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NONLINEAR FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS IN HILBERT SPACE

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ABSTRACT. Let X be a Hilbert space and let $\Omega \subset R^n$ be a bounded domain with smooth boundary $\partial\Omega$. We establish the existence and norm estimation of solutions for the parabolic partial functional integro-differential equation in X by using the fundamental solution.

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1. Introduction. Let X be a Hilbert space and let $\Omega \subset R^n$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following parabolic partial functional integro-differential equation.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \mathcal{A}_0 u(t,x) + \mathcal{A}_1 u(t-h,x) + \int_{-h}^{0} a(s)\mathcal{A}_2 u(t+s,x)\,ds \\ &\quad + \int_0^t \Big\{k(t,s)G(s,u(s-h),x) + H(t,s,u(s-h,x))\Big\}\,ds \\ &\quad + F(t,u(t-h,x)) + f(t,x), \quad 0 < t \le T,\ x \in \Omega,\end{aligned} \tag{1.1}$$

where $\mathcal{A}_i (i = 0,1,2)$ are elliptic differential operators, f is a forcing function, $h > 0$ is a delay time, $a(s)$ is a real scalar function on $[-h,0], G, H$, and F are nonlinear functions, and k is a kernel. The boundary condition attached to (1.1) is, e.g., given by the Dirichlet boundary condition

$$u|_{\partial\Omega} = 0, \quad 0 < t \le T, \tag{1.2}$$

and the initial condition is given by

$$u(\theta,x) = g(\theta,x), \quad \theta \in [-h,0],\ x \in \Omega. \tag{1.3}$$

From [4], the above mixed problems (1.1), (1.2), and (1.3) can be formulated abstractly as

$$\begin{aligned}\frac{du(t)}{dt} &= A_0 u(t) + A_1 u(t-h) + \int_{-h}^{0} a(s)A_2 u(t+s)\,ds \\ &\quad + \int_0^t \{k(t,s)G(s,u_s) + H(t,s,u_s)\}\,ds \\ &\quad + F(t,u_t) + f(t), \quad 0 < t \le T,\end{aligned} \tag{1.4}$$

$$u(\theta) = g(\theta), \quad \theta \in [-h,0], \tag{1.5}$$

where the state $u(x)$ of the system (1.5) lies in an appropriate Hilbert space and $A_i (i = 0,1,2)$ are unbounded operators associated with $\mathcal{A}_i (i = 0,1,2)$, respectively. Next, we explain the notation u_t in (1.5). Let $I = [-h,0]$. If a function $u(t)$ is continuous from $I \cup [0,T]$ into a Hilbert space X, then u_t is an element in $C = C([-h,0];X)$, which has the point-wise definition

$$u_t(\theta) = u(t+\theta) \quad \text{for } \theta \in I. \tag{1.6}$$

Let $\Delta_T = \{(s,t); 0 \le s \le t \le T\}$. We assume in (1.5) that $G : [0,T] \times C \to X$, $H : \Delta_T \times C \to X$, $F : [0,T] \times C \to X$ and the kernel $k : \Delta_T \to R$ (R denotes the set of real numbers) are continuous, $f : [0,T] \to V^*$ with some enlarged space $V^* \supset H$ and $g : [-h,0] \to V$ with some dense subspace $V \subset H$. It is assumed that the inclusions $V \subset H \subset V^*$ are continuous and V^* is the dual space of V.

Many authors [2, 8] studied the following delay differential equation:

$$\begin{gathered} \frac{du(t)}{dt} = A_0 u(t) + A_1 u(t-h) + \int_{-h}^{0} a(s) A_2 u(t+s)\, ds + f(t), \quad \text{a.e. } t \ge 0, \\ u(\theta) = g(\theta), \quad \theta \in [-h,0]. \end{gathered} \tag{1.7}$$

The fundamental solution is constructed in Tanabe [8]. In this paper, we establish the existence and norm estimation of solutions for the equation (1.5) by using the fundamental solution.

2. Preliminaries. Let H be a pivot complex Hilbert space and V be a complex Hilbert space such that V is dense in H and the inclusion map $i : V \to H$ is continuous. The norms of H, V, and the inner product of H are denoted by $|\cdot|$, $\|\cdot\|$, and $\langle \cdot,\cdot \rangle$, respectively. Identifying the antidual of H with H, we may consider that $V \subset H \subset V^*$. The norm of the dual space V^* is denoted by $\|\cdot\|_*$. We consider the following linear functional differential equation on the Hilbert space H.

$$\begin{gathered} \frac{du(t)}{dt} = A_0 u(t) + A_1 u(t-h) + \int_{-h}^{0} a(s) A_2 u(t+s)\, ds + f(t), \quad \text{a.e. } t \ge 0, \\ u(0) = g^0, \quad u(s) = g^1(s), \quad \text{a.e. } s \in [-h,0]. \end{gathered} \tag{2.1}$$

Let $a(u,v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

$$\operatorname{Re} a(u,u) \ge c_0 \|u\|^2 - c_1 |u|^2, \tag{2.2}$$

where $c_0 > 0$ and $c_1 \ge 0$ are real constants. Let A_0 be the operator associated with this sesquilinear form

$$\langle v, A_0 u \rangle = -a(u,v), \quad u,v \in V, \tag{2.3}$$

where $\langle \cdot,\cdot \rangle$ denotes the duality pairing between V and V^*. The operator A_0 is bounded linear from V into V^*. The realization of A_0 in H, which is the restriction of A_0 to the domain $D(A_0) = \{u \in V : A_0 u \in H\}$, is also denoted by A_0. It is proved in Tanabe [6] that A_0 generates an analytic semigroup $e^{tA_0} = T(t)$ both in H and V^* and that $T(t) : V^* \to V$ for each $t > 0$. Throughout this paper, it is assumed that each $A_i (i = 1,2)$ is bounded and linear from V to V^* (i.e., $A_i \in \mathcal{L}(V,V^*)$) such that A_i maps $D(A_0)$

endowed with the graph norm of A_0 to H continuously. The real valued scalar function $a(s)$ is assumed to be Hölder continuous on $[-h,0]$. We introduce a Stieltjes measure η given by

$$\eta(s) = -\chi_{(-\infty,-h]}(s)A_1 - \int_s^0 a(\xi)\,d\xi\,A_2 : V \longrightarrow V^*, \quad s \in [-h,0], \tag{2.4}$$

where $\chi_{(-\infty,-h]}$ denotes the characteristic function of $(-\infty,-h]$. Then the delay term in (2.1) is written simply as $\int_{-h}^0 d\eta(s)u(t+s)$. The fundamental solution $W(t)$ of (2.1) is defined as a unique solution of

$$W(t) = \begin{cases} T(t) + \int_0^t T(t-s) \int_{-h}^0 d\eta(\xi)\,W(\xi+s)\,ds, & t \ge 0, \\ 0, & t < 0, \end{cases} \tag{2.5}$$

and $W(t)$ is constructed by Tanabe [7] under the Hölder continuity of $a(s)$.

Theorem 2.1 [2]. *The fundamental solution $W(t)$ is strongly continuous in V,H, and V^*, and for each $t > 0$, $W(t) : V^* \to V$. Furthermore, $W(t)$ satisfies*

$$\frac{d}{dt}W(t) = A_0W(t) + \int_{-h}^0 d\eta(s)W(t+s), \quad \text{a.e. } t > 0. \tag{2.6}$$

For each $t > 0$, we define the operator valued function $U_t(\cdot)$ by

$$U_t(s) = \int_{-h}^s W(t-s+\xi)\,d\eta(\xi) : V \longrightarrow V, \quad \text{a.e. } s \in [-h,0]. \tag{2.7}$$

Let $T > 0$ be fixed. Associated with $U_t(\cdot)$, we consider the operator $\mathcal{U} : L^2(-h,0;V) \to L^2(0,T;V)$ defined by

$$\left(\mathcal{U}g^1\right)(t) = \int_{-h}^0 U_t(s)g^1(s)\,ds, \quad t \in [0,T] \tag{2.8}$$

for $g^1 \in L^2(-h,0;V)$.

Theorem 2.2 [8]. *Let $T > 0$ be fixed. Assume that $f \in L^2(0,T;V^*)$ and $g = (g^0,g^1) \in H \times L^2(-h,0;V)$. Then there exists a unique solution $u(t) = u(t;f,g)$ of (2.1) on $[0,T]$ satisfying*

$$u \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H). \tag{2.9}$$

Further, for each $T > 0$, there is a constant K_T such that

$$\int_0^T \|u(t)\|^2\,dt + \int_0^T \left\|\frac{du(t)}{dt}\right\|_*^2 dt \le K_T\left(|g^0|^2 + \int_{-h}^0 \|g^1(s)\|^2\,ds + \int_0^T \|f(t)\|_*^2\,dt\right). \tag{2.10}$$

This solution $u(t)$ is represented by

$$u(t;f,g) = W(t)g^0 + \left(\mathcal{U}g^1\right)(t) + \int_0^t W(t-s)f(s)\,ds. \tag{2.11}$$

In what follows, in order to consider the solutions in the state space $C = C([-h,0];H)$, we assume that $g = (g^0,g^1)$ is continuous in H, i.e.,

$$g(0) = g^0, \qquad g(\cdot) = g^1(\cdot) \in C([-h,0];H). \tag{2.12}$$

Let

$$\hat{u}(t;f,g) = \begin{cases} u(t;f,g), & t \in [0,T], \\ g(t), & t \in [-h,0]. \end{cases} \tag{2.13}$$

Then, by Theorem 2.2, we get

$$\hat{u}(\cdot;f,g) \in C([-h,T];H) \tag{2.14}$$

if (2.12) is satisfied.

3. Existence and uniqueness of functional integro-differential equations. Using the fundamental solution $W(t)$ in Section 2, we consider the following abstract functional integral equation.

$$\begin{aligned} v(t) &= u(t;f,g) \\ &\quad + \int_0^t W(t-s)\bigg[\int_0^s \{k(s,\tau)G(\tau,v_\tau) \\ &\quad + H(s,\tau,v_\tau)\}\,d\tau + F(s,v_s)\bigg]ds, \quad 0 < t \le T, \\ v(\theta) &= g(\theta), \quad \theta \in [-h,0], \end{aligned} \tag{3.1}$$

where $u(t;f,g)$ is given by (2.11).

We list the following hypotheses.

(A_1) The nonlinear functions $G:[0,T]\times C \to H$, $H:\Delta_T \times C \to H$, $F:[0,T]\times C \to H$, and the kernel $k:\Delta_T \to R$ (R denotes the set of real numbers) are continuous.

(A_2) Let $b_1, b_3 : [0,T] \to R$, $b_2 : \Delta_T \to R^+$ be continuous functions such that

$$\begin{aligned} |G(t,\phi) - G(t,\overline{\phi})|_X &\le b_1(t)\,|\phi - \overline{\phi}|_C; \\ |H(t,s,\phi) - H(t,s,\overline{\phi})|_X &\le b_2(t,s)\,|\phi - \overline{\phi}|_C; \\ |F(t,\phi) - F(t,\overline{\phi})|_X &\le b_3(t)\,|\phi - \overline{\phi}|_C \end{aligned} \tag{3.2}$$

for $t,s \in [0,T]$, $\phi, \overline{\phi} \in C$.

(A_3) The function $k(t,s)$ is Hölder continuous with exponent α, i.e., there exists a positive constant a such that

$$|k(t_1,s_1) - k(t_2,s_2)| \le a\left(|t_1 - t_2|^\alpha + |s_1 - s_2|^\alpha\right) \tag{3.3}$$

for $t_1, t_2, s_1, s_2 \in [0,T]$, $0 < \alpha \le 1$.

(A_4) For all $0 \le s \le t \le T$,

$$G(t,0) = 0, \qquad H(t,s,0) = 0, \qquad F(t,0) = 0. \tag{3.4}$$

Theorem 3.1. *Let $f \in L^2(0,T;V^*)$ and $g = (g(0), g(\cdot)) \in H \times L^2(-h,0;V)$ satisfy (2.12). Assume that the hypotheses (A_1)–(A_4) hold. Then there exists a time $t_1 > 0$ such that the functional integral equation (3.1) admits a unique solution $v(t)$ on $[0,t_1]$.*

Proof. We prove this theorem by using the method of successive approximations.

Set $v^0(t) = u(t;f,g)$, $t \ge 0$. Let $\hat{v}^0(t)$ be the extension of $v^0(t)$ on $[-h,T]$ by (2.13). Then, by the assumptions on f and g, we have $\hat{v}^0(t) \in C([-h,T];H)$. By hypotheses ($A_1$)–($A_4$), we define $\{\hat{v}^n\}_{n=0}^\infty \subset C([-h,T];H)$ successively by

$$\hat{v}^n(t) = u(t;f,g) + \int_0^t W(t-s)\Big[\int_0^s \{k(s,\tau)G(\tau,\hat{v}_\tau^{n-1}) + H(s,\tau,\hat{v}_\tau^{n-1})\}d\tau + F(s,v_s^{n-1})\Big]ds, \quad 0 < t \le T, \tag{3.5}$$

$$\hat{v}^n(\theta) = g(\theta), \quad \theta \in [-h,0]. \tag{3.6}$$

It is obvious that $M = \sup_{t\in[0,T]} \|W(t)\|_{L(H)}$ is finite and that

$$\hat{v}^{n+1}(\theta) - \hat{v}^n(\theta) = 0, \quad \theta \in [-h,0]. \tag{3.7}$$

For $0 \le t \le T$, we have, by (A_1)–(A_4) and the strong continuity of $W(t)$ on $[0,T]$,

$$\begin{aligned}
&|\hat{v}^{n+1}(t) - \hat{v}^n(t)| \\
&\quad = \Big| \int_0^t W(t-s)\Big[\int_0^s \{k(s,\tau)G(\tau,\hat{v}_\tau^n) + H(s,\tau,\hat{v}_\tau^n)\}d\tau + F(s,\hat{v}_s^n)\Big]ds \\
&\quad - \int_0^t W(t-s)\Big[\int_0^s \{k(s,\tau)G(\tau,\hat{v}_\tau^{n-1}) + H(s,\tau,\hat{v}_\tau^{n-1})\}d\tau + F(s,\hat{v}_s^{n-1})\Big]ds\Big| \\
&\quad \le M\int_0^t \Big[\int_0^s |k(s,\tau)|\,|G(\tau,\hat{v}_\tau^n) - G(\tau,\hat{v}_\tau^{n-1})|\,d\tau\, ds \\
&\qquad + \int_0^s |H(s,\tau,\hat{v}_\tau^n) - H(s,\tau,\hat{v}_\tau^{n-1})|\,d\tau\Big] + M\int_0^t |F(s,\hat{v}_s^n) - F(s,\hat{v}_s^{n-1})|\,ds \\
&\quad \le M\int_0^t \Big[\int_0^s \Big\{|k(s,\tau)|\,|b_1(\tau)|\,\|\hat{v}_\tau^n - \hat{v}_\tau^{n-1}\| + |b_2(s,\tau)|\,\|\hat{v}_\tau^n - \hat{v}_\tau^{n-1}\|\Big\}d\tau\Big]ds \\
&\quad + M\int_0^t |b_3(s)|\,\|\hat{v}_s^n - \hat{v}_s^{n-1}\|ds \\
&\quad \le M\int_0^t [KL_1 + L_2]\,\|\hat{v}_\tau^n - \hat{v}_\tau^{n-1}\|s\,ds + M\int_0^t L_3\|\hat{v}_s^n - \hat{v}_s^{n-1}\|ds \\
&\quad \le \Big[M(KL_1+L_2)\tfrac{1}{2}t^2 + ML_3t\Big]\|\hat{v}^n - \hat{v}^{n-1}\|_{C([-h,T];H)} \\
&\quad = (c_1t + c_2)t\,\|\hat{v}^n - \hat{v}^{n-1}\|_{C([-h,T];H)},
\end{aligned} \tag{3.8}$$

where $c_1 = (1/2)M(KL_1 + L_2)$ and $c_2 = ML_3$. We now choose a sufficiently small constant $t_1 > 0$ such that

$$L = (c_1t_1 + c_2)t_1 < 1. \tag{3.9}$$

Then by (3.6), (3.8), and (3.9), we get

$$\begin{aligned}
\|\hat{v}^{n+1} - \hat{v}^n\|_{C([-h,T];H)} &\le L\|\hat{v}^n - \hat{v}^{n-1}\|_{C([-h,T];H)} \\
&\cdots\cdots\cdots \\
&\le L^n\|\hat{v}^1 - \hat{v}^0\|_{C([-h,T];H)}.
\end{aligned} \tag{3.10}$$

This implies that $\{\hat{v}^n\}_{n=0}^\infty$ converges uniformly to some $\hat{v} \in C([-h,0];H)$. Therefore,

$$\lim_{n\to\infty} \sup_{t\in[0,t_1]} \|\hat{v}_t^n - \hat{v}_t\|_{C([-h,0];H)} = 0. \tag{3.11}$$

Hence, by letting $n\to\infty$ in (3.5), in view of (A_1)–(A_4) and (3.11), we get

$$\begin{aligned}\hat{v}(t) &= u(t;f,g)\\ &\quad+\int_0^t\Big[W(t-s)\Big[\int_0^s\{k(s,\tau)G(\tau,\hat{v}_\tau)\\ &\quad+H(s,\tau,\hat{v}_\tau)\}d\tau\Big]+F(s,\hat{v}_s)\Big]ds,\quad 0<t\le t_1,\\ \hat{v}(\theta)&=g(\theta),\quad \theta\in[-h,0].\end{aligned}\tag{3.12}$$

This shows the local existence of a solution $v(t)=\hat{v}(t)|_{[0,t_1]}$ of (3.1) on $[0,t_1]$. Let v_1 and v_2 be the solution of (3.1) on $[0,t_1]$. Then it is easy to see, similarly to the above, that

$$\|\hat{v}^1-\hat{v}^2\|_{C([-h,t_1];H)}\le L\|\hat{v}^1-\hat{v}^2\|_{C([-h,t_1];H)},\tag{3.13}$$

so that by $L<1$, $v_1(t)=v_2(t)$ on $[0,t_1]$. This proves the uniqueness. □

Since $k(s,\tau)G(\tau,\hat{v}_\tau)$, $H(s,\tau,\hat{v}^n)$, $F(s,v_s)\in L^2(0,t_1;H)\subset L^2(0,t_1;V^*)$, by Theorem 2.1, we see that the solution $v(t)$ of (3.1) satisfies

$$\begin{aligned}\frac{dv(t)}{dt} &= A_0v(t)+A_1v(t-h)+\int_{-h}^0 a(s)A_2v(t+s)ds\\ &\quad+\int_0^t\{k(t,s)G(s,v_s)+H(t,s,v_s)\}ds\\ &\quad+F(t,v_t)+f(t),\quad \text{a.e. } t\in[0,t_1],\\ v(\theta)&=g(\theta),\quad \theta\in[-h,0],\end{aligned}\tag{3.14}$$

and $v\in L^2(0,t_1;V)\cap W^{1,2}(0,t_1;V^*)$. In this sense, we call this v a mild solution of (1.5) on $[0,t_1]$. We give a norm estimation of the mild solution of (1.5) and establish the global existence of solutions with the aid of norm estimations. It is well known (cf. Lions and Magenes [3, Prop. 2.1, Thm. 3.1]) that the inclusion $L^2(0,T;V)\cap W^{1,2}(0,T;V^*)\subset C([0,T];H)$ is continuous, that is, there exists a constant c_0 such that

$$\|u\|_{C([0,T];H)}\le c_0\left(\|u\|_{L^2(0,T;V)}+\left\|\frac{du}{dt}\right\|_{L^2(0,T;V^*)}\right)\tag{3.15}$$

for all $u\in L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$.

Lemma 3.1 [5]. *Let $a(t)$, $b(t)$, and $c(t)$ be real valued nonnegative continuous functions defined on R^+, for which the inequality*

$$c(t)\le c_0+\int_0^t a(s)c(s)\,ds+\int_0^t a(s)\Big[\int_0^s b(\tau)c(\tau)\,d\tau\Big]ds\tag{3.16}$$

holds for all $t\in R^+$, where c_0 is a nonnegative constant. Then

$$c(t)\le c_0\left(1+\int_0^t a(s)\exp\Big[\int_0^s (a(\tau)+b(\tau))\,d\tau\Big]ds\right)\quad \textit{for all } t\in R^+.\tag{3.17}$$

Theorem 3.2. *Assume that the conditions in Theorem 3.1 hold. Then for any solution $v(t) = v(t;f,g)$ of (3.1) on $[-h,T]$, we have the estimate*

$$||v_t(\cdot;f,g)||_{C([0,T];H)} \le c\Big(|g(0)| + ||g||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}\Big)e^{Kt}, \tag{3.18}$$

where c is a positive constant which does not depend on v.

Proof. From hypotheses (A_1)–(A_4), we have

$$\begin{aligned}
&|v(t+\theta;f,g)| \\
&\quad\le |u(t+\theta;f,g)| + \bigg|\int_0^{t+\theta} W(t+\theta-s) \\
&\qquad\qquad \times\Big[\int_0^s \{k(s,\tau)G(\tau,v_\tau)+H(s,\tau,v_\tau)\}\,d\tau + F(s,v_s)\Big]ds\bigg| \\
&\quad\le ||u(\cdot;f,g)||_{C([0,T];H)} \\
&\qquad + M\int_0^{t+\theta}\Big[\int_0^s\{K|b_1(\tau)|\|v_\tau\| + |b_2(s,\tau)|\,\|v_\tau\|\}\,d\tau + |b_3(s)|\,\|v_s\|\Big]ds.
\end{aligned} \tag{3.19}$$

Hence, by (2.10) and (3.15),

$$\begin{aligned}
||v_t(\cdot;f,g)|| &= \sup_{\theta\in[-h,0]} |v(t+\theta;f,g)| \\
&\le K_T c_0\Big(|g(0)| + ||g||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}\Big) \\
&\quad + \int_0^t c_1||v_s(\cdot;f,g)||\,ds + \int_0^t\int_0^s c_2||v_\tau(\cdot;f,g)||\,d\tau\,ds \\
&\le c'\Big(|g(0)| + ||g||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}\Big) \\
&\quad + M\Big(\int_0^t ||v_s(\cdot;f,g)||\,ds + \int_0^t\int_0^s ||v_\tau(\cdot;f,g)||\,d\tau\,ds\Big).
\end{aligned} \tag{3.20}$$

By using Lemma 3.1, we get

$$\begin{aligned}
||v_t(\cdot;f,g)||_{C([0,T];H)} &\le c\Big(|g(0)| + ||g||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}\Big) \\
&\qquad \times\Big(1 + M\int_0^t \exp\Big(\int_0^s (M+1)\,d\tau\Big)ds\Big) \\
&\le c'\Big(|g(0)| + ||g||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}\Big) \\
&\qquad \times[1 + M\exp\{(M+1)T\}t] \\
&\le c\Big(|g(0)| + ||g||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}\Big)e^{Kt}
\end{aligned} \tag{3.21}$$

for some constants c and K. This completes the proof. □

By using Theorems 3.1, 3.2, we get the following theorem:

Theorem 3.3. *Assume that the conditions in Theorem 3.1 hold. Then there exists a unique solution $v(t)$ on $[0,T]$ of (3.1) which satisfies the estimate*

$$||v(\cdot;f,g)||_{C([0,T];H)} \le c\Big(|g(0)| + ||g||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}\Big)e^{KT} \tag{3.22}$$

for some constants c and K.

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