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ALMOST TRIANGULAR MATRICES OVER DEDEKIND DOMAINS

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ABSTRACT. Every matrix over a Dedekind domain is equivalent to a direct sum of matrices $A = (a_{i,j})$, where $a_{i,j} = 0$ whenever $j > i+1$.

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1. Introduction. Two $m\times n$ matrices A and B over a ring R are called equivalent if $B = PAQ$ for invertible matrices P and Q over R. From now on, assume that R denotes a Dedekind domain with quotient field K. If $I = \langle a,b\rangle$ is a non principal ideal in R, then, in contrast with the situation for Principal Ideal Domains, the 1×2 matrix $[a,b]$ is not equivalent over R to a matrix whose off diagonal entries are 0. Using the separated divisor theorem in the form given by Levy in [2], other facts about matrices over Dedekind domains in [2], and elementary properties of ideals in Dedekind domain [1], we show that any $m\times n$ matrix over a Dedekind domain is equivalent to a direct sum of matrices $A = (a_{i,j})$ with $a_{i,j} = 0$ when $j > i+1$. If the direct summand A has rank r, then the number of rows, respectively columns, of A is either r or $r+1$. The corresponding result for similarity of matrices over principal ideal rings is that every $n\times n$ matrix over a principal ideal ring is similar to an upper triangular matrix [3, p. 42].

2. Diagonalization of matrices. If A is an $m\times n$ matrix, then A can be viewed as an R-module homomorphism $A : R^n \to R^m$ by left multiplication. If M_A denotes the submodule of R^m generated by the columns of A, then M_A is the image of A in R^m and the isomorphism class of the cokernel $S_A = R^m/M_A$ of A determines the equivalence class of A.

SEPARATED DIVISOR THEOREM [2]. There is a chain of integral R-ideals $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_r$ and a fractional R-ideal H such that

$$S_A = \begin{cases} \oplus_{i=1}^{r} \frac{R}{L_i} \oplus H \oplus R^{m-r-1}, & m < r \\ \oplus_{i=1}^{r} \frac{R}{L_i}, & m = r, \end{cases} \tag{2.1}$$

where $H = \prod_{i=1}^{r} L_i$ if $r = n$ and $H \cong R$ if $r = 0$ or $r = m$.

The isomorphism class of S_A, the ideals $\{L_i\}_{i=1}^{r}$ (as sets), and the isomorphism class of H both determine and are determined by the equivalence class of A.

We also need the following elementary facts about ideals in Dedekind domains.

Lemma 1 [1, p. 150, 154]. *Let I, J be integral ideals in R. Then*

(1) *There is an α in the quotient field K of R such that αI is integral and $\alpha I + J = R$;*

(2) *There is an R-module isomorphism $\gamma : IJ \oplus R \to I \oplus J$, given by $\gamma(u,v) = (x_1 v - u, \alpha u - x_2 v)$, where α is as in* (1) *and $x_1 \in I$, $x_2 \in J$ are chosen with $\alpha x_1 - x_2 = 1$.*

Note. The R-linear homomorphism γ is given by the matrix $\begin{pmatrix} -1 & x_1 \\ \alpha & -x_2 \end{pmatrix}$, where $\alpha \in K$.

Theorem 2.2. *Every $m \times n$ matrix A over a Dedekind domain is equivalent to a direct sum of matrices (a_{ij}) with $a_{ij} = 0$ whenever $j > i+1$.*

Proof. An $m \times n$ matrix A is called indecomposable if A is not equivalent to a matrix of the form $\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$ for any matrices B_1, B_2. That is, A is not equivalent to a direct sum of matrices B_1, B_2. If $A = 0$, the result is clear. Assume that $A \neq 0$. It is sufficient to verify the result for indecomposable matrices. In this case, if r is the rank of A over the quotient field K of R, then [2, Lem. 2.1] asserts that $m = r$ or $r+1$ and $n = r$ or $r+1$. There are then four possible cases to check.

Case 1. Assume that $m = r$ and $n = r$. Then $S_A = \oplus_{i=1}^{r} R/L_i$, with $L_1, \dots, L_r$ integral R-ideals with $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_r$ and $\prod_{i=1}^{r} L_i \cong R$. Thus, $\prod_{i=1}^{r} L_i = \langle a \rangle$ is a principal ideal generated by $a \in R$. Let $\phi_0 : R^r \to \prod_{i=1}^{r} L_i \oplus R^{r-1}$ be given by $\phi_0(r_1, \dots, r_r) = (ar_1, r_2, \dots, r_r)$ and let $\phi_j : L_1 \oplus \cdots \oplus L_{j-1} \oplus \prod_{i=j}^{r} L_i \oplus R \oplus R^{r-j-1} \to L_1 \oplus \cdots \oplus L_j \oplus \prod_{i=j+1}^{r} L_i \oplus R^{r-j-1}$ be given by $\phi_j = I_{j-1} \oplus \gamma_j \oplus I_{r-j-1}$, where $\gamma_j : \prod_{i=j}^{r} L_i \oplus R \to L_j \oplus \prod_{i=j+1}^{r} L_i$ is the map given in Lemma 1 and I_{j-1}, I_{r-j-1} are the identity maps of indicated rank. Let $\phi : R^r \to L_1 \oplus \cdots \oplus L_r \subset R^r$ be given by $\phi = \phi_{r-1}\phi_{r-2}\cdots\phi_1\phi_0$. Then the matrix $[\phi]$ of ϕ, with respect to the standard bases for R^r, is: $[\phi] = [\phi_{r-1}] \cdots [\phi]\,[\phi_0]$.

While $[\phi_i]$ may have entries which are not in R, $[\phi]$ has all its entries in R since each L_j is integral. If we write

$$[\phi_j] = \begin{pmatrix} I_j & 0 & 0 & 0 \\ 0 & -1 & x_i^j & 0 \\ 0 & \alpha_j & -x_2^j & 0 \\ 0 & 0 & 0 & I_{r-j-1} \end{pmatrix}, \tag{2.2}$$

then a direct calculation shows that

$$[\phi] = \begin{pmatrix} -a & x_1^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a\alpha_1 & -x_2^1 & x_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a\alpha_1\alpha_2 & \alpha_2 x_2^1 & x_2^2 & x_1^3 & 0 & 0 & 0 & 0 & 0 \\ -a\alpha_1\alpha_2\alpha_3 & \alpha_2\alpha_3 x_2^2 & x_2^3 & x_1^4 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & & & & & & \vdots \\ -a\prod_{i=1}^{r-1}\alpha_i & & & \cdots & & & & \alpha_{r-2}x_2^{r-2} & x_2^{r-1} \end{pmatrix}. \tag{2.3}$$

Since $[\phi]$ has the same number of rows and columns and the same cokernel as A, $[\phi]$ is equivalent to A.

Remark. Assume that $L_i = \langle a_i \rangle$ is principal for each $i, i = 1,\ldots,r$ and $a_i \in R$. The isomorphism $\gamma_j : \prod_{i=j}^{r} L_i \oplus R \oplus \to L_j \oplus \prod_{i=j+1}^{r} L_i$ can be given as $\gamma_j(u,v) = (\alpha_j u, \beta_j v)$, where $\alpha_j = 1/\prod_{i=j+1} a_i$ and $\beta_j = \prod_{i=j+1}^{r} a_i$. In this case, $[\phi] = \operatorname{diag}\{a_1,\ldots,a_r\}$ with $a_i \mid a_{i+1}$ for $1 \le i \le r$. This is the only case which occurs if R is a PID.

Case 2. Assume that $m = r$ and $n = r+1$. Then $S_A = \oplus_{i=1}^{r} R/L_i$ with $L_i, 1 \le i \le r$ integral ideals and $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_r$. Let L_{r+1} be integral ideal with $\prod_{i=1}^{r+1} L = \langle a \rangle$ principal, then $\oplus_{i=1}^{r+1} L_i \cong R^n$ and there is a chain of R-homomorphisms

$$R^n \xrightarrow{\phi} L_1 \oplus \cdots \oplus L_r \oplus L_{r+1} \xrightarrow{\pi} L_1 \oplus \cdots \oplus L_r \subseteq R^r, \tag{2.4}$$

where π is the projection on $L_1 \oplus \cdots \oplus L_r$ along L_{r+1}. The matrix of $\pi \circ \phi$ is an $m \times n$ matrix obtained by deleting the last row of $[\phi]$ and, thus, has the same form as in Case 1. Since the cokernel of $\pi\phi$ is the same as A and $[\pi\phi]$ has the same number of rows and columns as $A, [\pi\phi]$ is equivalent to A.

Case 3. Assume that $m = r+1$ and $n = r$. Then $S_A = \oplus_{i=1}^{r} R/L_i \oplus H$, where L_i, $1 \le i \le r$ are integral ideals and $H \cong \prod_{i=1}^{r} L_i$. Choose $a \in R$ with $L_r H^{-1} a$ integral. Note that $L_r H^{-1} a$ is a submodule of $H^{-1} a$. From Case 1, we construct an R-isomorphism $\phi_r : R^r \to L_1 \oplus \cdots \oplus L_{r-1} \oplus L_r H^{-1} a \subset R^{r+1}$ whose matrix has the same form as that of $[\phi]$ in Case 1. By Lemma 1, there is a chain of isomorphisms $\psi : H^{-1}a \oplus H \to H^{-1}Ha \oplus R \to R \oplus R$ carrying $L_r H^{-1} a$ onto a submodule N of $R \oplus R$. By [1, Cor. 18.24], $(H^{-1}a \oplus H)/L_r H^{-1} a \cong R/L_r \oplus H$. Let $\Phi = (I_{r-1} \oplus \psi) \circ \phi_r : R^n \to R^m$. The matrix of Φ is $m \times n$ and the first $r = n$ rows are the same as $[\phi_r]$. The last row does not contribute any entries above the main diagonal. So, for each $j > i+1$, the i,jth entry of $[\Phi]$ is 0. Since the cokernel of $[\Phi]$ is S_A and $[\Phi]$ has the same number of rows and columns as $A, [\Phi]$ and A are equivalent.

Case 4. Let $S_A = \oplus_{i=1}^{r} R/L_i \oplus H$, where $L_1,\ldots,L_r$ are integral ideals with $L_1 \subseteq \cdots \subseteq L_r$ and by replacing H (if necessary) by an isomorphic copy, H is an integral ideal. By [1, Thm. 18.20], there is an integral ideal H_o with $H_o H$ principal and $H_o + H = R$. There is an $a \in R$ such that $J = (\prod_{i=1}^{r} L_i \cdot H_o)^{-1} a \subseteq H$. As in Case 1, there is an isomorphism $\phi_{r+1} : R^{r+1} \to L_1 \oplus \cdots \oplus L_{r-1} \oplus L_r H_o \oplus J$. View $L_i \le R$ for $1 \le i \le r, L_r H_o \le H_o$. As in Case 3, there is an isomorphism $\psi : H_o \oplus H \to R \oplus R$ with $\psi(L_r H_o) = N \le R \oplus R$ and $R \oplus R/N \cong R/L_r \oplus H$. Let $\Phi = (I_{r-1} \oplus \psi) \circ \phi_{r+1}$. Then $\Phi : R^{r+1} \to R^{r+1}$ and all the rows, except possibly the last two of $[\Phi]$, are the same as that of $[\phi]$ in Case 1. So, for each $j > i+1$, the i,jth entry of $[\Phi]$ is 0. Since the cokernel of Φ is S_A, $[\Phi]$ and A are equivalent. □

Remark. While we could have given explicit formula for the entries in the matrices constructed in Cases 2, 3, and 4 as in Case 1, these entries are not canonically determined by A as a result of the many choices made in their construction. In particular, the choices of α and x_1, x_2 in Lemma 1 are not canonically determined by the ideals I, J.

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