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RESEARCH NOTES

A SIMPLE CHARACTERIZATION OF COMMUTATIVE H^*-ALGEBRAS

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Abstract. Commutative H^*-algebras are characterized without postulating the existence of Hilbert space structure.

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1. Introduction. Let $\mathfrak{M}$ be the space of all maximal regular ideals in a commutative H^*-algebra A and let $x(M)$, $M \in \mathfrak{M}$, denote the Gelfand transform of x, Loomis [3] (in the sequel we use notation of Naimark [5]). Then it is easy to show (see Theorem 1 below) that the series $\sum x(M)\bar{y}(M)$ converges absolutely for all $x, y \in A$. Also, if we assume that each minimal self-adjoint idempotent in A has norm one, then it is true that for each bounded linear function f on $A(f \in A^*)$ there exists $a \in A$ such that $f(x) = \sum x(M)\bar{a}(M)$ for all $x \in A$.

In this note we show that these properties could be used to characterize commutative proper H^*-algebras of this kind. More specifically we show that each semi-single completely symmetric, Naimark [5], Banach algebra with the above properties is a proper H^*-algebra with respect to some Hilbertian norm which is equivalent to its original norm. Also, there is a characterization of *all* proper commutative H^*-algebras.

2. Characterizations. Let A be a complex commutative Banach algebra. We do not assume that A has an identity and so, because of this, we have to deal with regular maximal ideals. An ideal I in A is regular if the algebra A/I has an identity. If M is maximal regular ideal then it is closed and the algebra A/M is isomorphic to the complex field (Gelfand-Mazur theorem, complex case, Loomis [3, 22F]). It follows that there exists a continuous linear functional F_M, Loomis [3, 23B], such that $M = \{x \in A : F_M(x) = 0\}$, i.e., M is the kernel (null space) of F_M.

The Gelfand transform $x()$ (we use the Naimark's notion, Naimark [5], here) of x is defined by setting $x(M) = F_M(x)$ (Loomis uses the notion $x^\wedge$ in Loomis [3, 23B]), where M is a regular maximal ideal in A.

The algebra A is said to be semi-simple if $\cap_{M\in\mathfrak{M}} M = (0)$ (as it is stated above, $\mathfrak{M}$ denotes the space of all maximal regular ideals as A). Equivalent condition: mapping $x \to x()$ is one to one. The algebra A is said to be completely symmetric, Naimark [5], if it has an involution $x \to x^*$ such that $x^*(M) = \bar{x}(M)$ for all $M \in \mathfrak{M}$.

More details of Gelfand theory could be found in Gelfand-Raikov-Silov [2], Loomis [3], Mackey [4], Naimark [5], Simmons [7], and others.

A proper H^*-algebra is a Banach algebra A with an involution $x \to x^*$ and a scalar product $(\,,\,)$ such that $(x,x) = \|x\|^2$ and $(xy,z) = (y,x^*z) = (x,zy^*)$ for all $x,y,z \in A$. Note that A is semi-simple. For simplicity, a nonzero self-adjoint idempotent will be called projection (e.g., Saworotnow [6]). A projection e is minimal if it is not a sum of two projections whose product is zero.

A completely symmetric commutative Banach algebra is a Banach algebra with involution $x \to x^*$ such that $x^*(M) = \bar{x}(M)$ for all $x \in A$ and $M \in \mathfrak{M}$, Naimark [5, Sec. 14].

Theorem 1. *Each proper commutative H^*-algebra A is completely symmetric in the sense of Naimark* [5]. *Also, the series $\sum_{M\in\mathfrak{M}} |x(M)|^2$ converges for each $x \in A$ and if we assume that each minimal projection in A has norm one, then each bounded linear functional f on $A (f \in A^*)$ has the form $f(x) = \sum x(M)\bar{a}(M) (x \in A)$ for some $a \in A$.*

Proof. First and second parts of the theorem follow from Loomis [3, 27G]. For each $M \in \mathfrak{M}$ there exists a minimal projection e_M such that $x(M) = (x,e_M)\|e_M\|^{-2}$, $x = \sum_{M\in\mathfrak{M}} x(M)\times e_M$ and $e_{M_1}e_{M_2} = 0$ if $M_1 \neq M_2$ (Loomis [3] uses notation "e_α" instead of "e_M"). Note that $\|e_M\| \geq 1$ for each $M \in \mathfrak{M}$ $(\|e_M\| = \|e_M^2\| \leq \|e_M\|^2)$.

It follows that $\|x\|^2 = \sum_{M\in\mathfrak{M}} |x(M)|^2\|e_M\|^2 \geq \sum_{M\in\mathfrak{M}} |x(M)|^2$. The last part follows from Loomis [3, 10G]: If we assume that each minimal projection has norm one, then $\|x\|^2 = \sum_{M\in\mathfrak{M}} |x(M)|^2$ and $(x,a) = \sum_{M\in\mathfrak{M}} x(M)\bar{a}(M)$ for all $x,a \in A$ (and there exists $a \in A$ such that $f(x) = (x,a)$ for all $x \in A$). □

Now we have a characterization of those commutative H^*-algebra in which each minimal projection has norm one.

Theorem 2. *Let A be a semi-simple commutative completely symmetric Banach algebra. Assume further that the series $\sum_{M\in\mathfrak{M}} |x(M)|^2$ converges for each $x \in A$ and that for each bounded linear functional f on A there exists $a \in A$ such that $f(x) = \sum_{M\in\mathfrak{M}} x(M)\bar{a}(M)$ for all $x \in A$. Then there exists a Hilbertian norm $\|\;\|_2$ on A, equivalent to the original norm such that A is an H^*-algebra with respect to the scalar product $(\,,\,)$ associated with $\|\;\|_2$ and the original involution. Also, each minimal projection in A has norm* 1.

Proof. For each $x,y \in A$, define $(x,y) = \sum_{M\in\mathfrak{M}} x(M)\bar{y}(M)$. This series converges absolutely for all $x,y \in A$, since

$$\sum_{i=1}^{k} |x(M_i)\bar{y}(M_i)| \leq \frac{1}{2}\left(\sum_{i=1}^{k} |x(M_i)|^2 + \sum_{i=1}^{k} |y(M_i)|^2\right) \tag{2.1}$$

for each finite subset $\{M_1,\dots,M_k\}$ of $\mathfrak{M}$. Hence, the inner product $(\,,\,)$ is defined everywhere on A. Let $\|\;\|^2$ be the corresponding norm, $\|x\|_2^2 = (x,x)$ for all $x \in A$. Let us show that A is complete with respect to $\|\;\|_2$.

First, note that the completion A' of A with respect to $\|\;\|_2$ is a proper H^*-algebra (since $\|x^*\|_2 = \|x\|_2$ for all $x \in A$). Hence, A' is semi-simple. (It is a consequence of

Loomis [3, 27A].) So we can apply [5, Sec. 12, Thm. 1]: there exists $C > 0$ such that $\|x\|_2 \leq C\|x\|$ for all $x \in A$.

Now, let $\{a_n\}$ be a sequence of numbers of A such that $\lim_{m,n} \|a_n - a_m\|_2 = 0$. Then there exists $\mathfrak{N} > 0$ such that $\|a_n\|_2 \leq \mathfrak{N}$ for each n. For each fixed $x \in A$ define

$$f(x) = \lim_{m\to\infty} (x, a_m). \tag{2.2}$$

From $|(x, a_m)| < \|x\|_2 \|a_m\|_2 \leq \mathfrak{N}C\|x\|$ we conclude that f is a bounded linear functional on A. Hence, there exists $a \in A$ so that $f(x) = \sum_{M\in\mathfrak{M}} x(M)\bar{a}(M)$ for each $x \in A$. Let us show that $\|a - a_n\|_2 \to 0$. Let $\epsilon > 0$ be arbitrary, take n_0 so that $\|a_m - a_n\|_2 < \epsilon/2$ if $m, n > n_0$. Let $n > n_0$ and $x \in A$ be fixed. Then $\|a - a_n\|_2^2 = |(a - a_n, a - a_n)| \leq |(a - a_n, a - a_m)| + |(a - a_n, a_m - a_n)| \leq |f(a - a_n) - (a - a_n, a_m)| + \|a - a_n\|_2 \|a_m - a_n\|_2$.

Select $m > n_0$ so that

$$\left| f(a - a_n) - (a - a_n, a_m) \right| \leq \frac{\epsilon}{2} \|a - a_n\|_2. \tag{2.3}$$

Thus

$$\|a - a_n\|_2^2 \leq \frac{\epsilon}{2} \|a - a_n\|_2 + \frac{\epsilon}{2} \|a - a_n\|_2 = \epsilon \|a - a_n\|_2, \tag{2.4}$$

and this implies that $\|a - a_n\|_2 < \epsilon$ for each $n > n_0$. So, A is complete with respect to $\| \|_2$.

It follows from [5, Sec. 12, Thm. 1] that the norm $\| \|_2$ and the original norm $\| \|$ on A are equivalent.

It is also easy to see that A is an H^*-algebra with respect to the scalar product $(,)$ (and the original involution).

Let us show that every minimal projection in A has norm one. First note that the product of any two distinct minimal projections e_1 and e_2 is zero, $e_1 e_2 = 0$. It follows from the fact that $e = e_1 e_2$ is also a projection and that $ee_i = e_i$, $i = 1, 2$. This means that if $e \neq 0$, then both $e = e_1$ and $e = e_2$, which is impossible, since $e_1 \neq e_2$. Thus $e_{M_1} e_{M_2} = 0$ if $M_1 \neq M_2$ (as was remarked in a proof above). But this also means that every minimal projection e is of the form $e = e_{M'}$ for some $M' \in \mathfrak{M}$. It follows then that $e(M') = 1$ and $e(M) = 0$ if $M \neq M'$. Thus $\|e\|_2^2 = |e(M')|^2 = 1$. □

For the general case we have Theorems 3 and 4 below, which constitute a characterization of any proper commutative H^*-algebra. The characterization is stated in terms of multiplicative functionals (it could also be done in terms of ideals) (needless to say, Theorems 1 and 2 could be restated in terms of multiplicative functionals also).

Theorem 3. *For each proper commutative H^*-algebra A there exists a real valued function $k(q)$, defined on the set Q of all its continuous multiplicative linear functionals, with the following properties :*

(i) $k(q) \geq 1$ *for each* $q \in Q$.

(ii) *The series* $\sum_{q\in Q} |q(x)|^2 k(q)$ *converges for each* $x \in A$.

(iii) *For each* $f \in A^*$ *there exists* $\alpha \in A$ *such that* $f(x) = \sum_{q\in Q} q(x)\bar{q}(a)k(q)$ *for each* $x \in A$ *(A^* denotes the dual of A).*

Proof. It is easy consequence of Loomis [3, 27G] that for each nonzero member q of Q there exists a unique minimal projection e_q such that $q(x) = (x, e_q)\|e_q\|^{-2}$ and

$$x = \sum_{q \in Q} q(x) e_q \tag{2.5}$$

for each $x \in A$ (note that $\{e_q\}_{q \neq 0}$ is an orthogonal basis for A). We define the function $k(q)$ by setting $k(q) = \|e_q\|^2$ for each nonzero member q of Q and $k(0) = 1$. We leave it to the reader to verify that $k(q)$ has desired properties. □

Theorem 4. *Let A be a semi-simple commutative completely symmetric algebra and let Q be the set of all its continuous multiplicative linear functionals. Assume that there exists a real valued function $k(q)$ on Q with properties* (i), (ii), *and* (iii) *in Theorem 3.*

Then A is an H^*-algebra with respect to some Hilbert space norm $\| \|_2$ equivalent to the original norm of A, and the original involution.

Proof. Define the scalar product $(\, , \,)$ on A by setting

$$(x, y) = \sum_{q \in Q} q(x) q(y^*) k(q), \tag{2.6}$$

and take that corresponding norm $\| \|_2$ (with the property that $(x, x) = \|x\|_2^2$). Then we proceed as in the proof of Theorem 2. □

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