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CHARACTERIZATION ON SOME ABSOLUTE SUMMABILITY FACTORS OF INFINITE SERIES

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ABSTRACT. A general theorem concerning some absolute summability factors of infinite series is proved. This theorem characterizes as well as generalizes our previous result [4]. Other results are also deduced.

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1. Introduction. Let $\sum a_n$ be an infinite series with partial sum s_n. Let σ_n^δ and η_n^δ denote the nth Cesàro mean of order $\delta(\delta > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$, respectively. The series $\sum a_n$ is said to be summable $|C,\delta|_k$, $k \ge 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\sigma_n^\delta - \sigma_{n-1}^\delta\right|^k < \infty, \tag{1.1}$$

or, equivalently,

$$\sum_{n=1}^{\infty} n^{-1} \left|\eta_n\right|^k < \infty. \tag{1.2}$$

Let $\{p_n\}$ be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{1.3}$$

The series $\sum a_n$ is said to be summable $\left|\overline{N},p_n\right|_k$, $k \ge 1$, if (Bor [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|T_n - T_{n-1}\right|^k < \infty, \tag{1.4}$$

where

$$T_n = P_n^{-1} \sum_{v=0}^{n} p_v s_v. \tag{1.5}$$

For $p_n = 1$, $\left|\overline{N},p_n\right|_k$ summability is equivalent to $\left|C,1\right|_k$ summability. In general, the two summabilities are not comparable. Let $\{\varphi_n\}$ be any sequence of positive real constants. The series $\sum a_n$ is said to be summable $\left|\overline{N},p_n,\varphi_n\right|_k$, $k \ge 1$, if (Sulaiman [4])

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \left|T_n - T_{n-1}\right|^k < \infty. \tag{1.6}$$

Clearly,

$$\left|\overline{N}, p_n, \frac{P_n}{p_n}\right|_k = |\overline{N}, p_n|_k, \qquad |\overline{N}, 1, n|_k = |C, 1|_k. \tag{1.7}$$

Theorem 1.1 (Sulaiman [4]). *Let $\{p_n\}$, $\{q_n\}$, and $\{\varphi_n\}$ be sequences of real positive constants. Let t_n denote the $(\overline{N}, p_n)$-mean of the series $\sum a_n$. If*

$$\begin{gathered}\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^k\left(\frac{q_n}{Q_n}\right)^k \varphi_n^{k-1}|\epsilon_n|^k|\Delta t_{n-1}|^k < \infty,\\ \sum_{n=1}^{\infty}\varphi_n^{k-1}|\epsilon_n|^k|\Delta t_{n-1}|^k < \infty,\\ \sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^k \varphi_n^{k-1}|\Delta\epsilon_n|^k|\Delta t_{n-1}|^k < \infty,\end{gathered} \tag{1.8}$$

then the series $\sum a_n\epsilon_n$ is summable $|\overline{N}, q_n, \varphi_n|_k$, $k \ge 1$, where $\Delta f_n = f_n - f_{n+1}$ for any sequence $\{f_n\}$ and

$$Q_n = \sum_{v=0}^{n} q_v \longrightarrow \infty \quad \text{as } n \longrightarrow \infty \quad (q_{-1} = Q_{-1} = 0). \tag{1.9}$$

2. Lemmas

Lemma 2.1 (Bor [1]). *Let $k > 1$ and $A = (a_{nv})$ be an infinite matrix. In order that $A \in (\ell^k; \ell^k)$, it is necessary that*

$$a_{nv} = O(1) \qquad (\text{all } n, v). \tag{2.1}$$

Lemma 2.2. *Suppose that $\epsilon_n = O(f_n g_n)$, f_n, $g_n \ge 0$, $\{\epsilon_n / f_n g_n\}$ monotonic, $\Delta g_n = O(1)$, and $\Delta f_n = O(f_n / g_{n+1})$. Then $\Delta\epsilon_n = O(f_n)$.*

Proof. Let $k_n = (\epsilon_n / f_n g_n) = O(1)$. If (k_n) is nondecreasing, then

$$\begin{aligned}\Delta\epsilon_n &= k_n f_n g_n - k_{n+1} f_{n+1} g_{n+1}\\ &\le k_n f_n g_n - k_n f_{n+1} g_{n+1}\\ &= k_n \Delta(f_n g_n) = k_n(f_n \Delta g_n + g_{n+1}\Delta f_n),\\ |\Delta\epsilon_n| &= O(f_n|\Delta g_n|) + O(g_{n+1}|\Delta f_n|)\\ &= O(f_n) + O(f_n) = O(f_n).\end{aligned} \tag{2.2}$$

If (k_n) is nonincreasing, write $\nabla f_n = f_{n+1} - f_n$,

$$\begin{aligned}\nabla\epsilon_n &= k_{n+1} f_{n+1} g_{n+1} - k_n f_n g_n\\ &\le k_n \nabla(f_n g_n)\\ &= k_n(f_n \nabla g_n + g_{n+1}\nabla f_n),\\ |\Delta\epsilon_n| &= |\nabla\epsilon_n| = O(f_n|\nabla g_n|) + O(g_{n+1}|\nabla f_n|)\\ &= O(f_n|\Delta g_n|) + O(g_{n+1}|\Delta f_n|)\\ &= O(f_n) + O(f_n) = O(f_n).\end{aligned} \tag{2.3}$$

□

3. Main Result. We state and prove the following theorem:

Theorem 3.1. *Let* $\{p_n\}$, $\{q_n\}$, $\{\alpha_n\}$, *and* $\{\beta_n\}$ *be sequences of positive real numbers such that*

$$\left\{\frac{\beta_n q_n}{Q_n}\right\} \text{ is nonincreasing;} \tag{3.1}$$

$$p_n Q_n = O(P_n q_n); \tag{3.2}$$

$$\left\{\frac{P_n q_n}{p_n Q_n}\left(\frac{\beta_n}{\alpha_n}\right)^{1-(1/k)}\epsilon_n\right\} \text{ is monotonic;} \tag{3.3}$$

$$\Delta\left(\frac{Q_n}{q_n}\right) = O(1); \tag{3.4}$$

$$\Delta\left\{\frac{p_n}{P_n}\left(\frac{\alpha_n}{\beta_n}\right)^{1-(1/k)}\right\} = O\left\{\frac{p_n q_{n+1}}{P_n Q_{n+1}}\left(\frac{\alpha_n}{\beta_n}\right)^{1-(1/k)}\right\}. \tag{3.5}$$

Then the necessary and sufficient conditions that $\sum a_n\epsilon_n$ *be summable* $|\overline{N}, q_n, \beta_n|_k$, *whenever* $\sum a_n$ *is summable* $|\overline{N}, p_n, \alpha_n|_k$, $k \geq 1$, *are*

$$\epsilon_n = O\left\{\frac{p_n Q_n}{P_n q_n}\left(\frac{\alpha_n}{\beta_n}\right)^{1-(1/k)}\right\}, \tag{3.6}$$

$$\Delta\epsilon_n = \left\{\frac{p_n}{P_{n-1}}\left(\frac{\alpha_n}{\beta_n}\right)^{1-(1/k)}\right\}. \tag{3.7}$$

Proof. Write

$$\begin{aligned} T_n &= \beta_n^{1-(1/k)}\left(\frac{q_n}{Q_n Q_{n-1}}\right)\sum_{v=1}^{n} Q_{v-1} a_v \epsilon_v, \\ t_n &= \alpha_n^{1-(1/k)}\left(\frac{p_n}{P_n P_{n-1}}\right)\sum_{v=1}^{n} P_{v-1} a_v, \end{aligned} \tag{3.8}$$

$$\begin{aligned} T_n &= \beta_n^{1-(1/k)}\left(\frac{q_n}{Q_n Q_{n-1}}\right)\sum_{v=1}^{n} P_{v-1} a_v \frac{Q_{v-1}}{P_{v-1}}\epsilon_v \\ &= \beta_n^{1-(1/k)}\left(\frac{q_n}{Q_n Q_{n-1}}\right)\left[\sum_{v=1}^{n-1}\sum_{r=1}^{v}(P_{r-1}a_r)\Delta\left(\frac{Q_{v-1}}{P_{v-1}\epsilon_v}\right) + \sum_{r=1}^{n}(P_{r-1}a_r)\left(\frac{Q_{n-1}}{P_{n-1}}\epsilon_n\right)\right] \\ &= \beta_n^{1-(1/k)}\left(\frac{q_n}{Q_n Q_{n-1}}\right)\sum_{v=1}^{n-1}\frac{P_v P_{v-1}}{p_v}\alpha_v^{(1/k)-1} t_v\left\{\frac{-q_v}{P_{v-1}}\epsilon_v + \frac{p_v Q_v \epsilon_v}{P_{v-1}P_v} + \frac{Q_v}{P_v}\Delta\epsilon_v\right\} \\ &\quad + \left(\beta_n^{1-(1/k)}\frac{q_n}{Q_n Q_{n-1}}\right)\frac{P_n P_{n-1}}{p_n} t_n \alpha_n^{(1/k)-1}\frac{Q_{n-1}}{P_{n-1}}\epsilon_n \\ &= \beta_n^{1-(1/k)}\frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\left\{\frac{-q_v}{p_v}P_v\alpha_v^{(1/k)-1}t_v\epsilon_v \right.\\ &\qquad\qquad \left. + \alpha_v^{(1/k)-1}Q_v t_v\epsilon_v + \frac{P_v Q_v}{p_v}\alpha_v^{(1/k)-1}t_v\Delta\epsilon_v\right\} \\ &\quad + \frac{P_n q_n}{p_n Q_n}\alpha_n^{(1/k)-1}\beta_n^{1-(1/k)}t_n\epsilon_n. \end{aligned} \tag{3.9}$$

Let us denote the above form of T_n by $T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}$.

By Minkowski's inequality, in order to prove the sufficiency, it is sufficient to show that $\sum_{n=1}^{\infty}|T_{n,r}|^k<\infty, r=1,2,3,4$. Applying Hölder's inequality,

$$\begin{aligned}\sum_{n=2}^{m+1}|T_{n,1}|^k &= \sum_{n=2}^{m+1}\left|\beta_n^{1-(1/k)}\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{-q_v}{p_v}P_v\alpha_v^{(1/k)-1}t_v\epsilon_v\right|^k \\ &\le \sum_{n=2}^{m+1}\beta_n^{k-1}\left(\frac{q_n}{Q_n}\right)^k\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}q_v\left(\frac{P_v}{p_v}\right)^k\alpha_v^{1-k}|t_v|^k|\epsilon_v|^k\left\{\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right\}^{k-1} \\ &\le O(1)\sum_{v=1}^{m}q_v\left(\frac{P_v}{p_v}\right)^k\alpha_v^{1-k}|t_v|^k|\epsilon_v|^k\sum_{n=v+1}^{m+1}\beta_n^{k-1}\left(\frac{q_n}{Q_n}\right)^k\frac{1}{Q_{n-1}} \\ &= O(1)\sum_{v=1}^{m}q_v\left(\frac{P_v}{p_v}\right)^k\alpha_v^{1-k}|t_v|^k|\epsilon_v|^k\left(\beta_v\frac{q_v}{Q_v}\right)^{k-1}\sum_{n=v+1}^{m+1}\frac{q_n}{Q_nQ_{n-1}} \\ &= O(1)\sum_{v=1}^{m}\left(\frac{q_vP_v}{p_vQ_v}\right)^k\left(\frac{\beta_v}{\alpha_v}\right)^{k-1}|t_v|^k|\epsilon_v|^k,\end{aligned}$$

$$\begin{aligned}\sum_{n=2}^{m+1}|T_{n,2}|^k &= \sum_{n=2}^{m+1}\left|\beta_n^{1-(1/k)}\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\alpha_v^{(1/k)-1}Q_vt_v\epsilon_v\right|^k \\ &= \sum_{n=2}^{m+1}\beta_n^{k-1}\left(\frac{q_n}{Q_n}\right)^k\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\alpha_v^{1-k}\left(\frac{Q_v}{q_v}\right)^kq_v|t_v|^k|\epsilon_v|^k\left\{\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right\}^{k-1} \\ &\le O(1)\sum_{v=1}^{m}\alpha_v^{1-k}\left(\frac{Q_v}{q_v}\right)^kq_v|t_v|^k|\epsilon_v|^k\sum_{n=v+1}^{m+1}\beta_n^{k-1}\left(\frac{q_n}{Q_n}\right)^k\frac{1}{Q_{n-1}} \\ &= O(1)\sum_{v=1}^{m}\alpha_v^{1-k}\left(\frac{Q_v}{q_v}\right)^kq_v|t_v|^k|\epsilon_v|^k\left(\beta_v\frac{q_v}{Q_v}\right)^{k-1}\sum_{n=v+1}^{m+1}\frac{q_n}{Q_nQ_{n-1}} \\ &= O(1)\sum_{v=1}^{m}\left(\frac{\beta_v}{\alpha_v}\right)^{k-1}|t_v|^k|\epsilon_v|^k \\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\left(\frac{q_v}{Q_v}\right)^k\left(\frac{\beta_v}{\alpha_v}\right)^{k-1}|t_v|^k|\epsilon_v|^k,\end{aligned}$$

$$\begin{aligned}\sum_{n=2}^{m+1}|T_{n,3}|^k &= \sum_{n=2}^{m+1}\left|\beta_n^{1-(1/k)}\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{P_{v-1}}{p_v}Q_v\alpha_v^{(1/k)-1}t_v\Delta\epsilon_v\right|^k \\ &\le \sum_{n=2}^{m+1}\beta_n^{k-1}\left(\frac{q_n}{Q_n}\right)^k\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}q_v\left(\frac{P_{v-1}}{p_v}\right)^k\alpha_v^{1-k}\left(\frac{Q_v}{q_v}\right)^k \\ &\qquad\times|t_v|^k|\Delta\epsilon_v|^k\left\{\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right\}^{k-1} \\ &= O(1)\sum_{v=1}^{m}q_v\left(\frac{P_{v-1}}{p_v}\right)^k\left(\frac{Q_v}{q_v}\right)^k\alpha_v^{1-k}|t_v|^k|\Delta\epsilon_v|^k\sum_{n=v+1}^{m+1}\beta_n^{k-1}\left(\frac{q_n}{Q_n}\right)^k\frac{1}{Q_{n-1}} \\ &= O(1)\sum_{v=1}^{m}\frac{q_v}{Q_v}\left(\frac{P_{v-1}}{p_v}\right)^k\left(\frac{Q_v}{q_v}\right)^k\alpha_v^{1-k}|t_v|^k|\Delta\epsilon_v|^k\left(\beta_v\frac{q_v}{Q_v}\right)^{k-1}\end{aligned}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_{v-1}}{p_v}\right)^k\left(\frac{\beta_v}{\alpha_v}\right)^{k-1}|t_v|^k|\Delta\epsilon_v|^k,$$

$$\sum_{n=2}^{m+1}|T_{n,4}|^k = O(1)\sum_{n=1}^{m}\left(\frac{q_nP_n}{p_nQ_n}\right)^k\left(\frac{\beta_n}{\alpha_n}\right)^{k-1}|t_n|^k|\epsilon_n|^k. \tag{3.10}$$

Sufficiency of (3.6) and (3.7) follows.

Necessity of (3.6). Using the result of Bor in [2], the transformation from (t_n) into (T_n) maps ℓ^k into ℓ^k and, hence by Lemma 2.1 the diagonal elements of this transformation are bounded and so (3.6) is necessary.

Necessity of (3.7). This follows from Lemma 2.2 and the necessity of (3.6) by taking

$$f_n = \left(\frac{p_n}{P_n}\right)\left(\frac{\alpha_n}{\beta_n}\right)^{1-(1/k)}, \qquad g_n = \frac{Q_n}{q_n}. \tag{3.11}$$

□

4. Applications

Corollary 4.1. *Suppose that the conditions (3.1) and (3.2) are satisfied. Then the necessary and sufficient condition that* $\sum a_n$ *be summable* $|\overline{N},q_n,\beta_n|_k$*, whenever it is summable* $|\overline{N},p_n,\alpha_n|_k$*,* $k\ge 1$*, is*

$$\frac{P_nq_n}{p_nQ_n} = O\left\{\left(\frac{\alpha_n}{\beta_n}\right)^{1-(1/k)}\right\}. \tag{4.1}$$

Proof. The proof follows from Theorem 3.1 by putting $\epsilon_n = 1$ and noticing that we do not need the conditions (3.3), (3.4), and (3.5) as $\Delta\epsilon_n = 0$ for $\epsilon_n = 1$. □

Corollary 4.2. *Suppose that (3.2) and (3.4) are satisfied,* $\{(P_nq_n/p_nQ_n)^{(1/k)}\epsilon_n\}$ *is monotonic, and*

$$\Delta\left\{\frac{p_n}{P_n}\left(\frac{P_nq_n}{p_nQ_n}\right)^{1-(1/k)}\right\} = O\left\{\frac{p_nq_{n+1}}{P_nQ_{n+1}}\left(\frac{P_nq_n}{p_nQ_n}\right)^{1-(1/k)}\right\}. \tag{4.2}$$

Then the necessary and sufficient conditions that $\sum a_n\epsilon_n$ *be summable* $|\overline{N},q_n|_k$ *whenever* $\sum a_n$ *is summable* $|\overline{N},p_n|_k$*,* $k\ge 1$*, are*

$$\epsilon_n = O\left\{\frac{p_nQ_n}{P_nq_n}\right\}^{1/k}, \qquad \Delta\epsilon_n = \left\{\frac{p_n}{P_{n-1}}\left(\frac{P_nq_n}{p_nQ_n}\right)^{1-(1/k)}\right\}. \tag{4.3}$$

Proof. The proof follows from Theorem 3.1 by putting $\alpha_n = P_n/p_n$, $\beta_n = Q_n/q_n$. □

Corollary 4.3 (Bor and Thorpe [3]). *Suppose that* $p_nQ_n = O(P_nq_n)$ *and* $P_nq_n = O(p_nQ_n)$*. Then, the series* $\sum a_n$ *is summable* $|\overline{N},q_n|_k$ *if and only if it is summable* $|\overline{N},p_n|_k$*,* $k\ge 1$*.*

Proof. The proof follows from the sufficient part of Corollary 4.1. □

Remark. It may be noticed that (3.4) can be replaced by

$$Q_n \Delta q_n = O(q_n q_{n+1}), \tag{4.4}$$

as

$$\begin{aligned} \left| \Delta\left(\frac{Q_n}{q_n}\right) \right| &= \left| \frac{Q_n}{q_n} - \frac{Q_{n+1}}{q_{n+1}} \right| = \left| \frac{q_{n+1}Q_n - q_n(Q_n + q_{n+1})}{q_n q_{n+1}} \right| \\ &= \left| \frac{Q_n \Delta q_n}{q_n q_{n+1}} + 1 \right| \\ &\le 1 + \frac{Q_n |\Delta q_n|}{q_n q_{n+1}}. \end{aligned} \tag{4.5}$$

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