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A GENERALIZED HANKEL CONVOLUTION ON ZEMANIAN SPACES

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ABSTRACT. We define a new generalized Hankel convolution on the Zemanian distribution spaces of slow growth.

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1. Introduction. Zemanian (see [17, 19]) investigated the Hankel integral transformation, defined by

$$h_\mu(\phi)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\phi(x)\,dx, \quad y \in (0,\infty), \tag{1.1}$$

where J_μ represents the Bessel function of the first kind and of order μ, in spaces of generalized functions. Throughout this paper, μ is greater than $-1/2$.

In [17], it was introduced the space H_μ constituted by all those complex valued and smooth functions ϕ on $(0,\infty)$ such that

$$\gamma^\mu_{m,k}(\phi) = \sup_{x\in(0,\infty)} \left| x^m \left(\frac{1}{x}D\right)^k \left(x^{-\mu-1/2}\phi(x)\right)\right| < \infty \tag{1.2}$$

for every $m,k \in \mathbb{N}$. H_μ is endowed with the topology generated by the family $\{\gamma^\mu_{m,k}\}_{m,k\in\mathbb{N}}$ of seminorms and, thus, H_μ is a Fréchet space. The space $\mathbb{O}$ of multipliers of H_μ was characterized in [3] as follows. A smooth function f on $(0,\infty)$ is in $\mathbb{O}$ if and only if, for every $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $(1+x^2)^n((1/x)D)^k f(x)$ is a bounded function on $(0,\infty)$ (see [3, Thm. 2.3]). The Hankel transformation h_μ is an automorphism of H_μ (see [19, Thm. 5.4-1]). The dual space of H_μ is denoted by H'_μ as usual. The Hankel transformation is defined on H'_μ as the transpose of the Hankel transformation on H_μ. That is, for each $f \in H'_\mu$, the Hankel transform $h'_\mu f$ of f is given by

$$\langle h'_\mu f, \phi\rangle = \langle f, h_\mu \phi\rangle, \quad \phi \in H_\mu. \tag{1.3}$$

Thus, h'_μ is an automorphism of H'_μ when it is considered on H'_μ the weak* or the strong topology.

Zemanian [18] defined the spaces of functions $B_{\mu,a}$, $a \in (0,\infty)$ and B_μ as follows. Let $a \in (0,\infty)$. A smooth function ϕ on $(0,\infty)$ is in $B_{\mu,a}$ provided that $\phi \in H_\mu$ and $\phi(x) = 0$, $x \in (a,\infty)$. This space $B_{\mu,a}$ is equipped with the topology induced by H_μ on it. Thus, $B_{\mu,a}$ is a Fréchet space. Moreover, if $0 < a < b$, then $B_{\mu,a}$ is continuously contained in

$B_{\mu,b}$. The union space $B_\mu = \cup_{a>0} B_{\mu,a}$ is endowed with the inductive topology. B_μ is a dense subspace of H_μ. The dual space of B_μ is denoted by B'_μ. In [18, Thm. 1], the Hankel transform $h_\mu(B_\mu)$ of B_μ was characterized.

Haimo [12], Hirschman, Jr. [14], and Cholewinski [10] studied the convolution for a variant of the Hankel transformation, which is closely connected to h_μ. After straightforward manipulations in the convolution operators defined by the above mentioned authors, a convolution for the transformation h_μ can be obtained. Said convolution operation is defined as follows. Let f and g be measurable functions on $(0,\infty)$. The Hankel convolution $f * g$ of f and g is given by

$$(f*g)(x) = \int_0^\infty f(y)(\tau_x g)(y)\,dy, \tag{1.4}$$

where

$$(\tau_x g)(y) = \int_0^\infty g(z) D_\mu(x,y,z)\,dz \tag{1.5}$$

provided that the above integrals exist, and being

$$\begin{aligned} &D_\mu(x,y,z) \\ &\quad = \int_0^\infty t^{-\mu-1/2}(xt)^{1/2}J_\mu(xt)(yt)^{1/2}J_\mu(yt)(zt)^{1/2}J_\mu(zt)\,dt, \quad x,y,z \in (0,\infty). \end{aligned} \tag{1.6}$$

If $x^{\mu+1/2}f$ and $x^{\mu+1/2}g$ are in $L_1(0,\infty)$, the space of absolutely integrable functions on $(0,\infty)$, then $x^{\mu+1/2}(f*g) \in L_1(0,\infty)$ and the interchange formula

$$h_\mu(f*g)(x) = x^{-\mu-1/2}h_\mu(f)(x)h_\mu(g)(x), \quad x \in (0,\infty) \tag{1.7}$$

holds.

The study of the Hankel convolution on distribution spaces was started by Sousa-Pinto [11]. He defined the Hankel convolution of distributions of compact support on $(0,\infty)$ for $\mu = 0$. In the last years, the $*$ convolution was studied in different spaces of generalized functions by Betancor and Marrero (see [4, 5, 6, 7, 15]), Betancor and González [1], and Betancor and Rodríguez-Mesa (see [9, 8]).

The Hankel translation τ_x defines a continuous mapping from H_μ into itself for every $x \in (0,\infty)$ (see [15, Prop. 2.1]). Then the Hankel convolution $T*\phi$ of $T \in H'_\mu$ and $\phi \in H_\mu$ can be defined by

$$(T*\phi)(x) = \langle T, \tau_x\phi\rangle, \quad x \in (0,\infty). \tag{1.8}$$

In [15, Prop. 3.5], it was proved that $x^{-\mu-1/2}(T*\phi) \in \mathbb{O}$ for every $T \in H'_\mu$ and $\phi \in H_\mu$. The subspace $\mathbb{O}'_{\mu,*}$ of H'_μ consisting of the convolution operators in H_μ was characterized in [15, Prop. 4.2] as follows. A functional $T \in H'_\mu$ belongs to $\mathbb{O}'_{\mu,*}$ (i.e., $T*\phi \in H_\mu$ for every $\phi \in H_\mu$) if and only if $x^{-\mu-1/2}h'_\mu(T)$ is in $\mathbb{O}$. The convolution $S*T$ of $S \in H'_\mu$ and $T \in \mathbb{O}'_{\mu,*}$ is defined in [15].

Definition 1. Let $S \in H'_\mu$ and $T \in \mathbb{O}'_{\mu,*}$. The $*$-convolution $S*T$ of S and T is the element of H'_μ defined by

$$\langle S*T, \phi\rangle = \langle S, T*\phi\rangle, \quad \phi \in H_\mu. \tag{1.9}$$

If $S \in H'_\mu$ and $T \in \mathbb{O}'_{\mu,*}$, the following extension of the interchange formula (1.7):

$$h'_\mu(S*T)(x) = x^{-\mu-1/2} h'_\mu(S) h'_\mu(T) \tag{1.10}$$

holds.

In this paper, inspired in [13], we define the Hankel convolution in a subspace of $H'_\mu \times H'_\mu$ that contains $H'_\mu \times \mathbb{O}'_{\mu,*}$. The new convolution generalizes the one defined in [15, Def. 1].

Throughout this paper, C always denotes a suitable positive constant, which is not necessarily the same in each occurrence.

2. The generalized Hankel convolution on H'_μ.

Now, we are going to define a new generalized Hankel convolution on H'_μ. Let S and T be in H'_μ. Assume that

(P.1) $(S*\phi)(T*\psi) \in L_1(0,\infty)$ for every $\phi, \psi \in H_\mu$,

(P.2) $\int_0^\infty \tau_x(T*\phi)(y)(S*\psi)(y)\,dy = \int_0^\infty (T*\phi)(y)\tau_x(S*\psi)(y)\,dy$ for every $\phi,\psi \in H_\mu$ and $x \in (0,\infty)$.

When S and T satisfy properties (P.1) and (P.2), we say that the pair (S,T) has the (P)-property for the sake of simplicity.

Fixing $\psi \in H_\mu$, we define the linear mapping F_ψ from H_μ into the space $D'(0,\infty)$ of the distributions in $(0,\infty)$ by

$$F_\psi(\phi) = (S*\psi)(T*\phi), \quad \phi \in H_\mu. \tag{2.1}$$

F_ψ is a continuous mapping when $D'(0,\infty)$ is endowed with the weak* topology. Indeed, according to [15, Prop. 3.5], $x^{-\mu-1/2}(S*\phi) \in \mathbb{O}$ and $x^{-\mu-1/2}(T*\phi) \in \mathbb{O}$ for each $\phi \in H_\mu$. Also, $x^{-\mu-1/2}(T*\phi_n) \to 0$, as $n \to \infty$, in $\mathbb{O}$ provided that $\phi_n \to 0$, as $n \to \infty$, in H_μ. Hence, if $\phi_n \to 0$, as $n \to \infty$, in H_μ, then $F_\psi(\phi_n) \to 0$, as $n \to \infty$, in $D'(0,\infty)$. Then we conclude that F_ψ is continuous.

Therefore, since (S,T) satisfies (P.1), [16, Thm. 2] implies that F_ψ is a continuous mapping from H_μ into $L_1(0,\infty)$.

In other words, we have seen that the bilinear mapping

$$L : H_\mu \times H_\mu \longrightarrow L_1(0,\infty) \tag{2.2}$$

defined by

$$L(\phi,\psi) = (S*\psi)(T*\phi), \quad \psi,\phi \in H_\mu \tag{2.3}$$

is separately continuous. Then, since H_μ is a Fréchet space, the bilinear form $\mathcal{L}$, defined on $H_\mu \times H_\mu$ by

$$\mathcal{L}(\phi,\psi) = \int_0^\infty (S*\psi)(x)(T*\phi)(x)\,dx, \quad \psi,\ \phi \in H_\mu \tag{2.4}$$

is continuous.

Now, we introduce the linear mapping $\mathbb{L}$ from H_μ into H'_μ as follows. For every $\psi \in H_\mu$, $\mathbb{L}(\psi)$ denotes the element of H'_μ defined by

$$\langle \mathbb{L}(\psi), \phi \rangle = \mathcal{L}(\psi,\phi), \quad \phi \in H_\mu. \tag{2.5}$$

From [5, Lem. 2.2] and by taking into account that (S,T) satisfies (P.2), we have

$$\mathbb{L}(\tau_y\psi) = \tau_y(\mathbb{L}\psi), \quad \psi \in H_\mu. \tag{2.6}$$

Hence, according to [7, Prop. 1], there exists a unique $R \in H'_\mu$ such that

$$\mathbb{L}(\psi) = R * \psi, \quad \psi \in H_\mu. \tag{2.7}$$

Definition 2. Let S and $T \in H'_\mu$ such that the pair (S,T) satisfies the (P)-property. We define the Hankel convolution $S\#T$ of S and T as the unique element of H'_μ satisfying

$$\langle (S\#T) * \psi, \phi\rangle = \int_0^\infty (S*\psi)(x)(T*\phi)(x)\,dx, \quad \psi,\ \phi \in H_\mu. \tag{2.8}$$

Now, we show that Definition 2 applies to a wide class of generalized functions in H'_μ. Let $m \in \mathbb{Z}$. We consider the space Y_m that consists of all those complex valued and smooth functions f on $(0,\infty)$ such that

$$\sup_{x\in(0,\infty)} (1+x^2)^m x^{-\mu-1/2}|f(x)| < \infty. \tag{2.9}$$

According to [15, proof of Prop. 3.5], if $T \in H'_\mu$, then there exists $m \in \mathbb{Z}$ for which $T*\phi \in Y_m$, for each $\phi \in H_\mu$. We say that a functional $T \in H'_\mu$ is in $\mathbb{Y}_m$ when $T*\phi \in Y_m$ for every $\phi \in H_\mu$.

Proposition 2.1. *Let $S \in \mathbb{Y}_k$ and $T \in \mathbb{Y}_m$. Then (S,T) has the (P)-property provided that $m+k < \mu+1$.*

Proof. Let ϕ, $\psi \in H_\mu$. It is easy to see that

$$(S*\psi)(T*\phi) \in L_1(0,\infty). \tag{2.10}$$

Let $x \in (0,\infty)$. We can write ([14, (2), p. 308])

$$\tau_x(T*\phi)(y) = \int_{|x-y|}^{x+y} D_\mu(x,y,z)(T*\phi)(z)\,dz, \quad y \in (0,\infty). \tag{2.11}$$

Moreover, since $S \in \mathbb{Y}_k$ and $T \in \mathbb{Y}_m$, by taking into account [14, (2), p. 310], it follows that

$$\begin{aligned} &\int_0^\infty |(S*\psi)(y)| \int_{|x-y|}^{x+y} D_\mu(x,y,z)\,|(T*\phi)(z)|\,dz\,dy \\ &\quad\le C\int_0^\infty y^{\mu+1/2}\left(1+y^2\right)^{-k}\int_{|x-y|}^{x+y} D_\mu(x,y,z)z^{\mu+1/2}\left(1+z^2\right)^{-m} dz\,dy \\ &\quad\le Cx^{\mu+1/2}\int_0^\infty \left(1+y^2\right)^{-m-k} y^{2\mu+1}\,dy < \infty. \end{aligned} \tag{2.12}$$

Hence, Fubini theorem leads to

$$\int_0^\infty (S*\psi)(y)\tau_x(T*\phi)(y)\,dy = \int_0^\infty \tau_x(S*\psi)(y)(T*\phi)(y)\,dy. \tag{2.13}$$

Thus, we conclude that the pair (S,T) has the (P)-property. $\square$

In particular, from Proposition 2.1, we can immediately deduce the following.

Corollary 2.2. *If $S \in H'_\mu$ and $T \in \mathbb{O}'_{\mu,*}$, then (S,T) has the (P)-property.*

Proof. According to [15, Prop. 4.3], $T * \psi \in H_\mu$, for every $\phi \in H_\mu$. Hence, $T \in Y_m$ for every $m \in \mathbb{Z}$ and, from Proposition 2.1, we infer that (S,T) has the (P)-property. □

Now, we establish that the convolution $*$ defined by Definition 1 on $H'_\mu \times \mathbb{O}'_{\mu,*}$ (see [15]) is a special case of the convolution # given in Definition 2.

Proposition 2.3. *Let $S \in H'_\mu$ and $T \in \mathbb{O}'_{\mu,*}$. Then $S * T = S\#T$.*

Proof. By Corollary 2.2, the pair (S,T) has the (P)-property. Moreover, by invoking [15, Props. 3.5 and 4.3], we can write

$$\begin{aligned}\langle (S*T)*\psi,\phi\rangle &= \langle S*T,\psi*\phi\rangle = \langle S, T*(\psi*\phi)\rangle \\ &= \langle S,(T*\phi)*\psi\rangle = \langle S*\psi, T*\phi\rangle \\ &= \int_0^\infty (S*\psi)(x)(T*\phi)(x)\,dx, \quad \psi,\phi\in H_\mu.\end{aligned} \tag{2.14}$$

Thus, we conclude that $S * T = S\#T$. □

Next, some algebraic properties of the #-convolution are proved.

Proposition 2.4. *Let $S,T \in H'_\mu$ and $R \in \mathbb{O}'_{\mu,*}$. Assume that (S,T) satisfies the (P)-property. Then*

(i) $S\#T = T\#S$.

(ii) $(S\#T)\#R = S\#(T\#R)$.

(iii) $T\#\delta_\mu = T$, *where δ_μ represents the element of H'_μ defined by*

$$\langle \delta_\mu,\phi\rangle = 2^\mu\Gamma(\mu+1)\lim_{x\to 0^+} x^{-\mu-1/2}\phi(x), \quad \phi\in H_\mu. \tag{2.15}$$

(iv) $S_\mu(S\#T) = (S_\mu S)\#T = S\#(S_\mu T)$, *where S_μ denotes the Bessel operator $x^{-\mu-1/2}D\times x^{2\mu+1}Dx^{-\mu-1/2}$.*

Proof. (i) It is clear that (T,S) has the (P)-property. Moreover, according to [15, Prop. 3.5], for every $\psi,\phi \in H_\mu$,

$$\begin{aligned}\langle (S\#T)*\psi,\phi\rangle &= \langle S\#T,\psi*\phi\rangle = \langle (S\#T)*\phi,\psi\rangle \\ &= \int_0^\infty (S*\phi)(x)(T*\psi)(x)\,dx.\end{aligned} \tag{2.16}$$

Hence, $S\#T = T\#S$.

(ii) By virtue of Proposition 2.3, the pair $(S\#T,R)$ satisfies the (P)-property and $(S\#T)\#R = (S\#T)*R$. Moreover, $(S,T*R)$ has the (P)-property. Indeed, let $\psi,\phi \in H_\mu$. According to [15, Props. 4.3 and 4.7(i)], since (S,T) satisfies the (P)-property, we have

$$(S*\psi)((T*R)*\phi) = (S*\psi)(T*(R*\phi)) \in L_1(0,\infty), \tag{2.17}$$

and

$$\begin{aligned}
\int_0^\infty \tau_x\big((T*R)*\phi\big)(y)(S*\psi)(y)\,dy
&= \int_0^\infty \tau_x\big(T*(R*\phi)\big)(y)(S*\psi)(y)\,dy \\
&= \int_0^\infty \big(T*(R*\phi)\big)(y)\tau_x(S*\psi)(y)\,dy \\
&= \int_0^\infty \big((T*R)*\phi\big)(y)\tau_x(S*\psi)(y)\,dy, \quad x\in(0,\infty).
\end{aligned} \tag{2.18}$$

Also, we can write by [15, Props. 3.5 and 4.7(i)], for each $\phi,\psi\in H_\mu$,

$$\begin{aligned}
\big\langle\big((S\#T)*R\big)*\psi,\phi\big\rangle &= \big\langle(S\#T)*(R*\psi),\phi\big\rangle \\
&= \big\langle S\#T,(R*\psi)*\phi\big\rangle \\
&= \big\langle(S\#T)*\psi,R*\phi\big\rangle \\
&= \int_0^\infty (S*\psi)(x)\big(T*(R*\phi)\big)(x)\,dx \\
&= \int_0^\infty (S*\psi)(x)\big((T*R)*\phi\big)(x)\,dx.
\end{aligned} \tag{2.19}$$

Thus, we conclude that $(S\#T)*R = S\#(T*R)$.

(iii) It is immediately deduced from [15, Prop. 4.7(iv)] and Proposition 2.3.

(iv) Since (S,T) has the (P)-property, $(S_\mu S,T)$ and $(S,S_\mu T)$ also satisfy the same property. Indeed, let $\psi,\phi\in H_\mu$. Then, since the Bessel operator S_μ is a continuous operator from H_μ into itself [19, Lem. 5.3-3], by [15, Prop. 4.7(iii)],

$$\big((S_\mu S)*\psi\big)(T*\phi) = \big(S*(S_\mu\psi)\big)(T*\phi)\in L_1(0,\infty),$$

and

$$\begin{aligned}
\int_0^\infty \tau_x(T*\phi)(y)\big((S_\mu S)*\psi\big)(y)dy
&= \int_0^\infty \tau_x(T*\phi)(y)\big(S*(S_\mu\psi)\big)(y)\,dy \\
&= \int_0^\infty (T*\phi)(y)\tau_x\big(S*(S_\mu\psi)\big)(y)\,dy \\
&= \int_0^\infty (T*\phi)(y)\tau_x\big((S_\mu S)*\psi\big)(y)\,dy, \quad x\in(0,\infty).
\end{aligned} \tag{2.20}$$

Moreover, by [15, Prop. 2.2(ii)], we get

$$\begin{aligned}
\big\langle S_\mu(S\#T)*\psi,\phi\big\rangle = \big\langle S_\mu(S\#T),\psi*\phi\big\rangle &= \big\langle S\#T,(S_\mu\psi)*\phi\big\rangle \\
&= \int_0^\infty \big(S*(S_\mu\psi)\big)(x)(T*\phi)(x)\,dx \\
&= \int_0^\infty \big((S_\mu S)*\psi\big)(x)(T*\phi)(x)\,dx, \quad \psi,\phi\in H_\mu.
\end{aligned} \tag{2.21}$$

Hence, $S_\mu(S\#T) = (S_\mu S)\#T$.

To complete the proof of (iv), it is sufficient to take into account (i). □

Our next objective is to prove an interchange formula that relates the Hankel transformation h'_μ to the #-convolution.

First, we need to define the product $T \cdot S$ of T and S belonging to H'_μ.

As in [4], we say that a sequence $\{k_n\}_{n\in\mathbb{N}} \subset B_\mu$ is a Hankel approximated identity when the following three conditions hold for every $n \in N$:

(i) $k_n(x) \geq 0$, $x \in (0,\infty)$;

(ii) $k_n(x) = 0$, $x \notin ((1/n+1),(1/n))$;

(iii) $\int_0^{1/n} k_n(x)x^{\mu+1/2}\,dx = 2^\mu\Gamma(\mu+1)$.

Three useful properties of the Hankel approximated identities follow.

Proposition 2.5 ([2, Prop. 1] and [6, proof of Prop. 2.4, p. 1148]). *Let $\{k_n\}_{n\in\mathbb{N}}$ be a Hankel approximated identity. Then, we have*

(i) *For every $a > 0$, $y^{-\mu-1/2}h_\mu(k_n)(y) \to 1$, as $n \to \infty$, uniformly in $(0,a)$, and there exists $M > 0$ such that $|y^{-\mu-1/2}h_\mu(k_n)(y)| \leq M$, $n \in \mathbb{N}$ and $y \in (0,\infty)$.*

(ii) *For every $\phi \in H_\mu$, $k_n * \phi \to \phi$, as $n \to \infty$, in H_μ.*

(iii) *For every $T \in H'_\mu$, $T * k_n \to T$, as $n \to \infty$, in the strong topology of H'_μ.*

Let T and S be in H'_μ. We say that $R \in B'_\mu$ is the product $x^{-\mu-1/2}T\cdot S$ and we write $R = x^{-\mu-1/2}T\cdot S$ if for every Hankel approximated identity $\{k_n\}_{n\in\mathbb{N}}$, $x^{-\mu-1/2}(T*k_n)S \to R$ and $x^{-\mu-1/2}(S*k_n)T \to R$, as $n \to \infty$, in the weak* topology of B'_μ.

Note that if T, $S \in H'_\mu$ and there exists the product $x^{-\mu-1/2}T\cdot S$ of T and S, then also there exists the product $x^{-\mu-1/2}S\cdot T$ of S and T, and $x^{-\mu-1/2}T\cdot S = x^{-\mu-1/2}S\cdot T$. Moreover, if $T \in H_\mu$ and $S \in H'_\mu$, then

$$\langle x^{-\mu-1/2}T\cdot S,\phi\rangle = \langle S, x^{-\mu-1/2}T\phi\rangle, \quad \phi \in B_\mu. \tag{2.22}$$

Indeed, let $\{k_n\}_{n\in\mathbb{N}}$ be a Hankel approximated identity. Then we have, by Proposition 2.5(ii) and (iii),

$$\begin{aligned}
\langle x^{-\mu-1/2}(T*k_n)S,\phi\rangle &= \langle S, x^{-\mu-1/2}(T*k_n)\phi\rangle \longrightarrow \langle S, x^{-\mu-1/2}T\phi\rangle, \quad \text{as } n \longrightarrow \infty,\\
\langle x^{-\mu-1/2}(S*k_n)T,\phi\rangle &= \langle S*k_n, x^{-\mu-1/2}T\phi\rangle \longrightarrow \langle S, x^{-\mu-1/2}T\phi\rangle, \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{2.23}$$

for every $\phi \in B_\mu$.

Hence, the product that we have defined between two elements of H'_μ extends the usual product of a function in H_μ by a distribution in H'_μ.

Proposition 2.6 (The interchange formula). *Let $S,T \in H'_\mu$. Assume that the pair (S,T) has the (P)-property. Then, we have*

$$h'_\mu(S\#T) = x^{-\mu-1/2}h'_\mu(S)\cdot h'_\mu(T). \tag{2.24}$$

Proof. We only have to prove that, for every $\phi \in B_\mu$,

$$\langle x^{-\mu-1/2}(h'_\mu(S)*k_n)h'_\mu(T),\phi\rangle \longrightarrow \langle h'_\mu(S\#T),\phi\rangle, \quad \text{as } n \longrightarrow \infty, \tag{2.25}$$

where $\{k_n\}_{n\in\mathbb{N}}$ is a Hankel approximated identity.

Let $\phi \in B_\mu$ and let $\{k_n\}_{n\in\mathbb{N}}$ be a Hankel approximated identity. There exists $a > 0$ such that $\phi \in B_{\mu,a}$. Choose $b > a$ and $\chi \in B_\mu$ such that $\chi(x) = x^{\mu+1/2}$, $x \in (0,b)$.

According to [19, Thm. 5.4-1], $h_\mu(\phi) \in H_\mu$ and $h_\mu(\chi) \in H_\mu$. Hence, since (S,T) has the (P)-property, from Proposition 2.5(i), it follows that

$$\begin{aligned}\int_0^\infty &(S * h_\mu(\phi))(x)(T * h_\mu(\chi))(x)\,dx \\ &= \lim_{n\to\infty}\int_0^\infty (S * h_\mu(\phi))(x)(T * h_\mu(\chi))(x)x^{-\mu-1/2}h_\mu(k_n)(x)\,dx.\end{aligned}\tag{2.26}$$

Suppose that $\{\alpha_n\}_{n\in\mathbb{N}}$ is also a Hankel approximated identity. By [15, Prop. 4.5], we can write

$$\begin{aligned}\langle x^{-\mu-1/2}&h_\mu(\alpha_m)(S * h_\mu(\phi)), (T * h_\mu(\chi))x^{-\mu-1/2}h_\mu(k_n)\rangle \\ &= \langle h'_\mu(x^{-\mu-1/2}h_\mu(\alpha_m)(S * h_\mu(\phi))), h_\mu((T * h_\mu(\chi))x^{-\mu-1/2}h_\mu(k_n))\rangle \\ &= \langle (x^{-\mu-1/2}\phi h'_\mu(S)) * \alpha_m, (x^{-\mu-1/2}\chi h'_\mu(T)) * k_n\rangle, \quad n,\ m \in \mathbb{N}.\end{aligned}\tag{2.27}$$

Since $(T * h_\mu(\chi))x^{-\mu-1/2}h_\mu(k_n) \in H_\mu$ ([15, Prop. 3.5]), $n \in \mathbb{N}$, also $(x^{-\mu-1/2}\chi h'_\mu(T)) * k_n \in H_\mu$, $n \in \mathbb{N}$. Hence, by Proposition 2.5(iii), we have, for each $n \in \mathbb{N}$,

$$\begin{aligned}\langle (x^{-\mu-1/2}&\phi h'_\mu(S)) * \alpha_m, (x^{-\mu-1/2}\chi h'_\mu(T)) * k_n\rangle \\ &\longrightarrow \langle x^{-\mu-1/2}\phi h'_\mu(S), (x^{-\mu-1/2}\chi h'_\mu(T)) * k_n\rangle, \quad \text{as } m \longrightarrow \infty.\end{aligned}\tag{2.28}$$

Moreover, since (S,T) has the (P)-property and according to Proposition 2.5(i), one has, for every $n \in \mathbb{N}$,

$$\begin{aligned}\langle x^{-\mu-1/2}&h_\mu(\alpha_m)(S * h_\mu(\phi)), (T * h_\mu(\chi))x^{-\mu-1/2}h_\mu(k_n)\rangle \\ &= \int_0^\infty x^{-\mu-1/2}h_\mu(\alpha_m)(x)(S * h_\mu(\phi))(x)(T * h_\mu(\chi))(x)x^{-\mu-1/2}h_\mu(k_n)(x)\,dx \\ &\longrightarrow \int_0^\infty (S * h_\mu(\phi))(x)(T * h_\mu(\chi))(x)x^{-\mu-1/2}h_\mu(k_n)(x)\,dx \\ &= \langle S * h_\mu(\phi), (T * h_\mu(\chi))x^{-\mu-1/2}h_\mu(k_n)\rangle, \quad \text{as } m \to \infty.\end{aligned}\tag{2.29}$$

Hence, for every $n \in \mathbb{N}$,

$$\begin{aligned}\langle S * h_\mu(\phi), &(T * h_\mu(\chi))x^{-\mu-1/2}h_\mu(k_n)\rangle \\ &= \langle x^{-\mu-1/2}\phi h'_\mu(S), (x^{-\mu-1/2}\chi h'_\mu(T)) * k_n\rangle.\end{aligned}\tag{2.30}$$

On the other hand, since $\chi(x) = x^{\mu+1/2}$, $x \in (0,b)$, being $b > a$, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\langle x^{-\mu-1/2}&\phi h'_\mu(S), (x^{-\mu-1/2}\chi h'_\mu(T)) * k_n\rangle \\ &= \langle h'_\mu(S), x^{-\mu-1/2}\phi(h'_\mu(T) * k_n)\rangle \quad \text{for every } n \geq n_0.\end{aligned}\tag{2.31}$$

Moreover, according to [15, Prop. 3.5],

$$\begin{aligned}\langle h'_\mu(S\#T), \phi\rangle &= \langle S\#T, h_\mu(\phi)\rangle = \langle S\#T, h_\mu(x^{-\mu-1/2}\chi\phi)\rangle \\ &= \langle S\#T, h_\mu(\phi) * h_\mu(\chi)\rangle = \langle (S\#T) * h_\mu(\phi), h_\mu(\chi)\rangle.\end{aligned}\tag{2.32}$$

By combining (2.26), (2.30), and (2.32), it follows that

$$\begin{aligned}\langle h'_\mu(S\#T),\phi\rangle &= \langle (S\#T)*h_\mu(\phi),h_\mu(\chi)\rangle \\ &= \lim_{n\to\infty}\int_0^\infty \big(S*h_\mu(\phi)\big)(x)\big(T*h_\mu(\chi)\big)(x)x^{-\mu-1/2}h_\mu(k_n)(x)\,dx \\ &= \lim_{n\to\infty}\langle x^{-\mu-1/2}h'_\mu(S)\big(h'_\mu(T)*k_n\big),\phi\rangle.\end{aligned} \tag{2.33}$$

Thus, the proof is complete. □

Remark. Propositions 2.4 and 2.6 are extensions of [15, Props. 4.5 and 4.7].

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