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ON THE DEGREE OF APPROXIMATION BY GAUSS WEIERSTRASS INTEGRALS

HUZOOR H. KHAN and GOVIND RAM

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ABSTRACT. We obtain the degree of approximation of functions belonging to class $\text{Lip}(\psi(u,v);p)$, $p>1$ using the Gauss Weierstrass integral of the double Fourier series of $f(x,y)$.

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1. Introduction and results. Let the function $f(x,y)$ be integrable in the sense of Lebesgue over the square $(0,2\pi;0,2\pi)$ and periodic with period 2π in each variable outside the square. Let the double Fourier series of $f(x,y)$ be

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m,n}(x,y), \tag{1.1}$$

where

$$\begin{aligned} A_{0,0}(x,y) &= \frac{1}{4}a_{0,0}, \\ A_{m,0}(x,y) &= \frac{1}{2}(a_{m,0}\cos mx + b_{m,0}\sin mx), \\ A_{0,n}(x,y) &= \frac{1}{2}(a_{0,n}\cos nx + b_{0,n}\sin nx), \\ A_{m,n}(x,y) &= a_{m,n}\cos mx\cos ny + b_{m,n}\cos mx\sin ny \\ &\quad + c_{m,n}\sin mx\cos ny + d_{m,n}\sin mx\sin ny. \end{aligned} \tag{1.2}$$

Let

$$\begin{aligned} \phi(u,v) = \frac{1}{4}\Big\{ & f(x+u,y+v)+f(x-u,y+v)+f(x+u,y-v) \\ & + f(x-u,y-v)-4f(x,y)\Big\}. \end{aligned} \tag{1.3}$$

The series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{m,n}A_{m,n}(x,y) \tag{1.4}$$

is called the double Fourier series associated with the function $\phi(u,v)$ such that

$$\lambda_{m,n} = \begin{cases} \dfrac{1}{4} & \text{for } m = 0,\ n = 0, \\ \dfrac{1}{2} & \text{for } m = 0,\ n > 0;\ m > 0,\ n = 0, \\ 1 & \text{for } m > 0,\ n > 0. \end{cases} \tag{1.5}$$

Also, the coefficients in the series (1.4) are given by

$$a_{m,n} = \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(u,v) \cos mu \cos nv \, du \, dv \tag{1.6}$$

and three other similar expressions defining $b_{m,n}, c_{m,n}, d_{m,n}$.

We define the Gauss Weierstrass integral of $f(x,y)$ by

$$\begin{aligned} W_{m,n}(x,y) = W(x,y;\xi,\eta) &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{\exp\left((k^2\xi/4) + (\ell^2\eta/4)\right)} A_{k,\ell}(x,y) \\ &= \sqrt{\frac{\pi}{(\xi,\eta)}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{f(x+t,y+s)}{\exp\left((t^2/\xi) + (s^2/\eta)\right)} \, dt \, ds + o(\xi,\eta), \end{aligned} \tag{1.7}$$

where $o(\xi,\eta) \to 0$ as $\xi \to 0$, $\eta \to 0$, and (ξ,η) is the product of ξ and η.

A 2π periodic function $f(x,y)$ in each variable x and y is said to belong to the class $\operatorname{Lip}(\psi(u,v);p)$, $p > 1$, [2] if

$$|f(x+u,y+v) - f(x,y)| \le M\left(\frac{\psi(u,v)}{(u,v)^{1/p}}\right), \quad 0 < u < \pi,\ 0 < v < \pi, \tag{1.8}$$

where $\psi(u,v)$ is a positive increasing function of the variables u,v and M is a positive number independent of x,u and v.

Yoshimitsu, [1] proved a theorem for obtaining the degree of approximation of class of functions $\operatorname{Lip}(\alpha,\beta)$, $0 < \alpha < 1$ and $0 < \beta < 1$, by means of the first arithmetic means of double Fourier series. Siddiqi and Mohammadzadeh [3] extended the result in two directions in terms of a positive increasing function of two variables. Recently Khan, [2] extended the result of Yoshimitsu, Siddiqi et al. for the more general operator, the Jackson type operator, and more general class of functions, $\operatorname{Lip}(\psi(u,v);p)$, $p > 1$. The object of the present paper is to determine the degree of approximation for the functions belonging to the class $\operatorname{Lip}(\psi(u,v);p)$, $p > 1$, by means of Gauss Weierstrass integral of the double Fourier series of $f(x,y)$.

Our theorem states as follows.

Theorem 1.1. *Let $f(x,y)$ be a continuous function of period 2π with respect to each variable x and y belonging to $\operatorname{Lip}(\psi(u,v);p)$, $p > 1$ class, then*

$$|W(x,y;\xi,\eta) - f(x,y)| = \mathbb{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{(1/p)-(1/2)}}\right), \tag{1.9}$$

provided

$$\left[\int_0^{\xi} \int_0^{\eta} \left(\frac{\psi(t,s)}{(t,s)^{1/p}}\right)^p dt \, ds\right]^{1/p} = \mathbb{O}(\psi(\xi,\eta)), \tag{1.10}$$

$$\left[\int_0^{\xi}\int_{\eta}^{\pi}\left(\frac{\psi(t,s)}{(t,s)^{2+(1/p)}}\right)^p dt\,ds\right]^{1/p}=\mathbb{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^2}\right). \tag{1.11}$$

2. Proof of the theorem. Using (1.7), we get

$$\begin{aligned}
&W(x,y;\xi,\eta)-f(x,y)\\
&\quad=\sqrt{\frac{\pi}{(\xi,\eta)}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\phi(t,s)e^{-((t^2/\xi)+(s^2/\eta))}\,dt\,ds+R(x,y;\xi,\eta)\\
&\quad=4\sqrt{\frac{\pi}{(\xi,\eta)}}\int_0^{\pi}\int_0^{\pi}\phi(t,s)e^{-((t^2/\xi)+(s^2/\eta))}\,dt\,ds+\mathbb{O}(\xi,\eta) \qquad (2.1)\\
&\quad=4\sqrt{\frac{\pi}{(\xi,\eta)}}\left(\int_0^{\xi}\int_0^{\eta}+\int_0^{\xi}\int_{\eta}^{\pi}+\int_{\xi}^{\pi}\int_0^{\eta}+\int_{\xi}^{\pi}\int_{\eta}^{\pi}\right)\phi(t,s)e^{-((t^2/\xi)+(s^2/\eta))}\,dt\,ds\\
&\quad=I_{1,2}(x,y)+I_{1,3}(x,y)+I_{4,2}(x,y)+I_{4,3}(x,y).
\end{aligned}$$

Applying Hölder's inequality of two variables and the fact that $\phi(u,v)\in \operatorname{Lip}(\psi(u,v);p)$, $p>1$ for $I_{1,2}(x,y)$, we get

$$\begin{aligned}
|I_{1,2}(x,y)|\le 4\sqrt{\frac{\pi}{(\xi,\eta)}}&\left[\int_0^{\xi}\int_0^{\eta}|\phi(t,s)|^p\,dt\,ds\right]^{1/p}\\
&\cdot\left[\int_0^{\xi}\int_0^{\eta}\left(e^{-((t^2/\xi)+(s^2/\eta))}\right)^{p'}dt\,ds\right]^{1/p'},
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
\frac{1}{p'}&=\frac{p-1}{p}\\
&\le 4\sqrt{\frac{\pi}{(\xi,\eta)}}\left[\int_0^{\xi}\int_0^{\eta}\left(\frac{\psi(t,s)}{(t,s)^{1/p}}\right)^p dt\,ds\right]^{1/p}\left[\int_0^{\xi}\int_0^{\eta}\left(e^{-(t^2/\xi)+(s^2/\eta)}\right)^{p'}dt\,ds\right]^{1/p'}\\
&=\mathbb{O}\left(\frac{\psi(\xi,\eta)}{(\sqrt{\xi},\sqrt{\eta})}\right)\left(\int_0^{\xi}\int_0^{\eta}e^{-(t^2p'/\xi)}e^{-(s^2p'/\eta)}\,dt\,ds\right)^{1/p'} \quad \text{using condition (1.10)}\\
&=\mathbb{O}\left(\frac{\psi(\xi,\eta)}{(\sqrt{\xi},\sqrt{\eta})}\right)\left(\int_0^{\sqrt{\xi p'}}e^{-u^2}\frac{\sqrt{\xi}}{\sqrt{p'}}\,du\int_0^{\sqrt{\eta p'}}e^{-v^2}\frac{\sqrt{\eta}}{\sqrt{p'}}\,dv\right)^{1/p'}\\
&=\mathbb{O}\left(\frac{\psi(\xi,\eta)}{(\sqrt{\xi},\sqrt{\eta})}\right)\mathbb{O}(\xi,\eta)^{1/p'}=\mathbb{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{(1/p)-(1/2)}}\right),
\end{aligned} \tag{2.3}$$

where $u=t\sqrt{p'/\xi}$ and $v=s\sqrt{p'/\eta}$.

Applying Hölder's inequality of two variables to $I_{1,3}(x,y)$ and using the fact that $\phi(u,v)\in \operatorname{Lip}(\psi(u,v);p)$, $p>1$, we get

$$\begin{aligned}|I_{1,3}(x,y)| &\le \frac{4\sqrt{\pi}}{\left(\sqrt{\xi},\sqrt{\eta}\right)}\\ &\quad\cdot\left[\left(\int_0^{\xi}\int_{\eta}^{\pi}\left|\frac{\phi(t,s)}{(t,s)^2}\right|^p dt\,ds\right)^{1/p}\left(\int_0^{\xi}\int_{\eta}^{\pi}\frac{e^{-(t^2/\xi)+(s^2/\eta)p'}}{(t,s)^{-2p'}}dt\,ds\right)^{1/p'}\right]\\ &\le \frac{4\sqrt{\pi}}{\left(\sqrt{\xi},\sqrt{\eta}\right)}\left\{\left[\int_0^{\xi}\int_{\eta}^{\pi}\left(\frac{\psi(t,s)}{(t,s)^{2+(1/p)}}\right)^p dt\,ds\right]^{1/p}\right.\\ &\quad\left.\cdot\left(\int_0^{\xi}e^{-t^2p'/\xi}\cdot t^{2p'}dt\int_{\eta}^{\pi}e^{-s^2p'/\eta}s^{2p'}ds\right)^{1/p'}\right\}\\ &= \mathbb{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{5/2}}\right)\left\{\left[\int_0^{\sqrt{\xi p'}}e^{-u^2}\left(\frac{\sqrt{\xi}}{\sqrt{p}}u\right)^{2p}\left(\frac{\sqrt{\xi}}{\sqrt{p}}\right)du\right]^{1/p}\right.\\ &\quad\left.\cdot\left[\int_{\sqrt{\eta p'}}^{(\pi/\sqrt{\eta p'})}e^{-v^2}\left(\frac{\sqrt{\eta}}{\sqrt{p'}}v\right)^{2p'}\left(\frac{\sqrt{\eta}}{\sqrt{p'}}\right)dv\right]\right\}^{1/p'}\quad\text{using condition (1.11)}\\ &= \mathbb{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{5/2}}\right)\mathbb{O}(\xi,\eta)^{2+(1/p')} = \mathbb{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{(1/p)-(1/2)}}\right).\end{aligned} \tag{2.4}$$

Similarly, we can prove that

$$|I_{4,2}(x,y)| = \mathbb{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{(1/p)-(1/2)}}\right),\qquad |I_{4,3}(x,y)| = \mathbb{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{(1/p)-(1/2)}}\right). \tag{2.5}$$

Finally, we get

$$W(x,y;\xi,\eta)-f(x,y) = \mathbb{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{(1/p)-(1/2)}}\right). \tag{2.6}$$

Remark 2.1. It may also be remarked that by giving different values to $\psi(u,v)$, we get some interesting results:

(i) If $\psi(u,v) = u^{\alpha} * v^{\beta}$, [1], $0<\alpha<1$, $0<\beta<1$, then we have

$$\left|W(x,y;\xi,\eta)-f(x,y)\right| = \mathbb{O}\left(\xi^{\alpha+(1/2)-(1/p)} * \eta^{\beta+(1/2)-(1/p)}\right). \tag{2.7}$$

(ii) If $\psi(u,v) = J(u,v)(u,v)^{1/p}$, [3], where $J(u,v)$ is a positive increasing function of variables u and v, then

$$\left|W(x,y;\xi,\eta)-f(x,y)\right| = \mathbb{O}\left(\frac{J(\xi,\eta)}{(\xi,\eta)^{-1/2}}\right). \tag{2.8}$$

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Khan and Ram: Department of Mathematics, Aligarh Muslim University, Aligarh-202 002, India