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ONE-SIDED LEBESGUE BERNOULLI MAPS OF THE SPHERE OF DEGREE n^2 AND $2n^2$

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ABSTRACT. We prove that there are families of rational maps of the sphere of degree n^2 $(n = 2,3,4,\dots)$ and $2n^2$ $(n = 1,2,3,\dots)$ which, with respect to a finite invariant measure equivalent to the surface area measure, are isomorphic to one-sided Bernoulli shifts of maximal entropy. The maps in question were constructed by Böettcher (1903–1904) and independently by Lattès (1919). They were the first examples of maps with Julia set equal to the whole sphere.

Keywords and phrases. One-sided Bernoulli, rational maps of the sphere, Julia sets.

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1. Introduction. We consider rational maps of the sphere with Julia set equal to the whole sphere. Böettcher [3, 4] and Lattès [7] independently constructed the first examples of such maps. A map is said to be one-sided Bernoulli of maximal entropy if it can be modeled by a fair d-sided die toss, where d is the degree of the map. Freire et al. [5] and independently Lyubich [8] conjectured that all rational maps of the sphere of degree greater than 1 are one-sided Bernoulli with respect to a particular measure, namely the measure of maximal entropy. Mañé [9] and Lyubich [8] proved that, for every rational map of the sphere of degree greater than 1, the maximal entropy measure is unique. Mañé [10] showed that for every rational map, there is some forward iterate which is one-sided Bernoulli with respect to the maximal entropy measure. However, the conjecture that all rational maps are one-sided Bernoulli is yet unknown. In [12], Parry discusses some of the reasons why proving that a map is one-sided Bernoulli is particularly difficult.

In this paper, we examine two families of rational maps which were constructed by Böettcher and Lattès, and we construct direct isomorphisms between these families of maps and one-sided Bernoulli shifts with respect to the maximal entropy measure. For the maps we examine here, the maximal entropy measure is equivalent to one of the most natural measures on the sphere, the normalized surface area measure. Furthermore, Zdunik [16] proved that the only maps that could possibly be one-sided Bernoulli with respect to a measure equivalent to the surface area measure are maps constructed in the fashion discussed here, i.e., as factors of toral endomorphisms.

Let R_ω, for $\omega = n$ or $n+ni$, denote the families of maps we consider where $n \in \mathbb{N}$, and $\omega\overline{\omega} \geq 2$. In Section 2, we describe how the maps R_ω from these two families are constructed. In Section 3, we construct a toral map isomorphic to R_ω which is used throughout the rest of this paper. In Section 4, the one-sided Lebesgue Bernoulli property is introduced and the method used to prove R_ω is one-sided Bernoulli is presented. Then Sections 5 and 6 are devoted to proofs concerning the two families of maps, R_n and R_{n+ni}, respectively, using an integral lattice. In the last section, we extend the results from Sections 5 and 6 to hold for any appropriate lattice.

2. Basic construction. Let $\mathbb{C}_\infty$ denote the complex sphere and m denote the normalized surface area measure of $\mathbb{C}_\infty$. Let $\alpha, \beta \in \mathbb{C} - \{0\}$ be such that α/β is not a real number. We define a lattice of points in the complex plane by $\Gamma = [\alpha, \beta] = \{\alpha i + \beta j : i, j \in \mathbb{Z}\}$. Given a lattice Γ and a complex number ω satisfying $\omega\Gamma \subset \Gamma$, the map $T_\omega(z) = \omega z \pmod{\Gamma}$ gives an endomorphism of the complex torus $\mathbb{C}/\Gamma$. In this paper, we restrict to maps T_ω, where $\omega = n$, $n \in \mathbb{Z} - \{0, 1, -1\}$, or $\omega = n + ni$, $n \in \mathbb{Z} - \{0\}$.

Let $\wp_\Gamma$ denote the Weierstrass elliptic function defined on a lattice Γ by

$$\wp_\Gamma(z) = \frac{1}{z^2} + \sum_{\zeta \in \Gamma - \{0\}} \left[\frac{1}{(z+\zeta)^2} - \frac{1}{\zeta^2} \right]. \tag{2.1}$$

It is easy to see that $\wp_\Gamma$ is an even, onto, 2-fold branched cover of $\mathbb{C}_\infty$. Until Section 7 of this paper, we use $\Gamma = [1, i]$, and we denote $\wp = \wp_{[1,i]}$. We define a map $R_\omega : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $R_\omega = \wp T_\omega \wp^{-1}$. The map R_ω is well defined since $T_\omega \Gamma \subset \Gamma$, and hence the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}/\Gamma & \xrightarrow{T_\omega} & \mathbb{C}/\Gamma \\ \downarrow \wp & & \downarrow \wp \\ \mathbb{C}_\infty & \xrightarrow{R_\omega} & \mathbb{C}_\infty. \end{array} \tag{2.2}$$

Since R_ω is locally a composition of analytic maps, R_ω is analytic. Further, all analytic maps of $\mathbb{C}_\infty$ are rational, so R_ω is a rational map with degree $d = \omega\overline{\omega}$. Thus, for $\omega = n$, we have the $\deg(R_\omega) = n^2$ and for $\omega = n + ni$, $\deg(R_\omega) = 2n^2$.

It can be shown explicitly via algebraic identities that $R_2(z) = (z^2+1)^2/[4z(z^2-1)]$ as done in [2]. Also, Ueda [15] has proved that the rational map R_{1+i} of the sphere is given by the equation $R_{1+i}(z) = (1/2i)(z - (1/z))$, a degree 2 map.

We represent the torus $\mathbb{C}/\Gamma$ by a period parallelogram $\Omega_0 = \{x + iy : x, y \in \mathbb{R}, 0 \le x,\ y < 1\}$, the unit square without the top or right boundaries. Let $\mathcal{B}_0$ be the Lebesgue measurable sets in Ω_0, and let *leb* be normalized 2-dimensional Lebesgue measure restricted to Ω_0. We use the σ-algebra $\mathcal{B}$ and measure ν on $\mathbb{C}_\infty$ induced by the factor map $\wp$. Thus $\mathcal{B}$ is precisely the σ-algebra of Borel sets on $\mathbb{C}_\infty$ and the measure ν is given by $\nu(A) = leb(\wp|_{\Omega_0}^{-1} A)$ for all $A \in \mathcal{B}$. By definition, since T_ω is measure-preserving with respect to *leb*, R_ω is measure-preserving with respect to ν. We note that the measure ν is equivalent to m, the normalized surface area measure on $\mathbb{C}_\infty$.

Since $\wp$ is an even map, $\wp(u) = \wp(-u)$ for all $u \in \mathbb{C}$. Therefore, the map R_ω has critical values at precisely the points $\wp(u)$ where $u \equiv -u \pmod{\Gamma}$, as R_ω fails to have d distinct preimages at these points. There are exactly 4 points in the period parallelogram Ω_0 where $u \equiv -u \pmod{\Gamma}$. These are

$$V = \left\{ \frac{1}{2}, \frac{i}{2}, \frac{1+i}{2}, 0 \right\}. \tag{2.3}$$

Therefore, the critical values (i.e., the images under R_ω of the critical points) of the rational map R_ω are

$$e_1 = \wp\left(\frac{1}{2}\right), \qquad e_2 = \wp\left(\frac{i}{2}\right), \qquad e_3 = \wp\left(\frac{1+i}{2}\right), \quad \wp(0) = \infty. \tag{2.4}$$

The dynamical behavior of a rational map R_ω depends on the postcritical set of R_ω, which is defined as

$$P(R_\omega) = \overline{\bigcup_{j>0}\left\{R_\omega^j(c) : c \text{ is a critical point of } R_\omega\right\}}. \tag{2.5}$$

For $\omega = n$ or $\omega = n+ni$, $P(R_\omega) = \{e_1, e_2, e_3, \infty\}$. The postcritical orbit of the map R_ω depends on whether n is even or odd.

For any $\omega = n$ or $\omega = n+ni$ with n even,

$$e_1 \longrightarrow \infty \longrightarrow \infty \cdots, \qquad e_2 \longrightarrow \infty \longrightarrow \infty \cdots, \qquad e_3 \longrightarrow \infty \longrightarrow \infty \cdots. \tag{2.6}$$

If $\omega = n$ and n is odd, then

$$e_1 \longrightarrow e_1 \cdots, \qquad e_2 \longrightarrow e_2 \cdots, \qquad e_3 \longrightarrow e_3 \cdots, \qquad \infty \longrightarrow \infty \cdots. \tag{2.7}$$

If $\omega = n+ni$ and n is odd, then

$$e_1 \longrightarrow e_3 \longrightarrow \infty \longrightarrow \infty \cdots, \qquad e_2 \longrightarrow e_3 \longrightarrow \infty \longrightarrow \infty \cdots. \tag{2.8}$$

In all cases, R_ω has a finite postcritical set, the critical values are periodic or preperiodic, and none of the critical points of R_ω are periodic (i.e., there are no critical points c of R_ω and no integers $r > 0$ such that $R_\omega^r(c) = c$). Therefore, R_ω is ergodic and conservative (see [14]) as well as exact (see [1]) with respect to m.

3. Realization of R_ω on the torus. In order to study the behavior of R_ω on the sphere, we analyze the behavior of an isomorphic map on a subset Ω_ω of the topological closure of the period parallelogram Ω_0 as defined below. Due to the differences in the structure of $P(R_\omega)$ for n even and n odd, we define Ω_ω separately for n even and n odd as follows:

$$\begin{aligned} &\text{If } n \text{ is even, } \Omega_\omega = \left\{x+iy \in \overline{\Omega}_0 : 0 \le y \le \frac{1}{2}\right\}. \\ &\text{If } n \text{ is odd, } \Omega_\omega = \{x+iy \in \overline{\Omega}_0 : y \le x\}. \end{aligned} \tag{3.1}$$

See Figure 3.1 for an illustration of Ω_ω.

We next discuss some properties associated with Ω_ω. For all $u \in \Omega_0$, we define an operation prime ($'$) by $u' = 1+i-u \in \Omega_0$. Note that $u' = -u \pmod{\Gamma}$ and $u'' = u$.

Lemma 3.1. *There is a set of leb-measure one in Ω_0 on which the following property holds:*

$$u \in \Omega_\omega \textit{ if and only if } u' \in \Omega_0 - \Omega_\omega. \tag{3.2}$$

Proof. Let $\partial\Omega_\omega$ denote the boundary of Ω_ω, and $\Omega_\omega^0 = \Omega_\omega - \partial\Omega_\omega$ denote the interior of Ω_ω. Clearly, $\partial\Omega_\omega$ has *leb*-measure 0, so it suffices to prove this lemma for the

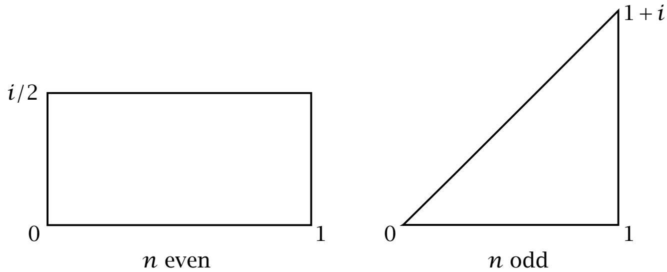

FIGURE 3.1. Illustration of Ω_ω.

points in Ω_ω^0. Suppose $u \in \Omega_\omega^0$. Then the midpoint of the line segment connecting u and u' in $\mathbb{C}$ is $(u+u')/2 = (1+i)/2$ which is the center of Ω_0. Thus $u' \in \Omega_0 - \Omega_\omega$ by the symmetry of Ω_0.

The other direction works similarly.

Let $\theta_\omega = \wp|_{\Omega_\omega}$. We recall that $\wp$ is a two-to-one (except at 4 points) surjective map, and $\wp(u) = \wp(u')$. From Lemma 3.1, θ_ω is measure-theoretically one-to-one. Therefore, $\theta_\omega^{-1}(z)$ has a unique value for every $z \in \mathbb{C}_\infty$ in a set of ν-measure 1.

Define $\hat{T}_\omega : \Omega_\omega \to \Omega_\omega$ such that $\hat{T}_\omega(u) = \theta_\omega^{-1} R_\omega \theta_\omega(u)$ and the following diagram commutes:

$$\begin{array}{ccc} \Omega_\omega & \xrightarrow{\hat{T}_\omega} & \Omega_\omega \\ \downarrow{\scriptstyle \theta_\omega} & & \downarrow{\scriptstyle \theta_\omega} \\ \mathbb{C}_\infty & \xrightarrow{R_\omega} & \mathbb{C}_\infty. \end{array} \tag{3.3}$$

Let $\mathcal{B}_\omega$ be the Lebesgue measurable sets in Ω_ω. We define $\mu(A) = 2\,leb(A)$ for all $A \in \mathcal{B}_\omega$. Thus, $\hat{T}_\omega$ is well defined on a set of μ-measure 1 and $\hat{T}_\omega$ is measure preserving with respect to μ. Using the construction defined above, we are able to prove the following proposition which enables us to study the behavior of R_ω by analyzing $\hat{T}_\omega$ on Ω_ω. □

PROPOSITION 3.2. *We have* $(\Omega_\omega, \mathcal{B}_\omega, \mu, \hat{T}_\omega) \cong (\mathbb{C}_\infty, \mathcal{B}, \nu, R_\omega)$.

PROOF. Let U be a set of *leb*-measure 1 in Ω_0 given in Lemma 3.1, and define $U_\omega = U \cap \Omega_\omega$. Then θ_ω is one-to-one on U_ω. By the definitions of ν and μ, $1 = \nu(\wp(U_\omega)) = \mu(\theta_\omega^{-1}(\wp(U_\omega))) = \mu(\Omega_\omega) = \nu(\mathbb{C}_\infty)$.

Combining these facts with the construction of $\hat{T}_\omega$ above, we have shown that $\theta_\omega : U_\omega \to \wp(U_\omega)$ is a measure-theoretic isomorphism, proving the proposition. □

4. The one-sided Lebesgue Bernoulli property. Let $X_d^+ = \prod_{j=0}^{\infty}\{0,1,2,\dots,d-1\}$ be the collection of one-sided sequences on d symbols. Let $\mathcal{C}$ be the σ-algebra generated by cylinder sets of X_d^+ and let σ be the one-sided shift on X_d^+ given by $\sigma(x_0,x_1,x_2,\dots) = (x_1,x_2,\dots)$. Let p_d be the $(1/d,1/d,\dots,1/d)$ Bernoulli measure on X_d^+, i.e., $p_d([j_0,j_1,\dots,j_{k-1}]) = 1/d^k$, where $[j_0,j_1,\dots,j_{k-1}] = \{x \in X_d^+ : x_0 = j_0,\ x_1 = j_1,\dots,\ x_{k-1} = j_{k-1}\}$ is a cylinder set.

We say that a map R_ω is one-sided Bernoulli with respect to the measure ν if $(\mathbb{C}_\infty, \mathcal{B}, \nu, R_\omega)$ is isomorphic to $(X_d^+, \mathcal{C}, p_d, \sigma)$. Due to Proposition 3.2, this is equivalent to satisfying $(\Omega_\omega, \mathcal{B}_\omega, \mu, \hat{T}_\omega) \cong (X_d^+, \mathcal{C}, p_d, \sigma)$.

To prove that $\hat{T}_\omega$ is one-sided Bernoulli with respect to μ, we construct a partition $\mathcal{P}_\omega$ of the space Ω_ω into d congruent atoms: $A_0, A_1, \dots, A_{d-1}$. Then we code the system using a map $\varphi_\omega : \Omega_\omega \to X_d^+$ such that $\varphi_\omega(u) = c_{i_1}, c_{i_2}, c_{i_3}, \dots$ where $\hat{T}_\omega^j(u) \in A_{i_j}$. Notice that $\hat{T}_\omega^j(\hat{T}_\omega(u)) = \hat{T}_\omega^{j+1}(u) \in A_{c_{j+1}}$. Using this property, it is easy to show that $\sigma\varphi_\omega(u) = \varphi_\omega(\hat{T}_\omega(u))$.

The map φ_ω is the standard method used to code a system from a partition. The join of two partitions η and ξ is defined to be $\eta \vee \xi = \{A_j \cap B_k : A_j \in \eta,\ B_k \in \xi\}$. We define $\mathcal{P}_\omega^j = \bigvee_{r=0}^{j} T_\omega^{-r}(\mathcal{P}_\omega) = \mathcal{P}_\omega \vee T_\omega^{-1}(\mathcal{P}_\omega) \vee \cdots \vee T_\omega^{-j}(\mathcal{P}_\omega)$, and we use Proposition 4.1 to show that the map φ_ω is an isomorphism. A proof of Proposition 4.1 can be found in Petersen [13]; see Barnes [1] for the noninvertible case.

Proposition 4.1. *If $\mathcal{P}_\omega$ generates under $\hat{T}_\omega$, then the map $\varphi_\omega : \Omega_\omega \to X_d^+$ is an isomorphism.*

To show that $\mathcal{P}_\omega$ is a generating partition, i.e., $\widehat{\bigvee_{j=0}^{\infty} \mathcal{P}_\omega^j}$ is the Borel σ-algebra (mod 0), we apply Theorem 4.2, whose proof can be found in Mañé [11].

Theorem 4.2. *Let $(X, \mathcal{D}, \mu)$ be a probability space, where X is a locally complete separable metric space and $\mathcal{D}$ is the Borel σ-algebra of X. Let $\mathcal{P}_\omega^j$, $j \ge 1$ be a sequence of partitions such that*

$$\lim_{j\to\infty}\left(\sup_{P\in\mathcal{P}_\omega^j} \operatorname{diam}(P)\right) = 0. \tag{4.1}$$

Then $\widehat{\bigvee_{j=0}^{\infty} \mathcal{P}_\omega^j} = \mathcal{D}$ (mod 0).

5. The family R_n.

Recall that $T_2(u) = 2u \pmod{\Gamma}$ and $R_2(z) = \wp T_2 \wp^{-1}(z)$. We prove the following for R_2.

Theorem 5.1. *For $\Gamma = [1, i]$, the rational map $R_2(z)$ with measure ν on $\mathbb{C}_\infty$ is isomorphic to the one-sided $(1/4, 1/4, 1/4, 1/4)$ Bernoulli shift on 4 symbols with measure p_4; i.e., $(\mathbb{C}_\infty, \mathcal{B}, \nu, R_2) \cong (X_4^+, \mathcal{C}, p_4, \sigma)$.*

Proof. In order to use the methods described in Section 4, we define a partition, $\mathcal{P}_2$ of Ω_2 by $\mathcal{P}_2 = \{A_k : 0 \le k \le 3\}$, where

$$\begin{aligned}
A_0 &= \left\{x + iy \in \Omega_2 : x, y \in \mathbb{R},\ 0 \le x < \frac{1}{2},\ 0 \le y < \frac{1}{4}\right\}, \\
A_1 &= \left\{x + iy \in \Omega_2 : x, y \in \mathbb{R},\ \frac{1}{2} \le x \le 1,\ 0 \le y < \frac{1}{4}\right\}, \\
A_2 &= \left\{x + iy \in \Omega_2 : x, y \in \mathbb{R},\ 0 \le x < \frac{1}{2},\ \frac{1}{4} \le y \le \frac{1}{2}\right\}, \\
A_3 &= \left\{x + iy \in \Omega_2 : x, y \in \mathbb{R},\ \frac{1}{2} \le x \le 1,\ \frac{1}{4} \le y \le \frac{1}{2}\right\}.
\end{aligned} \tag{5.1}$$

To determine the partitions $\mathcal{P}_2^j$, we analyze images of the boundaries of the atoms of $\mathcal{P}_2$ under $\hat{T}_2^{-1}$. All of these boundaries are horizontal or vertical lines. Notice that any vertical line in Ω_2 can be expressed by $l_{v,k} = \{u \in \Omega_2 : \text{Re}(u) = k\}$, for some real number k. $\hat{T}_2^{-1}(l_{v,k})$ is a pair of vertical lines described by $\{u \in \Omega_2 : \text{Re}(u) = k/2$ or $\text{Re}(u) = (k+1)/2\}$. Similarly, any horizontal line in Ω_2 is of the form $l_{h,k} = \{u \in \Omega_2 : \text{Im}(u) = k\}$ for some $k \in \mathbb{R}$ and also has 2 lines in its image under $\hat{T}_2^{-1}$. One of these lines is the horizontal line satisfying $l_1 = \{u \in \Omega_2 : \text{Im}(u) = k/2\}$. The other line is $l_2 = \{u \in \Omega_2 : \text{Im}(u) = (1-k)/2\}$. This comes from the fact that $2l_2 = \text{Im}(u) = 1-k \in \Omega_0 - \Omega_2$, and for all $u \in \{z : \text{Im}(z) = 1-k\}$, $u' \in \{z : \text{Re}(z) = k\}$. In Figure 5.1, we illustrate the partitions $\mathcal{P}_2$ and $\mathcal{P}_2^1$.

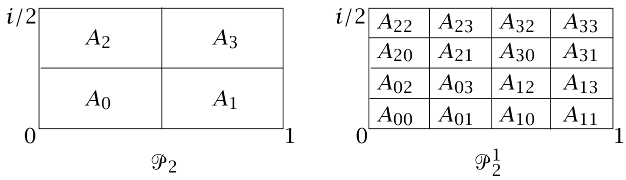

FIGURE 5.1. Partitions for $\hat{T}_2$.

As j approaches ∞, the sequence of partitions consists of rectangles with smaller and smaller diameter. We can calculate

$$\lim_{j\to\infty}\left(\sup_{P\in\mathcal{P}_2^j}\text{diam}(P)\right) = \lim_{j\to\infty}\frac{\sqrt{5}}{2^{4+2j}} = 0. \tag{5.2}$$

Therefore, by Proposition 4.1 and Theorem 4.2, φ_2 is an isomorphism. □

We extend the same type of argument to handle all $n \in \mathbb{Z} - \{0,1,-1\}$.

THEOREM 5.2. *For* $\Gamma = [1,i]$, *the rational map* $R_n(z)$, $n \in \mathbb{Z} - \{0,1,-1\}$ *with measure* ν *on* $\mathbb{C}_\infty$ *is isomorphic to the one-sided* $(1/n^2,\dots,1/n^2)$ *Bernoulli shift on* n^2 *symbols with measure* p_{n^2}; *i.e.,* $(\mathbb{C}_\infty,\mathcal{B},\nu,R_2) \cong (X_{n^2}^+,\mathcal{C},p_{n^2},\sigma)$.

PROOF. First we consider the case where n is even. By increasing the value of the even number n, the partition becomes an $n\times n$ grid instead of a 2×2 grid. More precisely, $\mathcal{P}_n = \{A_k : 0 \le k \le n^2-1\}$, where

$$\begin{aligned} A_k = \Big\{x+iy \in \Omega_n : \frac{k(\text{mod}\, n)}{n} \le x < \frac{k(\text{mod}\, n)+1}{n}, \\ \frac{(k/n)-(k/n)(\text{mod}\,1)}{2n} \le y \le \frac{(k/n)-(k/n)(\text{mod}\,1)+1}{2n}\Big\}. \end{aligned} \tag{5.3}$$

The behavior of $\hat{T}_n$ on $\mathcal{P}_n^j$ is similar to that of $\hat{T}_2$ on $\mathcal{P}_2^j$ in that each atom of the partition $\mathcal{P}_n^j$ is a rectangle which decreases in diameter as j increases. For $|n| > 2$, the diameter of any atom of any partition $\mathcal{P}_n^j$ is strictly less than the diameter of an atom of $\mathcal{P}_2^j$. Thus

$$\lim_{j\to\infty}\left(\sup_{P\in\mathcal{P}_n^j}\operatorname{diam}(P)\right)<\lim_{j\to\infty}\frac{\sqrt{5}}{2^{4+2j}}=0, \tag{5.4}$$

and φ_n is an isomorphism when n is even.

Now suppose n is odd. Then Ω_n is a triangle instead of a rectangle, but it still has half the *leb*-measure of the period parallelogram Ω_0. As in the even case, the key to the proof comes from defining the appropriate partition $\mathcal{P}_n$.

We define $\mathcal{P}_n=\{A_k : 0\le k\le n^2-1\}$, where $s_k=(k/n)-(k/n)(\operatorname{mod}1)$ and

$$\begin{aligned} A_k=\Bigg\{&x+iy\in\Omega_n \text{ or } (x+iy)'\in\Omega_n : \frac{k(\operatorname{mod} n)}{n}\le x<\frac{k(\operatorname{mod} n)+1}{n};\\ &y-x<\frac{s_k-2[k(\operatorname{mod} n)]}{2n},\ y>\frac{s_k}{2n} \text{ if } s_k \text{ is even};\\ &y-x>\frac{s_k-1-2[k(\operatorname{mod} n)]}{2n},\ y<\frac{s_k+1}{2n} \text{ if } s_k \text{ is odd}\Bigg\}. \end{aligned} \tag{5.5}$$

See Figure 5.2 for illustration of $\mathcal{P}_3$.

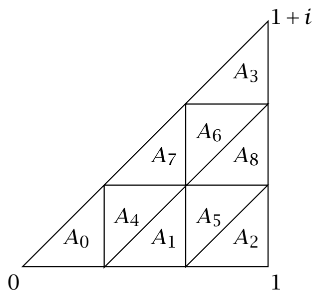

FIGURE 5.2. Partition $\mathcal{P}_3$.

This partition consists of right triangles which are contained in $(1/n)\times(1/n)$ squares. Furthermore, we can calculate

$$\lim_{j\to\infty}\left(\sup_{P\in\mathcal{P}_n^j}\operatorname{diam}(P)\right)=\lim_{j\to\infty}\frac{\sqrt{2}}{n^{j+1}}=0, \tag{5.6}$$

and φ_n is an isomorphism. □

6. The family R_{n+ni}. We next extend Theorem 5.2 to the family of rational maps of the sphere $R_{n+ni}(z)=\wp\circ T_{n+ni}\circ\wp^{-1}(z)$, where $T_{n+ni}(u)=(n+ni)(u)\ (\operatorname{mod}\Gamma)$.

THEOREM 6.1. *Let $\Gamma=[1,i]$. The rational map $R_{n+ni}(z)$ with measure ν on $\mathbb{C}_\infty$ is isomorphic to the one-sided $(1/d,\dots,1/d)$ Bernoulli shift on $d=2n^2$ symbols with measure p_d; i.e., $(\mathbb{C}_\infty,\mathcal{B},\nu,R_{n+ni})\cong(X_d^+,\mathcal{C},p_d,\sigma)$.*

PROOF. We begin by proving Theorem 6.1 in the case where $n=1$ and then extend the result to the maps of higher degree. We define a partition $\mathcal{P}_{1+i}=\{A_0,A_1\}$ of Ω_{1+i}

as follows:

$$A_0 = \{x+iy \in \Omega_{1+i} : x,y \in \mathbb{R},\ y \le x,\ y > -x+1\},$$
$$A_1 = \{x+iy \in \Omega_{1+i} : x,y \in \mathbb{R},\ y \le x,\ y \le -x+1\}. \tag{6.1}$$

For $u = a+bi \in \mathbb{C}$, we have $T_{1+i}(u) = T_{1+i}(a+ib) = (a-b)+(a+b)i$. Thus the action of T_{1+i} on a point u is to rotate it by $\pi/4$ and increase its magnitude by $\sqrt{2}$, and T_{1+i}^{-1} rotates the point by $-\pi/4$ and shrinks its magnitude by $\sqrt{2}$. We show illustrations of $\mathcal{P}_{1+i}$, $\mathcal{P}_{1+i}^1$, and $\mathcal{P}_{1+i}^2$ in Figure 6.1.

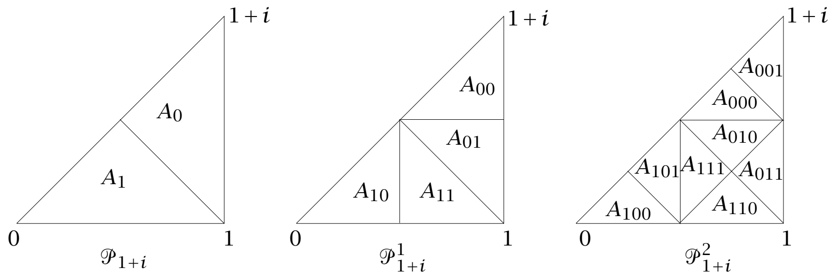

FIGURE 6.1. Partitions for $\hat{T}_{1+i}$.

The partition $\bigvee_{r=0}^{j} \hat{T}_{1+i}^{-r}(\mathcal{P}_{1+i})$ cuts every $P \in \mathcal{P}_{1+i}^{j-1}$ into two congruent triangles. Further, we can calculate for any $P \in \mathcal{P}_{1+i}^{j}$ the $\operatorname{diam}(P) = 1/\sqrt{2}^{j}$. Therefore,

$$\lim_{j\to\infty}\left(\sup_{P\in\mathcal{P}_{1+i}^{j}} \operatorname{diam}(P)\right) = \lim_{j\to\infty}\frac{1}{\sqrt{2}^{j}} = 0. \tag{6.2}$$

Applying Proposition 4.1 and Theorem 4.2, φ_{1+i} is an isomorphism, and we have completed the proof in the case $n = 1$. □

Notice that the composition $T_n \circ T_{1+i} = T_{n+ni}$ gives a degree $2n^2$ map of the torus. By using the above arguments and Theorem 5.2, we see that R_{n+ni} is isomorphic to the one-sided $(1/2n^2,\dots,1/2n^2)$ Bernoulli shift on $2n^2$ symbols.

7. Changing the lattice. We prove that the above isomorphism extends to other tori $\mathbb{C}/\Gamma'$, where $T_\omega\Gamma' \subset \Gamma'$. Suppose $\Gamma' = [\gamma,\delta]$. Then we can find real numbers x_1, y_1, x_2, and y_2 such that

$$\gamma = x_1\alpha + y_1\beta, \qquad \delta = x_2\alpha + y_2\beta. \tag{7.1}$$

The period parallelogram for $\mathbb{C}/\Gamma'$ becomes $\Omega_0' = \{c\gamma + d\delta \in \mathbb{C} : 0 \le c,\ d < 1\}$. We define a linear map $L : \Omega_0 \to \Omega_0'$ by

$$L(u) = L(a+ib) = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}. \tag{7.2}$$

We use L to define a measure-theoretic isomorphism between $(\Omega_\omega, \mathcal{B}_\omega, \mu, T_\omega)$ and $(Y_\omega, \mathcal{B}_Y, \mu_Y, T_{\omega,L})$, where $Y_\omega = L(\Omega_\omega) \subset \Omega_0'$, $\mathcal{B}_Y$ is the Borel sets on Y_ω, μ_Y is normalized Lebesgue measure in Y_ω, and $T_{\omega,L} = LT_\omega L^{-1}$. We state this result in the following lemma.

Lemma 7.1. *We have* $(\Omega_\omega, \mathcal{B}_\omega, \mu, T_\omega) \cong (Y_\omega, \mathcal{B}_Y, \mu_Y, T_{\omega,Y})$.

Lemma 7.1 provides us with the following corollary.

Corollary 7.2. *If* $T_\omega : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma$ *is given by* $T_\omega(u) = \omega u \pmod{\Gamma}$, *where* $\omega = n$, $n \in \mathbb{Z} - \{0, 1, -1\}$ *or* $\omega = n + ni$, $n \in \mathbb{Z} - \{0\}$, *then the rational map* $R_\omega(z) = \wp_\Gamma \circ T_\omega \circ \wp_\Gamma^{-1}(z)$ *with measure* ν *on* $\mathbb{C}_\infty$ *is isomorphic to the one-sided* $(1/d, \ldots, 1/d)$ *Bernoulli shift on* $d = \omega\overline{\omega}$ *symbols with measure* p_d; *i.e.,* $(\mathbb{C}_\infty, \mathcal{B}_\omega, \nu, R_\omega) \cong (X_d^+, \mathcal{C}, p_d, \sigma)$.

This technique is further developed by Koss [6] to analyze other classes of critically finite rational maps with parabolic orbifolds.

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