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THE COXETER GROUP D_n

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ABSTRACT. We show that the Coxeter group D_n is the split extension of $n-1$ copies of Z_2 by S_n for a given action of S_n described in the paper. We also find the centre of D_n and some of its other important subgroups.

Keywords and phrases. Coxeter group, split extension.

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1. Introduction. The Coxeter group D_n [5] has the presentation

$$D_n = \langle x_1, x_2, \ldots, x_n \mid x_i^2 = e, 1 \le i \le n;\ (x_i x_{i+1})^3 = e, 2 \le i \le n-1; \\ (x_i x_j)^2 = e, |i-j| \ne 1 \text{ and } (i,j) \ne (1,3);\ (x_1 x_2)^2 = (x_1 x_3)^3 = e \rangle. \tag{1.1}$$

and the graph given in Figure 1.1.

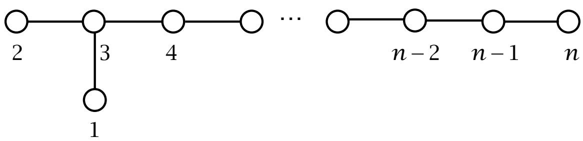

FIGURE 1.1.

In [1], we have shown that D_4 is solvable with derived length 4 and that its order is 192. In this paper, we explain the algebraic structure of D_n and find its centre. We also find the derived series; and the growth series of D_n for $4 \le n \le 8$.

2. The structure of D_n. In [4], we have shown that the Coxeter group B_n whose graph is given in Figure 2.1

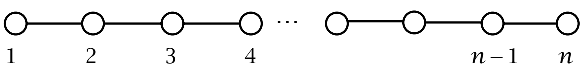

FIGURE 2.1.

is the wreath product of Z_2 by S_n, that is, B_n is the split extension of Z_2^n by S_n. Let Z_2^n have the presentation

$$H = Z_2^n = \langle a_1, a_2, \ldots, a_n \mid a_i^2 = e, 1 \le i \le n;\ (a_i a_j)^2 = e, 1 \le i < j \le n \rangle. \tag{2.1}$$

Let K be the even subgroup of H, that is, the subgroup consisting of even words in H. It is easy to find the following presentation for K:

$$K = \langle b_2, b_3, \dots, b_n \mid b_i^2 = e, 2 \le i \le n;\ (b_i b_j)^2 = e, 2 \le i < j \le n \rangle, \tag{2.2}$$

where $b_i = a_1 a_i$, $2 \le i \le n$. Thus K is Z_2^{n-1}. In the extension $B_n \cong Z_2^n \rtimes S_n$, the action of S_n on Z_2^n is a natural one that can be explained as follows. Let S_n have the presentation

$$\begin{aligned} S_n = \langle x_1, x_2, \dots, x_{n-1} \mid x_i^2 = e, 1 \le i \le n-1;\ (x_i x_{i+1})^3 = e, 1 \le i \le n-2; \\ (x_i x_j)^2 = e, 1 \le i < j-1 \le n-2 \rangle, \end{aligned} \tag{2.3}$$

where x_i is the transposition $(i\, i+1)$. The action of S_n on H is given by

$$(a_1, a_2, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n)^{x_i} = (a_1, a_2, \dots, a_{i-1}, a_{i+1}, a_i, \dots, a_n). \tag{2.4}$$

Using this action we compute the action of S_n on K as follows:

$$\begin{aligned} (b_2, b_3, \dots, b_n)^{x_1} &= (a_1a_2, a_1a_3, \dots, a_1a_n)^{x_1} \\ &= (a_2a_1, a_2a_3, \dots, a_2a_n) \\ &= (b_2^{-1}, b_2^{-1}b_3, \dots, b_2^{-1}b_n), \\ (b_2, b_3, \dots, b_i, b_{i+1}, \dots, b_n)^{x_i} &= (a_1a_2, \dots, a_1a_i, a_1a_{i+1}, \dots, a_1a_n)^{x_i} \\ &= (a_1a_2, \dots, a_1a_{i+1}, a_1a_i, \dots, a_1a_n) \\ &= (b_2, \dots, b_{i+1}, b_i, \dots, b_n), \quad 2 \le i \le n-1. \end{aligned} \tag{2.5}$$

We use this action to construct a split extension E of $K = Z_2^{n-1}$ by S_n. A presentation for this extension is given by the method in [2], $E = \langle$generators of K, generators of $S_n \mid$ relations of K, relations of S_n, the action of S_n on $K\rangle$.

We change the action of S_n on K to the following relations:

$$b_2^{x_1} = b_2^{-1}, \tag{2.6}$$

$$b_i^{x_1} = b_2^{-1} b_i, \quad 3 \le i \le n, \tag{2.7}$$

$$b_i^{x_i} = b_{i+1}, \quad 2 \le i \le n-1, \tag{2.8}$$

$$b_{i+1}^{x_i} = b_i, \quad 2 \le i \le n-1, \tag{2.9}$$

$$b_j^{x_i} = b_j, \quad 2 \le j \le n,\ 2 \le i \le n-1,\ j \ne i,\ j \ne i+1. \tag{2.10}$$

We will use Tietze transformations to show that E is isomorphic to D_n. But before that we observe the following. The relation (2.8) implies that $b_i = b_2^{x_2 x_3 \cdots x_{i-1}}$, $3 \le i \le n$. Let $u_i = x_2 x_3 \cdots x_i$.

Lemma 2.1. *The following identities hold in the group S_{n-1}:*

(i) $u_k x_i = x_{i+1} u_k$, *if* $2 \le i \le k$,

(ii) $u_k x_i = u_{k-1}$, *if* $i = k$,

(iii) $u_k x_i = u_{k+1}$, *if* $i = k+1$,

(iv) $u_k x_i = x_i u_k$, *if* $i > k+1$,

(v) $u_k u_i = (x_3 x_4 \cdots x_{i+1}) u_k$, *if* $2 \le i < k$,

(vi) $u_k u_i = (x_3 x_4 \cdots x_i) u_{k-1}$, *if* $i \ge k$.

Proof. We will make use of the relations of S_{n-1}.

(i)

$$\begin{aligned} u_k x_i &= (x_2 x_3 \cdots x_i x_{i+1} \cdots x_k) x_i \\ &= x_2 x_3 \cdots x_i x_{i+1} x_i \cdots x_k \\ &= x_2 x_3 \cdots x_{i+1} x_i x_{i+1} \cdots x_k \\ &= x_{i+1} x_2 x_3 \cdots x_i x_k = x_{i+1} u_k. \end{aligned} \tag{2.11}$$

(ii), (iii), and (iv) are clear, while (v) and (vi) are applications of (i) to (iv). □

We reduce relation (2.6) to (2.10) as follows. Relation (2.6) easily becomes

$$(x_1 b_2)^2 = e. \tag{2.12}$$

Using Lemma 2.1, (2.7) becomes

$$(b_2 x_1 x_2)^3 = e. \tag{2.13}$$

Using $b_i = b_2^{x_2 x_3 \cdots x_{i-1}}$, (2.8) becomes redundant. Relation (2.9), using Lemma 2.1, becomes redundant. Using Lemma 2.1, (2.10) becomes

$$(x_i b_2)^2 = e, \quad 3 \le i \le n. \tag{2.14}$$

The relation $b_i^2 = e = (b_i b_j)^2$ become redundant for $i \ge 3$. Let $c = x_1 b_2$. Then (2.12) becomes $c^2 = e$. Relation (2.13) becomes $(c x_2)^3 = e$. Relation (2.14) becomes $(c x_i)^2 = e$ for $i \ge 3$. The relation $b_2^2 = e$ becomes $(c x_1)^2 = e$. Therefore a presentation for E is

$$\begin{aligned} E = \langle x_1, x_2, \ldots, x_{n-1}, c \mid\ & x_i^2 = e, 1 \le i \le n-1;\ c^2 = e; \\ & (x_i x_j)^2 = e, 1 \le i < j-1 \le n-2;\ (x_i x_{i+1})^3 = e, 1 \le i \le n-2; \\ & (c x_2)^3 = e; (c x_i)^2 = e, 1 \le i \le n-1 \text{ and } i \ne 2 \rangle. \end{aligned} \tag{2.15}$$

Consequently, we have proved the following theorem.

Theorem 2.2. *The group D_n is the split extension of $n-1$ copies of Z_2 by S_n.*

Remark 2.3. Let us describe the relation between the Coxeter groups B_n and D_n. It is easy to show that D_n is the subgroup of B_n consisting of all elements of even length in B_n. Thus D_n is a subgroup of B_n of index 2. On the other hand, let S_n be the symmetric group of degree n generated by y_i, $1 \le i \le n-1$. We consider the map $\theta : D_n \to S_n$ defined by $\theta(x_1) = y_2$, $\theta(x_2) = y_2$, and $\theta(x_i) = y_{i-1}$, $3 \le i \le n$. Using the Reidemeister-Schreier process, it is possible to show that $\theta^{-1}(S_{n-1}) \cong B_{n-1}$. Hence B_{n-1} is a subgroup of D_n of index n. We observe the graph given in Figure 2.2.

The orders of B_n and D_n are $|B_n| = 2^n n!$ and $|D_n| = 2^{n-1} n!$, respectively. It is also easy to see that $B_n' \cong D_n'$.

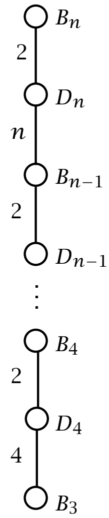

FIGURE 2.2.

3. The derived series of D_n. In [1], we showed that D_4 is solvable of derived length 4. For $n > 4$ we use the Reidemeister-Shreier process to find the following presentation for D'_n:

$$\begin{aligned} D'_n = \langle b_2, b_3, \ldots, b_n \mid b_2^2 = b_3^3 = b_i^2 = e,\ 4 \le i \le n; \\ (b_i b_{i+1})^3 = e,\ 2 \le i \le n-1;\ (b_i b_j)^2 = e,\ 2 \le i < j-1 \le n-1 \rangle. \end{aligned} \tag{3.1}$$

The group D'_n / D''_n is trivial. Hence D'_n is a complete group. We thus have the following theorem.

THEOREM 3.1. *D_n is solvable of derived length 4 if $n = 4$. If $n > 4$, then D_n is not solvable.*

4. The centre of D_n. We use the structure of D_n explained in Section 2 to prove the following theorem.

THEOREM 4.1. *The centre of D_n is Z_2 if n is even and trivial if n is odd.*

PROOF. In Section 2, we showed that D_n is the split extension of Z_2^{n-1} by S_n. This means the existence of an epimorphism $\theta : D_n \to S_n$, where $\ker\theta = Z_2^{n-1}$. It follows that $\theta(Z(D_n)) \subseteq Z(\theta(D_n)) = Z(S_n) = \{e\}$. Hence $Z(D_n) \subseteq \ker\theta = z_2^{n-1}$. We use the previous notation where $S_n = \langle x_1, x_2, \ldots, x_{n-1} \rangle$ and $Z_2^{n-1} = \langle b_2, b_3, \ldots, b_n \rangle$ and the previous action. We let $w \in Z(D_n) \Rightarrow w \in Z_2^{n-1}$ and $w^{x_i} = w$ for $1 \le i \le n-1$. Also $w = b_2^{\epsilon_2} b_3^{\epsilon_3} \cdots b_n^{\epsilon_n}$, where $\epsilon_i = 0$ or 1 since $b_i^2 = e$. Using the action of S_n on Z_2^{n-1}, we get

$$\left(b_2^{\epsilon_2} b_3^{\epsilon_3} \cdots b_i^{\epsilon_i} b_{i+1}^{\epsilon_{i+1}} \cdots b_n^{\epsilon_n}\right)^{x_i} = b_2^{\epsilon_2} b_3^{\epsilon_3} \cdots b_i^{\epsilon_i} b_{i+1}^{\epsilon_{i+1}} \cdots b_n^{\epsilon_n}. \tag{4.1}$$

Letting $2 \le i \le n-1$, we get

$$b_2^{\epsilon_2} b_3^{\epsilon_3} \cdots b_{i+1}^{\epsilon_i} b_i^{\epsilon_{i+1}} \cdots b_n^{\epsilon_n} = b_2^{\epsilon_2} b_3^{\epsilon_3} \cdots b_i^{\epsilon_i} b_{i+1}^{\epsilon_{i+1}} \cdots b_n^{\epsilon_n}, \tag{4.2}$$

and so $b_{i+1}^{\epsilon_i} b_i^{\epsilon_{i+1}} = b_i^{\epsilon_i} b_{i+1}^{\epsilon_{i+1}}$. This implies $\epsilon_i = \epsilon_{i+1}$ and so $\epsilon_2 = \epsilon_3 = \cdots = \epsilon_n$. Hence $w = b_2 b_3 \cdots b_n$ or $w = b_2^0 b_3^0 \cdots b_n^0 = e$. Now, we consider the action of x_1 on w in the following two cases:

(a) If n is even, we get $(b_2 b_3 \cdots b_n)^{x_1} = b_2^{n-2} b_2 b_3 \cdots b_n = b_2 b_3 \cdots b_n$ since $b_2^2 = e$. Hence $b_2 b_3 \cdots b_n$ is in the centre of D_n. Since $(b_2 b_3 \cdots b_n)^2 = e$, we get $Z(D_n) = Z_2$.

(b) If n is odd, $(b_2 b_3 \cdots b_n)^{x_1} = b_3 b_4 \cdots b_n$ and $b_2 b_3 \cdots b_n$ does not commute with x_1. Thus w is $b_2^0 b_3^0 \cdots b_n^0 = e$ and $Z(D_n) = \{e\}$. □

5. The growth series. The growth series, in the sense of Milnor and Gromov, of D_n for $4 \le n \le 8$ were computed as follows [3]:

$$\gamma(D_4) = (1+t)^4 (1+t^2)^2 (1-t+t^2)(1+t+t^2), \tag{5.1}$$

$$\gamma(D_5) = (1+t)^4 (1+t^2)^2 (1+t^4)(1-t+t^2)(1+t+t^2)(1+t+t^2+t^3+t^4), \tag{5.2}$$

$$\begin{aligned} \gamma(D_6) = {} & (1+t)^6 (1+t^2)^2 (1+t^4)(1-t+t^2)^2 (1+t+t^2)^2 \\ & \times (1-t+t^2-t^3+t^4)(1+t+t^2+t^3+t^4), \end{aligned} \tag{5.3}$$

$$\begin{aligned} \gamma(D_7) = {} & (1+t)^6 (1+t^2)^3 (1+t^4)(1-t+t^2)^2 (1+t+t^2)^2 (1-t^2+t^4) \\ & \times (1-t+t^2-t^3+t^4)(1+t+t^2+t^3+t^4)(1+t+t^2+t^3+t^4+t^5+t^6), \end{aligned} \tag{5.4}$$

$$\begin{aligned} \gamma(D_8) = {} & (1+t)^8 (1+t^2)^4 (1+t^4)^2 (1-t+t^2)^2 (1+t+t^2)^2 \\ & \times (1-t^2+t^4)(1-t+t^2-t^3+t^4)(1+t+t^2+t^3+t^4) \\ & \times (1-t+t^2-t^3+t^4-t^5+t^6)(1+t+t^2+t^3+t^4+t^5+t^6). \end{aligned} \tag{5.5}$$

We make two observations about these growth polynomials. First, each growth series is a product of cyclotomic polynomials. Second, the value of the series at 1 is the order of the corresponding group and the degree of the growth series equals the length of the element of maximal length.

We have not yet succeeded in finding the growth series of D_n for general n.

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