\times n}, \qquad N_i^n = -\Big\langle B\Big(\sum_1^n \xi_j^n e_j\Big), e_i\Big\rangle - \Big\langle b\Big(\sum_1^n \dot{\xi}_j^n e_j\Big), e_i\Big\rangle, \tag{3.5}$$

where "$\cdot$" denotes time derivate.

Equation (3.4) is a set of second-order ordinary differential equation and its local solution ξ^n exists on $I_n = [0, T_n], 0 < T_n \le T$.

Set $u_n(t) = \sum_1^n \xi_j^n e_j$ $(t \in I_n)$. Equation (3.4) is equivalent to

$$\langle \ddot{u}_n, e_i\rangle = -\langle Bu_n, e_i\rangle - \langle b(\dot{u}_n), e_i\rangle + \langle f, e_i\rangle, \quad i = 1, 2, \ldots, n. \tag{3.6}$$

Multiplying equation (3.6) by $\dot{\xi}_i^n$, summing from $i = 1$ to $i = n$, and integrating over $[0, t]$ $(t \le I_n)$, we get

$$\begin{aligned}
|\dot{u}_n(t)|^2_{L^2(\Omega)} + \langle Bu_n(t), u_n(t)\rangle + 2\int_0^t \langle b(\dot{u}_n), \dot{u}_n\rangle d\tau &\\
= 2\int_0^t \langle f, \dot{u}_n\rangle d\tau + (y_n, y_n) + \langle Bx_n, x_n\rangle&,\\
|b(\dot{u}_n)|^2_{L^2(0,t;L^2(\Omega))} = \int_0^t |b(\dot{u}_n)|^2_{L^2(\Omega)} d\tau &\le c\int_0^t\int_\Omega (1 + |\dot{u}_n|)^2 dx\, d\tau\\
&\le \frac{c}{2}\int_0^t \Big(|\Omega| + |\dot{u}_n(t)|^2_{L^2(\Omega)}\Big) d\tau\\
&\le c_4 + \frac{c}{2}\int_0^t |\dot{u}_n(\tau)|^2_{L^2(\Omega)}\, d\tau,
\end{aligned} \tag{3.7}$$

where $|\Omega|$ denotes the Lebesgue measure of domain Ω.

$$\begin{aligned}
\int_0^t \langle b(\dot{u}_n), \dot{u}_n\rangle d\tau &\le |b(\dot{u}_n)|_{L^2(0,t;L^2(\Omega))} \cdot |\dot{u}_n|_{L^2(0,t;L^2(\Omega))}\\
&\le \frac{1}{2}\Big(|b(\dot{u}_n)|^2_{L^2(0,t;L^2(\Omega))} + |\dot{u}_n|^2_{L^2(0,t;L^2(\Omega))}\Big)\\
&\le \frac{1}{2}\Big\{c_4 + \Big(\frac{c}{2} + 1\Big)\int_0^t |\dot{u}_n(\tau)|^2_{L^2(\Omega)} d\tau\Big\},
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\int_0^t \langle f, \dot{u}_n\rangle d\tau &\le |f|_{L^2(0,T;L^2(\Omega))} \cdot |\dot{u}_n|_{L^2(0,t;L^2(\Omega))}\\
&\le \frac{1}{2}\Big(|f|^2_{L^2(0,T;L^2(\Omega))} + |\dot{u}_n|^2_{L^2(0,t;L^2(\Omega))}\Big).
\end{aligned} \tag{3.9}$$

From (3.7), $\exists c_5 > 0$ such that

$$\begin{aligned}
|\dot{u}_n(t)|^2_{L^2(\Omega)} &+ c_2|u_n(t)|^2_{H_0^1(\Omega)}\\
&\le c_5 + c_3|u_n(t)|^2_{L^2(\Omega)} + \frac{1}{2}\Big(\frac{c}{2} + 1\Big)\int_0^t |u_n(\tau)|^2_{L^2(\Omega)} d\tau.
\end{aligned} \tag{3.10}$$

We note that

$$\begin{gathered} u_n(t) = u_n(0) + \int_0^t \dot{u}_n \, d\tau, \\ |u_n(t)|_{L^2(\Omega)} \le |u_n(0)|_{L^2(\Omega)} + \int_0^t |\dot{u}_n|_{L^2(\Omega)} \, d\tau. \end{gathered} \tag{3.11}$$

By Hölder inequality, $\exists c_6, c_7 > 0$ such that

$$|u_n(t)|^2_{L^2(\Omega)} \le c_6 + c_7 \int_0^t |\dot{u}_n(\tau)|^2_{L^2(\Omega)} d\tau. \tag{3.12}$$

From (3.10) and (3.12), we obtain: $\exists c_8, c_9 > 0$ such that

$$|\dot{u}_n(t)|^2_{L^2(\Omega)} + c_2 |u_n(t)|^2_{H^1_0(\Omega)} \le c_8 + c_9 \int_0^t |\dot{u}_n(\tau)|^2_{L^2(\Omega)} d\tau, \quad (t \in I_n). \tag{3.13}$$

Hence,

$$|\dot{u}_n(t)|^2_{L^2(\Omega)} \le c_8 + c_9 \int_0^t |\dot{u}_n(\tau)|^2_{L^2(\Omega)} d\tau, \quad (t \in I_n). \tag{3.14}$$

By Gronwall's inequality, we have

$$|\dot{u}_n(t)|^2_{L^2(\Omega)} \le c_8 \exp(c_9 t), \quad (t \in I_n). \tag{3.15}$$

Therefore, from (3.12), (3.15), and (3.16), there exists $c_{10} > 0$ such that

$$|\dot{u}_n(t)|_{L^2(\Omega)} \le C_{10}, \qquad |u_n(t)|_{L^2(\Omega)} \le C_{10}, \qquad |u_n(t)|_{H^1_0(\Omega)} \le C_{10}, \quad (t \in I_n), \tag{3.16}$$

where $c_4, c_5, c_6, c_7, c_8, c_9, c_{10}$ are positive constants independent of n and T_n. The estimate (3.16) implies that we can prolongate the solution of equation (3.4) to the interval $[0, T]$, i.e., $I_n = [0, T]$ $(\forall n)$.

From (3.6), we see that, for every $\eta \in \operatorname{span}\{e_1, e_2, \dots, e_n\}$,

$$\begin{aligned} |\langle \ddot{u}_n, \eta \rangle| \le\; & |f(t)|_{L^2(\Omega)} \cdot |\eta|_{L^2(\Omega)} + |b(\dot{u}_n)|_{L^2(\Omega)} \cdot |\eta|_{L^2(\Omega)} \\ & + |B| \cdot |u_n|_{H^1_0(\Omega)} \cdot |\eta|_{H^1_0(\Omega)}, \end{aligned} \tag{3.17}$$

where $|B|$ is the norm of the linear continuous operator B.

$$\begin{aligned} |\ddot{u}_n(t)|_{H^{-1}(\Omega)} &= \sup_{|\eta|_V = 1} |\langle \ddot{u}_n(t), \eta \rangle| = \sup_{\substack{\eta \in \operatorname{span}\{e_1, \dots, e_n\} \\ |\eta|_V = 1}} |\langle \ddot{u}_n(t), \eta \rangle| \\ &\le c_{11} \Big(|f(t)|_{L^2(\Omega)} + |b(\dot{u}_n)|_{L^2(\Omega)} \Big) + |B| \cdot |u_n(t)|_{H^1_0(\Omega)}, \end{aligned} \tag{3.18}$$

where c_{11} is the imbedding constant which $H^1_0(\Omega)$ imbeds in $L^2(\Omega)$.

$$|b(\dot{u}_n)(t)|^2_{L^2(\Omega)} \le c \int_\Omega (1 + |\dot{u}_n(t)|)^2 dx \le \frac{c}{2} \Big(|\Omega| + |\dot{u}_n(t)|^2_{L^2(\Omega)} \Big). \tag{3.19}$$

This shows that $\{b(\dot{u}_n)\}$ is also a bounded subset of $L^\infty(0, T; L^2(\Omega))$. Hence, (3.18) implies that $\{\ddot{u}_n\}$ is a bounded subset of $L^2(0, T; H^{-1}(\Omega))$.

Therefore, there exists a subsequence of $\{u_n\}$ (still denoted by itself) and a function u such that $u \in L^\infty(0,T;H_0^1(\Omega)), \dot{u} \in L^\infty(0,T;L^2(\Omega)), \ddot{u} \in L^2(0,T;H^{-1}(\Omega))$ satisfying

$$\begin{aligned} u_n &\longrightarrow u \quad \text{weakly-star in } L^\infty(0,T;H_0^1(\Omega)),\\ \dot{u}_n &\longrightarrow \dot{u} \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)),\\ \ddot{u}_n &\longrightarrow \ddot{u} \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)),\\ b(\dot{u}_n) &\longrightarrow g \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)). \end{aligned} \tag{3.20}$$

Furthermore, $\dot{u}_n(t,x) \to \dot{u}(t,x)$, a.e. $(t,x) \in [0,T]\times\Omega$.

It is well known that the space $W(V)$, defined by $W(V) = \{u \in L^2(0,T;V), \dot{u} \in L^2(0,T;V')\}$ with the norm $|u|_W = |u|_{L^2(0,T;V)} + |\dot{u}|_{L^2(0,T;V')}$, is continuously imbedded in $C([0,T];\, L^2(\Omega))$. It is obvious that $u \in C(0,T;L^2(\Omega))$, $\dot{u} \in C(0,T;H^{-1}(\Omega))$. Hence, $u(0)$, $\dot{u}(0)$ make sense.

For $\lambda \in L^2(0,T)$, by (3.6), we have

$$\begin{aligned} \int_0^T \langle \ddot{u}_n, \lambda e_i\rangle dt = &-\int_0^T \langle B(u_n),\lambda e_i\rangle dt - \int_0^T \langle b(\dot{u}_n),\lambda e_i\rangle dt \\ &+ \int_0^T \langle f(t),\lambda e_i\rangle dt, \quad i = 1,2,\dots,n. \end{aligned} \tag{3.21}$$

For every given positive integer i, let $n \to \infty$ in (3.21). Then, it follows that

$$\begin{aligned} \int_0^T \langle \ddot{u}, \lambda e_i\rangle dt = &-\int_0^T \langle B(u),\lambda e_i\rangle dt - \int_0^T \langle g,\lambda e_i\rangle dt \\ &+ \int_0^T \langle f(t),\lambda e_i\rangle dt, \quad i = 1,2,\dots. \end{aligned} \tag{3.22}$$

Therefore,

$$\ddot{u}(t) + B(u) + g(t) = f(t) \quad \text{in } L^2(0,T;H^{-1}(\Omega)). \tag{3.23}$$

In the following, we show that

$$g(x,t) \in \varphi(\dot{u}(x,t)), \quad \text{a.e. } (x,t) \in Q_T = \Omega\times[0,T]. \tag{3.24}$$

Since $\dot{u}_n(x,t) \to \dot{u}(x,t)$, a.e. $(x,t) \in Q_T$, by Eropob's theorem [12], for every $\delta > 0$, there exists a subset $Q_\delta \subseteq Q_T = \Omega\times[0,T], |Q_\delta| \le \delta$,

$$\dot{u}_n(x,t) \longrightarrow \dot{u}(x,t) \quad \text{uniformly in } Q_T/Q_\delta. \tag{3.25}$$

That is, for every $\varepsilon > 0$, there exists a positive integer $\overline{N}$, when $n \ge \overline{N}$,

$$\left|\dot{u}_n(x,t) - \dot{u}(x,t)\right| \le \varepsilon \quad \forall (x,t) \in Q_T/Q_\delta. \tag{3.26}$$

It is obvious that

$$\underline{b}_\varepsilon(\dot{u}(x,t)) \le b(\dot{u}_n(x,t)) \le \overline{b}_\varepsilon(\dot{u}(x,t)) \quad \forall (x,t) \in Q_T/Q_\delta. \tag{3.27}$$

For every $\mu \in L^1(0,T;L^2(\Omega))$, $\mu \ge 0$

$$\begin{aligned} \int_{Q_T\setminus Q_\delta} g(x,t)\mu(x,t)\,dx\,dt &= \lim_{n\to\infty}\int_{Q_T\setminus Q_\delta} b(\dot{u}_n(x,t))\mu(x,t)\,dx\,dt \\ &\le \int_{Q_T\setminus Q_\delta} \overline{b}_\varepsilon(\dot{u}(x,t))\mu(x,t)\,dx\,dt, \end{aligned} \tag{3.28}$$

$$\int_{Q_T\setminus Q_\delta} g(x,t)\mu(x,t)\,dx\,dt \le \limsup_{\varepsilon\to 0^+}\int_{Q_T\setminus Q_\delta} \overline{b}_\varepsilon(\dot u(x,t))\mu(x,t)\,dx\,dt \le \int_{Q_T\setminus Q_\delta} \overline{b}(\dot u(x,t))\mu(x,t)\,dx\,dt. \tag{3.29}$$

Analogously, we can obtain

$$\int_{Q_T\setminus Q_\delta} g(x,t)\mu(x,t)\,dx\,dt \ge \int_{Q_T\setminus Q_\delta} \underline{b}(\dot u(x,t))\mu(x,t)\,dx\,dt. \tag{3.30}$$

Hence,

$$g(x,t)\in\varphi(\dot u(x,t)),\quad \text{a.e. } (x,t)\in Q_T/Q_\delta. \tag{3.31}$$

Letting $\delta\longrightarrow 0^+$, we get

$$g(x,t)\in\varphi(\dot u(x,t)),\quad \text{a.e. } (x,t)\in Q_T=\Omega\times[0,T]. \tag{3.32}$$

Let $\lambda\in C^1[0,T],\lambda(T)=0$. Integrating by parts the left-hand sides of equations (3.21) and (3.22) gives

$$\begin{aligned} -\langle \dot u_n(0),\lambda(0)e_i\rangle-\int_0^T\langle \dot u_n,\dot\lambda e_i\rangle dt &= \text{ the right of (3.21)},\\ -\langle \dot u(0),\lambda(0)e_i\rangle-\int_0^T\langle \dot u,\dot\lambda e_i\rangle dt &= \text{ the right of (3.22)}. \end{aligned} \tag{3.33}$$

Making a comparison between the two equations of (3.33), we get

$$\lim_{n\to\infty}\langle \dot u_n(0)-\dot u(0),e_i\rangle=0,\quad i=1,2,\dots. \tag{3.34}$$

Therefore,

$$\dot u_n(0)\longrightarrow \dot u(0) \text{ weakly in } H^{-1}(\Omega). \tag{3.35}$$

The uniqueness of the limit implies that $\dot u(0)=u_1$ (in $H^{-1}(\Omega)$).

Let $\lambda\in C^2[0,T]$, $\lambda(T)=0$, $\dot\lambda(T)=0$. Analogously, integrating by parts the left-hand sides of equations (3.33), and making a comparison with the obtained results again gives: $u(0)=u_0$(in $L^2(\Omega)$). This completes the proof. □

Theorem 2. *Let $f\in L^2(0,T;L^2(\Omega))$, $u_0\in H_0^1(\Omega)$, $u_1\in L^2(\Omega)$. Assume that b satisfies (1)′ $b(\xi)\xi\ge-\delta$ for almost everywhere $\xi\in R$, and $\exists\bar c>0, |b(\xi)|\le\bar c(1+|\xi|^p)$, a.e. $\xi\in R$, if $n>2$, $0<p\le(2n)/(n-2)$; if $n\le 2$, $0\le p<\infty$, and condition (2) of Theorem 1 is valid. Then there exists a function v, defined in $\Omega\times[0,T]$, satisfying*

$$v\in L^\infty(0,T;H_0^1(\Omega)),\qquad \dot v\in L^\infty(0,T;L^2(\Omega)), \tag{3.36}$$

and

$$\begin{gathered} \ddot v+B(v)+\overline{g}(t)=f(t)\quad \text{in } L^1(0,T;H^{-1}(\Omega)+L^1(\Omega)),\\ \overline{g}(x,t)\in\varphi(\dot v(x,t)),\quad \text{a.e. } (x,t)\in Q_T=\Omega\times[0,T],\\ v(0)=u_0,\qquad \dot v(0)=u_1. \end{gathered} \tag{3.37}$$

Proof. Analogously to Theorem 1, we still may get (3.7), where $\{e_n\}_{n=1}^{\infty}$ is a basis of $H_0^1(\Omega)\cap L^{\infty}(\Omega)$ satisfying $(e_i,e_j)=\delta_{ij}$. Under assumption $(1)'$, $\int_0^t\langle b(\dot{u}_n),\dot{u}_n\rangle d\tau\geq-\delta$. From (3.7), we have

$$|\dot{u}_n(t)|_{L^2(\Omega)}^2+c_2|u_n(t)|_{H_0^1(\Omega)}^2\leq c_4+c_3|u_n(t)|_{L^2(\Omega)}^2+2\int_0^t\langle f,\dot{u}_n\rangle d\tau. \tag{3.38}$$

It is easy to see that equations (3.12), (3.13), (3.15), and (3.16) are still true and the solution of equation (3.4) may still be extended to the interval $[0,T]$.

By Sobolev Imbedding Theorem, we have, for a.e. $t\in[0,T]$, if $n>2$, then

$$H_0^1(\Omega)\subset L^{p^*}(\Omega)\subset L^p(\Omega),\qquad p^*=\frac{2n}{n-2}, \tag{3.39}$$

and if $n=2$, then

$$H_0^1(\Omega)\subset L^q(\Omega)\quad\forall 1\leq q\leq\infty, \tag{3.40}$$

so

$$|u_n(t)|_{L^p(\Omega)}\leq c_{11}|u_n(t)|_{H_0^1(\Omega)}\leq c_{11}c_9; \tag{3.41}$$

if $n=1$, then

$$H_0^1(\Omega)\subset C(\overline{\Omega}),\ \text{and ditto,}\ |u_n(t)|_{C(\overline{\Omega})}=\max_{x\in\overline{\Omega}}|u_n(x,t)|\leq c_{11}c_9, \tag{3.42}$$

where $\overline{\Omega}$ denotes the Closure of Ω and c_{11} is the imbedding constant which $H_0^1(\Omega)$ imbeds in $L^p(\Omega)$ or $C(\overline{\Omega})$. Everyway, we always have that $b(\dot{u}_n)\in L^{\infty}(0,T;L^1(\Omega))$ and $\{b(\dot{u}_n)\}$ is a bounded subset of $L^{\infty}(0,T;L^1(\Omega))$.

Therefore, there exists a subsequence of $\{u_n\}$, still denoted by itself, and a function v, such that $v\in L^{\infty}(0,T;H_0^1(\Omega)),\dot{v}\in L^{\infty}(0,T;L^2(\Omega))$, satisfying

$$\begin{aligned} u_n&\longrightarrow v\quad\text{weakly-star in } L^{\infty}(0,T;H_0^1(\Omega)),\\ \dot{u}_n&\longrightarrow \dot{v}\quad\text{weakly-star in } L^{\infty}(0,T;L^2(\Omega)),\\ b(\dot{u}_n)&\longrightarrow \overline{g}\quad\text{weakly-star in } L^{\infty}(0,T;L^1(\Omega)).\end{aligned} \tag{3.43}$$

Since the dual of the space $H_0^1(\Omega)\cap L^{\infty}(\Omega)$ is the space $H^{-1}(\Omega)+L^1(\Omega)$, by (3.6), it is easy to obtain

$$\ddot{v}(t)+B(v)+\overline{g}(t)=f(t)\quad\text{in } L^1(0,T;H^{-1}(\Omega)+L^1(\Omega)). \tag{3.44}$$

The rest is analogous to that of Theorem 1.

This completes the proof. □

4. Decay of solution

Theorem 3. *Let $T=+\infty$, $f\equiv 0$. Suppose that $\langle Bw,w\rangle\geq 0,\forall w\in H_0^1(\Omega)$. If $(b(w),w)\geq\mu_0|w|_{L^2(\Omega)}^2$, then, under the conditions of Theorem 2, the solution in Theorem 2, obtained from the regularized equation (3.4), satisfies*

$$|\dot{u}(t)|_{L^2(\Omega)}^2\leq\mu_1\exp(-\mu_2 t),\quad \text{a.e. } t\geq 0, \tag{4.1}$$

where μ_0, μ_1, and μ_2 are positive constants.

Proof. Let u_n be a solution of (3.4), i.e., u_n satisfies (3.6) and (3.7). Since $(b(w),w) \geq \mu_0|w|^2_{L^2(\Omega)}$, by (3.7), we have

$$|\dot{u}_n(t)|^2_{L^2(\Omega)} + \langle Bu_n(t), u_n(t)\rangle \leq c_{12} - 2\mu_0 \int_0^t |\dot{u}_n(\tau)|^2_{L^2(\Omega)} d\tau, \quad t \in [0,+\infty), \tag{4.2}$$

where c_{12} is a positive constant independent of n.

If $\langle Bw, w\rangle \geq 0, \forall w \in H_0^1(\Omega)$ and $\langle Bu_n(t), u_n(t)\rangle \geq 0$, then, by Gronwall inequality,

$$|\dot{u}_n(t)|^2_{L^2(\Omega)} \leq c_{12} \exp(-2\mu_0 t), \quad \text{a.e. } t \geq 0. \tag{4.3}$$

Since

$$|\dot{u}_n(t)|_{L^2(\Omega)} \leq c_9, \quad \dot{u}_n \longrightarrow \dot{u} \text{ weakly-star in } L^\infty(0,\infty; L^2(\Omega)), \tag{4.4}$$

it is easy to obtain that $\dot{u}_n(t) \to \dot{u}(t)$ weakly in $L^2(\Omega)$ for almost everywhere $t \geq 0$. But $L^2(\Omega)$ is a real Hilbert space, hence, $|\dot{u}(t)|_{L^2(\Omega)} \leq \underline{\lim}_{n\to\infty} |\dot{u}_n(t)|_{L^2(\Omega)}$, a.e. $t \geq 0$ (see [4]). Finally, we get

$$|\dot{u}(t)|^2_{L^2(\Omega)} \leq c_{12} \exp(-2\delta_0 t), \quad (\text{a.e. } t \geq 0). \tag{4.5}$$

□

Remark 1. If $Bu = -\Delta u$, $\varphi(u) = |u|^p u$ then (1.1) is the equation which was ever considered by J. L. Lions [6]. J. L. Lions ever obtained the existence and uniqueness. But at this case, the result of decay of solution is true since the conditions of Theorem 3 is satisfied.

Remark 2. When $Bu = -\Delta u$ and φ denotes the friction potential, equation (1.1) was considered by P. D. Panagiotopoulos under stronger conditions [8].

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Xingming: Shanghai University, Shanghai Institute of Applied Mathematics and Mechanics, Shanghai 200072, China