Internat. J. Math. & Math. Sci.
Vol. 22, No. 4 (1999) 775–784
S 0161-1712⟨99⟩22775-5

THE ABEL-TYPE TRANSFORMATIONS INTO ℓ

MULATU LEMMA

(Received 24 November 1997 and in revised form 16 January 1998)

ABSTRACT. Let t be a sequence in (0,1) that converges to 1, and define the Abel-type matrix $A_{\alpha,t}$ by $a_{nk} = \binom{k+\alpha}{k} t_n^{k+1}(1-t_n)^{\alpha+1}$ for $\alpha > -1$. The matrix $A_{\alpha,t}$ determines a sequence-to-sequence variant of the Abel-type power series method of summability introduced by Borwein in [1]. The purpose of this paper is to study these matrices as mappings into ℓ. Necessary and sufficient conditions for $A_{\alpha,t}$ to be ℓ-ℓ, G-ℓ, and G_w-ℓ are established. Also, the strength of $A_{\alpha,t}$ in the ℓ-ℓ setting is investigated.

Keywords and phrases. ℓ-ℓ methods, ℓ-stronger, G-G methods, G_w-G_w methods.

1991 Mathematics Subject Classification. Primary 40A05; Secondary 40C05.

1. Introduction and background. The Abel-type power series method [1], denoted by $A_\alpha, \alpha > -1$, is the following sequence-to-function transformation: if

$$\sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k x^k < \infty \quad \text{for } 0 < x < 1 \tag{1.1}$$

and

$$\lim_{x\to 1^-} (1-x)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k x^k = L, \tag{1.2}$$

then we say that u is A_α-summable to L. In order to study this summability method as a mapping into ℓ, we must modify it into a sequence to sequence transformation. This is achieved by replacing the continuous parameter x with a sequence t such that $0 < t_n < 1$ for all n and $\lim t_n = 1$. Thus, the sequence u is transformed into the sequence $A_{\alpha,t}u$ whose nth term is given by

$$(A_{\alpha,t}u)_n = (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k t_n^k. \tag{1.3}$$

This transformation is determined by the matrix $A_{\alpha,t}$ whose nkth entry is given by

$$a_{nk} = \binom{k+\alpha}{k} t_n^k (1-t_n)^{\alpha+1}. \tag{1.4}$$

The matrix $A_{\alpha,t}$ is called the Abel-type matrix. The case $\alpha = 0$ is the Abel matrix introduced by Fridy in [5]. It is easy to see that the $A_{\alpha,t}$ matrix is regular and, indeed, totally regular.

2. Basic notations. Let $A = (a_{nk})$ be an infinite matrix defining a sequence-to-sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \tag{2.1}$$

where $(Ax)_n$ denotes the nth term of the image sequence Ax. The sequence Ax is called the A-transform of the sequence x. If X and Z are sets of complex number sequence, then the matrix A is called an X-Z matrix if the image Au of u under the transformation A is in Z whenever u is in X.

Let y be a complex number sequence. Throughout this paper, we use the following basic notations:

$$\begin{aligned}
\ell &= \left\{y : \sum_{k=0}^{\infty} |y_k| \text{ converges}\right\}, \\
\ell^p &= \left\{y : \sum_{k=0}^{\infty} |y_k|^p \text{converges}\right\}, \\
d(A) &= \left\{y : \sum_{k=0}^{\infty} a_{nk} y_k \text{ converges for each } n \geq 0\right\}, \\
\ell(A) &= \{y : A_y \in \ell\}, \\
G &= \{y : y_k = O(r^k) \text{ for some } r \in (0,1)\}, \\
G_w &= \{y : y_k = O(r^k) \text{ for some } r \in (0,w), 0 < w < 1\}, \\
c(A) &= \{y : y \text{ is summable by } A\}.
\end{aligned} \tag{2.2}$$

3. The main results. Our first result gives a necessary and sufficient condition for $A_{\alpha,t}$ to be ℓ-ℓ.

Theorem 1. *Suppose that $-1 < \alpha \leq 0$. Then the matrix $A_{\alpha,t}$ is ℓ-ℓ if and only if $(1-t)^{\alpha+1} \in \ell$.*

Proof. Since $-1 < \alpha \leq 0$ and $0 < t_n < 1$, we have

$$\sum_{n=0}^{\infty} |a_{nk}| = \binom{k+\alpha}{k} \sum_{n=0}^{\infty} t_n^k (1-t_n)^{\alpha+1} \leq \sum_{n=0}^{\infty} (1-t_n)^{\alpha+1} \quad \text{for each } k. \tag{3.1}$$

Thus, if $(1-t)^{\alpha+1} \in \ell$, Knopp-Lorentz theorem [6] guarantees that $A_{\alpha,t}$ is an ℓ-ℓ matrix. Also, if $A_{\alpha,t}$ is an ℓ-ℓ matrix, then by Knopp-Lorentz theorem, we have

$$\sum_{n=0}^{\infty} |a_{n,o}| < \infty, \tag{3.2}$$

and this yields $(1-t)^{\alpha+1} \in \ell$. □

Remark 1. In Theorem 1, the implication that $A_{\alpha,t}$ is ℓ-$\ell \Rightarrow (1-t)^{\alpha+1} \in \ell$ is also true for any $\alpha > 0$, however, the converse implication is not true for any $\alpha > 0$. This is demonstrated in Theorem 4 below.

Corollary 1. *If $-1 < \alpha \le 0$ and $< 0 < t_n < w_n < 1$, then $A_{\alpha,w}$ is an ℓ-ℓ matrix whenever $A_{\alpha,t}$ is an ℓ-ℓ matrix.*

Proof. The corollary follows easily by Theorem 1. □

Corollary 2. *If $-1 < \alpha < \beta \le 0$, then $A_{\beta,t}$ is an ℓ-ℓ matrix whenever $A_{\alpha,t}$ is an ℓ-ℓ matrix.*

Corollary 3. *If $-1 < \alpha \le 0$ and $A_{\alpha,t}$ is an ℓ-ℓ matrix, then $1/\log(1-t) \in \ell$.*

Corollary 4. *If $-1 < \alpha \le 0$, then $\arcsin(1-t)^{\alpha+1} \in \ell$ if and only if $A_{\alpha,t}$ is an ℓ-ℓ matrix.*

Corollary 5. *Suppose that $-1 < \alpha \le 0$ and $w_n = 1/t_n$. Then the zeta matrix z_w* [2] *is ℓ-ℓ whenever $A_{\alpha,t}$ is an ℓ-ℓ matrix.*

Corollary 6. *Suppose that $-1 < \alpha \le 0$ and $t_n = 1-(n+2)^{-q}, 0 < q < 1$: then $A_{\alpha,t}$ is not an ℓ-ℓ matrix.*

Proof. Since $(1-t)^{\alpha+1}$ is not in ℓ, the corollary follows easily by Theorem 1. □

Before considering our next theorem, we recall the following result which follows as a consequence of the familiar Hölder's inequality for summation. The result states that if x and y are real number sequences such that $x \in \ell^p, y \in \ell^q$, $p > 1$, and $(1/p)+(1/q) = 1$, then $xy \in \ell$.

Theorem 2. *If $A_{\alpha,t}$ is an ℓ-ℓ matrix, then*

$$\sum_{n=0}^{\infty} \log \frac{(2-t_n)}{(n+1)} < \infty. \tag{3.3}$$

Proof. Since $\log(2-t_n) \sim (1-t_n)$, it suffices to show that

$$\sum_{n=0}^{\infty} \frac{(1-t_n)}{(n+1)} < \infty. \tag{3.4}$$

If $A_{\alpha,t}$ is an ℓ-ℓ matrix, then, by Theorem 1, we have $(1-t)^{\alpha+1} \in \ell$. If $-1 < \alpha \le 0$, it is easy to see that if $(1-t)^{\alpha+1} \in \ell$, then we have $(1-t) \in \ell$ and, consequently, the assertion follows. If $\alpha > 0$, then the theorem follows using the preceding result by letting $x_n = 1-t_n$, $y_n = 1/(n+1)$, $p = \alpha+1$, and $q = (\alpha+1)/\alpha$. □

Theorem 3. *Suppose that $t_n = (n+1)/(n+2)$. Then $A_{\alpha,t}$ is an ℓ-ℓ matrix if and only if $\alpha > 0$.*

Proof. If $A_{\alpha,t}$ is an ℓ-ℓ matrix, then, by Theorem 1, it follows that $(1-t)^{\alpha+1} \in \ell$ and this yields $\alpha > 0$. Conversely, suppose that $\alpha > 0$. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty} |a_{nk}| &= \binom{k+\alpha}{k} \sum_{n=0}^{\infty} \left(\frac{n+1}{n+2}\right)^k (n+2)^{-(\alpha+1)} \\ &= \binom{k+\alpha}{k} \sum_{n=0}^{\infty} (n+1)^k (n+2)^{-(k+\alpha+1)} \\ &\le M\binom{k+\alpha}{k} \int_0^{\infty} (x+1)^k (x+2)^{-(k+\alpha+1)} dx\end{aligned} \tag{3.5}$$

for some $M > 0$. This is possible as both the summation and the integral are finite since $\alpha > 0$. Now, we let

$$g(k) = \int_0^\infty (x+1)^k (x+2)^{-(k+\alpha+1)} dx, \tag{3.6}$$

and we compute $g(k)$ using integration by parts repeatedly. We have

$$g(k) = \frac{1}{k+\alpha} \cdot 2^{-(k+\alpha)} + h_1(k), \tag{3.7}$$

where

$$\begin{aligned} h_1(k) &= \frac{k}{k+\alpha} \quad \int_0^\infty (x+1)^{k-1}(x+2)^{-(k+\alpha)} dx \\ &= \frac{k \cdot 2^{-(k+\alpha-1)}}{(k+\alpha)(k+\alpha-1)} + h_2 k \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} h_2(k) &= \frac{k(k-1)}{(k+\alpha)(k+\alpha-1)} \int_0^\infty (x+1)^{k-2}(x+2)^{-(k+\alpha-1)} dx \\ &= \frac{k(k-1) \cdot 2^{-(k+\alpha-2)}}{(k+\alpha)(k+\alpha-1)(k+\alpha-2)} + h_3(k). \end{aligned} \tag{3.9}$$

It follows that

$$h_3(k) = \frac{k(k-1)(k-2) \cdot 2^{-(k+\alpha-3)}}{(k+\alpha)(k+\alpha-1)(k+\alpha-2)(k+\alpha-3)} + h_4(k), \tag{3.10}$$

where

$$\begin{aligned} h_4(k) &= \frac{k(k-1)(k-2)(k-3)}{(k+\alpha)(k+\alpha-1)(k+\alpha-2)(k+\alpha-3)(k+\alpha-4)} \\ &\quad \times \int_0^\infty (x+1)^{k-4}(x+2)^{-(k+\alpha-3)} dx. \end{aligned} \tag{3.11}$$

Continuing this process, we get

$$h_k(k) = \frac{k(k-1)(k-2)\cdots 2^{-\alpha}}{(k+\alpha)(k+\alpha-1)(k+\alpha-2)\cdots\alpha} = \frac{2^{-\alpha}}{\alpha\binom{k+\alpha}{k}}. \tag{3.12}$$

It is easy to see that $g(k)$ can be written using summation notation as

$$\begin{aligned} g(k) &= \frac{2^{-\alpha}}{\alpha\binom{k+\alpha}{k}} \sum_{i=0}^{k} \binom{i+\alpha-1}{i} 2^{-i} \\ &\le \frac{2^{-\alpha}}{\alpha\binom{k+\alpha}{k}} \sum_{i=0}^{\infty} \binom{i+\alpha-1}{i} 2^{-i} \\ &= \frac{2^{-\alpha}}{\alpha\binom{k+\alpha}{k}} 2^{\alpha} = \frac{1}{\alpha\binom{k+\alpha}{k}}. \end{aligned} \tag{3.13}$$

Consequently, we get

$$\sum_{n=0}^{\infty}|a_{nk}| \le M\binom{k+\alpha}{k}g(k) \le \frac{M\binom{k+\alpha}{k}}{\alpha\binom{k+\alpha}{k}} = \frac{M}{\alpha}. \tag{3.14}$$

Thus by the Knopp-Lorentz theorem [6], $A_{\alpha,t}$ is an ℓ-ℓ matrix. □

Corollary 7. *Suppose $t_n = (n+1)/(n+2)$. Then $A_{\alpha,t}$ is an ℓ-ℓ matrix if and only if $(1-t)^{\alpha+1} \in \ell$.*

Theorem 4. *Suppose $\alpha > 0$ and $t_n = 1-(n+2)^{-q}, 0 < q < 1$. Then $A_{\alpha,t}$ is not an ℓ-ℓ matrix.*

Proof. If $(1-t)^{\alpha+1}$ is not in ℓ, then by Theorem 1, $A_{\alpha,t}$ is not ℓ-ℓ. If $(1-t)^{\alpha+1} \in \ell$, then we prove that $A_{\alpha,t}$ is not ℓ-ℓ by showing that the condition of the Knopp-Lorentz theorem [6] fails to hold. For convenience, we let $q = 1/p$ and $2^{1/p} = R$, where $p > 1$. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty}|a_{nk}| &= \binom{k+\alpha}{k}\sum_{n=0}^{\infty}\left(1-(n+2)^{-1/p}\right)^k(n+2)^{(-1/p)(\alpha+1)} \\ &= \binom{k+\alpha}{k}\sum_{n=0}^{\infty}\left((n+2)^{1/p}-1\right)^k(n+2)^{(-1/p)(k+\alpha+1)} \\ &\ge M\binom{k+\alpha}{k}\int_0^{\infty}\left((x+2)^{1/p}-1\right)^k(x+2)^{(-1/p)(k+\alpha+1)}dx\end{aligned} \tag{3.15}$$

for some $M > 0$. This is possible as both the summation and integral are finite since $(1-t)^{\alpha+1} \in \ell$. Now, let us define

$$g(k) = \int_0^{\infty}\left((x+2)^{1/p}-1\right)^k(x+2)^{(-1/p)(k+\alpha+1)}dx. \tag{3.16}$$

Using integration by parts repeatedly, we can easily deduce that

$$\begin{aligned}g(k) &= \frac{p(R-1)^kR^{-(k+\alpha+1-p)}}{k+\alpha+1-p}+\frac{pk(R-1)^{k-1}(R)^{-(k+\alpha-p)}}{(k+\alpha+1-p)(k+\alpha-p)} \\ &\quad+\cdots+\frac{pk(k-1)(k-2)\cdots(R)^{-(\alpha+1-p)}}{(k+\alpha+1-p)(k+\alpha-p)(k+\alpha-1-p)\cdots(\alpha+1-p)}.\end{aligned} \tag{3.17}$$

This implies that

$$\begin{aligned}g(k) &> \frac{pk(k-1)(k-2)\cdots R^{-(\alpha+1-p)}}{(k+\alpha+1-p)(k+\alpha-p)(k+\alpha-1-p)\cdots(\alpha+1-p)} \\ &= \frac{pR^{-(\alpha+1-p)}}{(\alpha+1-p)\binom{k+\alpha+1-p}{k}}.\end{aligned} \tag{3.18}$$

Now, we have

$$\sum_{n=0}^{\infty}|a_{nk}| \geq M_1\binom{k+\alpha}{k}g(k)$$
$$> \frac{pM_1\binom{k+\alpha}{k}R^{-(\alpha+1-p)}}{(\alpha+1-p)\binom{k+\alpha+1-p}{k}} > \frac{M_2k^{\alpha}}{k^{\alpha+1-p}} = M_2k^{p-1}. \tag{3.19}$$

Thus, it follows that

$$\sup_k\left\{\sum_{n=0}^{\infty}|a_{nk}|\right\} = \infty, \tag{3.20}$$

and hence $A_{\alpha,t}$ is not ℓ-ℓ. □

In case $t_n = 1-(n+2)^{-q}$, it is natural to ask whether $A_{\alpha,t}$ is an ℓ-ℓ matrix. For $-1 < \alpha \leq 0$, it is easy to see that $A_{\alpha,t}$ is ℓ-ℓ if and only if $\alpha > (1-q)/q$, by Theorem 1. For $\alpha > 0$, the answer to this question is given by the next theorem, which gives a necessary and sufficient condition for the matrix to be ℓ-ℓ.

Theorem 5. *Suppose that $\alpha > 0$ and $t_n = 1-(n+2)^{-q}$. Then $A_{\alpha,t}$ is an ℓ-ℓ matrix if and only if $q \geq 1$.*

Proof. Suppose that $q \geq 1$. Let $q = 1/p$, $2^{1/p} = R$ and $(R-1)/R = S$, where $0 < p \leq 1$. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty}|a_{nk}| &= \binom{k+\alpha}{k}\sum_{n=0}^{\infty}\left(1-(n+2)^{-1/p}\right)^k(n+2)^{(-1/p)(\alpha+1)}\\ &= \binom{k+\alpha}{k}\sum_{n=0}^{\infty}\left((n+2)^{1/p}-1\right)^k(n+4)^{(-1/p)(k+\alpha+1)}\\ &\leq M\binom{k+\alpha}{k}\int_0^{\infty}\left((x+2)^{1/p}-1\right)^k(x+2)^{(-1/p)(k+\alpha+1)}\,dx\end{aligned} \tag{3.21}$$

for some $M > 0$. This is possible as both the summation and the integral are finite since $(1-t)^{\alpha+1} \in \ell$ for $\alpha > 0$. Now, let us define

$$g(k) = \int_0^{\infty}\left((x+2)^{1/p}-1\right)^k(x+2)^{(-1/p)(k+\alpha+1)}\,dx. \tag{3.22}$$

Using integration by parts repeatedly, we can easily deduce that

$$\begin{aligned}g(k) &= \frac{p(R-1)^kR^{-(k+\alpha-p+1)}}{k+\alpha-p+1} + \frac{pk(R-1)^{k-1}(R)^{-(k+\alpha-p)}}{(k+\alpha-p+1)(k+\alpha-p)}\\ &\quad + \cdots + \frac{pk(k-1)(k-2)\cdots R^{-(\alpha-p+1)}}{(k+\alpha-p+1)(k+\alpha-p)\cdots(\alpha-p+1)}.\end{aligned} \tag{3.23}$$

Now, from the hypotheses that $q \geq 1$ and $\alpha > 0$, it follows that

$$\begin{aligned} g(k) &\le \frac{(R-1)^{k+\alpha}R^{-(k+\alpha)}}{k+\alpha}+\frac{k(R-1)^{k+\alpha-1}R^{-(k+\alpha-1)}}{(k+\alpha)(k+\alpha-1)} \\ &\quad+\cdots+\frac{k(k-1)(k-2)\cdots R^{-(\alpha)}}{(k+\alpha)(k+\alpha-1)\cdots(\alpha)} \\ &\le \frac{S^{k+\alpha}}{k+\alpha}+\frac{kS^{k+\alpha-1}}{(k+\alpha)(k+\alpha-1)}+\cdots+\frac{k(k-1)(k-2)\cdots S^{\alpha}}{(k+\alpha)(k+\alpha-1)\cdots\alpha}. \end{aligned} \tag{3.24}$$

By writing the right-hand side of the preceding inequality using the summation notation, we obtain

$$\begin{aligned} g(k) &\le \frac{S^{\alpha}}{\alpha\binom{k+\alpha}{k}}\sum_{i=0}^{k}\binom{i+\alpha-1}{i}S^i \\ &\le \frac{S^{\alpha}}{\alpha\binom{k+\alpha}{k}}\sum_{i=0}^{\infty}\binom{i+\alpha-1}{i}S^i \\ &= \frac{S^{\alpha}}{\alpha\binom{k+\alpha}{k}}S^{-\alpha}=\frac{1}{\alpha\binom{k+\alpha}{k}}. \end{aligned} \tag{3.25}$$

Consequently, we have

$$\sum_{n=0}^{\infty}|a_{nk}| \le M\binom{k+\alpha}{k}g(k) \le \frac{M\binom{k+\alpha}{k}}{\alpha\binom{k+\alpha}{k}}=\frac{M}{\alpha}. \tag{3.26}$$

Thus, by Knopp-Lorentz theorem [6], $A_{\alpha,t}$ is an ℓ-ℓ matrix .

Conversely, if $A_{\alpha,t}$ is an ℓ-ℓ matrix, then it follows, by Theorems 3 and 4, that $q \ge 1$. □

Corollary 8. *Suppose that $t_n = 1-(n+2)^{-q}$, $w_n = 1-(n+2)^{-p}$ and $q < p$. Then $A_{\alpha,W}$ is an ℓ-ℓ matrix whenever $A_{\alpha,t}$ is an ℓ-ℓ matrix.*

Proof. The result follows immediately from Theorems 1 and 5. □

Corollary 9. *Suppose that $\alpha > 0$, $t_n = 1-(n+2)^{-q}, w_n = 1-(n+2)^{-p}$ and $(1/q)+(1/p) = 1$. Then both $A_{\alpha,t}$ and $A_{\alpha,w}$ are ℓ-ℓ matrices.*

Proof. The hypotheses imply that both q and p are greater than 1, and hence the corollary follows easily by Theorem 5. □

Theorem 6. *The following statements are equivalent:*

(1) *$A_{\alpha,t}$ is a G_w-ℓ matrix;*
(2) $(1-t)^{\alpha+1} \in \ell$;
(3) $\arcsin(1-t)^{\alpha+1} \in \ell$;
(4) $((1-t)^{\alpha+1})/(\sqrt{1-(1-t)^{2(\alpha+1)}}) \in \ell$;
(5) *$A_{\alpha,t}$ is a G-ℓ matrix.*

Proof. We get (1) ⇒ (2) by [9, Thm. 1.1] and (2) ⇒ (3) ⇒ (4) ⇒ (5) follow easily from the following basic inequality

$$x < \arcsin x < \frac{x}{\sqrt{(1-x^2)}}, \quad 0 < x < 1, \tag{3.27}$$

and by [4, Thm. 1]. The assertion that (5) ⇒ (1) follows immediately as G_w is a subset of G. □

Corollary 10. *Suppose that $t_n = 1-(n+2)^{-q}$. Then $A_{\alpha,t}$ is a G-ℓ matrix if and only if $\alpha > (1-q)/q$. For $q = 1$, $A_{\alpha,t}$ is a G-ℓ matrix if and only if it is an ℓ-ℓ matrix.*

Proof. The proof follows using Theorems 3 and 6. □

Theorem 7. *The following statements are equivalent:*
(1) *$A_{\alpha,t}$ is a G_w-G matrix ;*
(2) *$(1-t)^{\alpha+1} \in G$;*
(3) *$\arcsin(1-t)^{\alpha+1} \in G$;*
(4) *$A_{\alpha,t}$ is a G-G matrix.*

Proof. (1) ⇒ (2) follows by [9, Thm. 2.1] and (2) ⇒ (3) ⇒ (4) follows easily from (3.27) and [4, Thm. 4]. The assertion that (4) ⇒ (1)follows immediately as G_w is a subset of G. □

Corollary 11. *If $A_{\alpha,t}$ is a G_w-G_w matrix, then it is a G-G matrix.*

Our next few results suggest that the Abel-type matrix $A_{\alpha,t}$ is ℓ-stronger than the identity matrix (see [7, Def. 3]). The results indicate how large the sizes of $\ell(A_{\alpha,t})$ and $d(A_{\alpha,t})$ are.

Theorem 8. *Suppose that $-1 < \alpha \le 0$, $A_{\alpha,t}$ is an ℓ-ℓ matrix, and the series $\sum_{k=0}^{\infty} x_k$ has bounded partial sums. Then it follows that $x \in \ell(A_{\alpha,t})$.*

Proof. Since, for $-1 < \alpha \le 0$, $\binom{k+\alpha}{k}$ is decreasing, the theorem is proved by following the same steps used in the proof of [7, Thm. 4]. □

Remark 2. Although the preceding theorem is stated for $-1 < \alpha \le 0$, the conclusion is also true for $\alpha > 0$ for some sequences. This is demonstrated as follows: let x be the bounded sequence given by

$$x_k = (-1)^k. \tag{3.28}$$

Let Y be the $A_{\alpha,t}$-transform of the sequence x. Then it follows that the sequence Y is given by

$$\begin{aligned} Y_n &= (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \\ &= (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} (-1)^k t_n^k \\ &= \frac{(1-t_n)^{\alpha+1}}{(1+t_n)^{\alpha+1}} \end{aligned} \tag{3.29}$$

which implies that

$$Y_n < (1-t_n)^{\alpha+1}. \tag{3.30}$$

Hence, if $A_{\alpha,t}$ is an ℓ-ℓ matrix, then by Theorem 1, $(1-t)^{\alpha+1} \in \ell$, and so $x \in \ell(A_{\alpha,t})$.

Corollary 12. *Suppose that $-1 < \alpha \le 0$, $A_{\alpha,t}$ is an ℓ-ℓ matrix. Then $\ell(A_{\alpha,t})$ contains the class of all sequences x such that $\sum_{k=0}^{\infty} x_k$ is conditionally convergent.*

Remark 3. In fact, we can give a further indication of the size of $\ell(A_{\alpha,t})$ by showing that if $A_{\alpha,t}$ is an ℓ-ℓ matrix, then it also contains an unbounded sequence. To verify this, consider the sequence x given by

$$x_k = (-1)^k \frac{k+\alpha+1}{\alpha+1}. \tag{3.31}$$

Let Y be the $A_{\alpha,t}$-transform of the sequence x. Then we have

$$\begin{aligned} Y_n &= (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \\ &= (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} (-1)^k \frac{k+\alpha+1}{\alpha+1} t_n^k \\ &= \frac{(1-t_n)^{\alpha+1}}{(1+t_n)^{\alpha+2}} \end{aligned} \tag{3.32}$$

and, consequently,

$$Y_n < (1-t_n)^{\alpha+1}. \tag{3.33}$$

Hence, if $A_{\alpha,t}$ is an ℓ-ℓ matrix, then by Theorem 1, $(1-t)^{\alpha+1} \in \ell$, and so $x \in \ell(A_{\alpha,t})$. This example clearly indicates that $A_{\alpha,t}$ is a rather strong method in the ℓ-ℓ setting for any $\alpha > -1$.

The ℓ-ℓ strength of the $A_{\alpha,t}$ matrices can also be demonstrated by comparing them with the familiar Norland matrices (N_p) [3]. By using the same techniques used in the proof of [3, Thm. 8], we can show that the class of the $A_{\alpha,t}$ matrix summability methods is ℓ-stronger than the class of N_p matrix summability methods for some p.

When discussing the ℓ-ℓ strength of $A_{\alpha,t}$, or the size of $\ell(A_{\alpha,t})$, it is very important that we also determine the domain of $A_{\alpha,t}$. The following proposition, which can be easily proved, gives a characterization of the domain of $A_{\alpha,t}$.

Proposition 1. *The complex number sequence x is in the domain of the matrix $A_{\alpha,t}$ if and only if*

$$\limsup_k |x_k|^{1/k} \le 1. \tag{3.34}$$

Remark 4. Proposition 1 can be used as a powerful tool in making a comparison between the ℓ-ℓ strength of the $A_{\alpha,t}$ matrices and some other matrices as shown by the following examples.

Example 1. The $A_{\alpha,t}$ matrix is not ℓ-stronger than the Borel matrix B[8, p. 53]. To demonstrate this, consider the sequence x given by

$$x_k = (-3)^k. \tag{3.35}$$

Then we have

$$(Bx)_n = \sum_{k=0}^{\infty} e^{-n} \frac{n^k}{k!} (-3)^k = e^{-4n}. \tag{3.36}$$

Thus, we have $Bx \in \ell$ and hence $x \in \ell(B)$, but by Proposition 1, $x \notin \ell(A_{\alpha,t})$. Hence, $A_{\alpha,t}$ is not ℓ-stronger than B.

Example 2. The $A_{\alpha,t}$ matrix is not ℓ-stronger than the familiar Euler-Knopp matrix E_r for $r \in (0,1)$. Also, E_r is not ℓ-stronger than $A_{\alpha,t}$. To demonstrate this, consider the sequence x defined by

$$x_k = (-q)^k \qquad \text{and} \qquad r = \frac{1}{q}, \tag{3.37}$$

where $q > 1$. Let Y be the E_r-transform of the sequence x . Then it is easy to see that the sequence Y is defined by

$$Y_n = \left(\frac{-1}{q}\right)^n. \tag{3.38}$$

Since $q > 1$, we have $Y \in \ell$ and hence $x \in \ell(E_r)$, but $x \notin \ell(A_{\alpha,t})$ by Proposition 1. Hence, $A_{\alpha,t}$ is not ℓ-stronger than E_r. To show that E_r is not ℓ-stronger than $A_{\alpha,t}$, we let $-1 < \alpha \leq 0$ and consider the sequence x that was constructed by Fridy in his example of [5, p. 424] . Here, we have $x \notin \ell(E_r)$, but $x \in \ell(A_{\alpha,t})$ by Theorem 8. Thus, E_r is not ℓ-stronger than $A_{\alpha,t}$.

Acknowledgement. I would like to thank professor J. Fridy for several helpful suggestions that significantly improved the exposition of these results. I also want to thank my wife Mrs. Tsehaye Dejene for her great encouragement.

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Lemma: Department of mathematics, Savannah state university, Savannah, Georgia 31404, USA