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SOME MAXIMUM PRINCIPLES FOR SOLUTIONS OF A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS IN $\Omega \subset \mathbb{R}^n$

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ABSTRACT. We find maximum principles for solutions of semilinear elliptic partial differential equations of the forms: (1) $\Delta^2 u + \alpha f(u) = 0$, $\alpha \in \mathbb{R}^+$ and (2) $\Delta\Delta u + \alpha(\Delta u)^k + gu = 0$, $\alpha \le 0$ in some region $\Omega \subset \mathbb{R}^n$.

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1. Introduction. In [1], Chow and Dunninger proved the following result: let $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$ be a nonconstant solution of

$$\begin{aligned} \Delta^2 u + \alpha u &= 0 \quad \text{in } \Omega C \mathbb{R}^n,\ \alpha \in \mathbb{R}^+, \\ \Delta u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Then u satisfies a maximum principle. In this paper, we extend this result to solutions of semilinear partial differential equations of the forms:

(1) $\Delta^2 u + \alpha f(u) = 0$, where α is a positive constant and $f(u)$ is a positive, nondecreasing, differentiable function.

(2) $\Delta\Delta u + \alpha(\Delta u)^k + gu = 0$, where α is a nonpositive constant, k is an odd integer, and $g > 0$ is a twice continuously differentiable function.

2. The maximum principles. The new results are in the following two theorems.

THEOREM 2.1. *Let $u = u(x_1, x_2, \dots, x_n)$ be a nonconstant solution of*

$$\Delta^2 u + \alpha f(u) = 0, \tag{2.1}$$

where α is a positive constant, and $f(u)$ is a positive, nondecreasing, differentiable function; and if $\Delta u = 0$ on $\partial\Omega$, then u attains its maximum on $\partial\Omega$.

PROOF. Let $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$ be a solution of (2.1), then u satisfies the equations

$$\Delta u = h, \qquad \Delta h = \alpha f(u). \tag{2.2}$$

Now define

$$L(x) = 2\alpha \int_0^{u(x)} f(s)\,ds + h^2. \tag{2.3}$$

□

Denoting one of the variables, x_k, by β and differentiating (2.3) twice with respect to β, we get

$$L_\beta = 2\alpha f(u(x))u_\beta + 2hh_\beta$$
$$L_{\beta\beta} = 2\alpha f'(u(x))u_\beta^2 + 2\alpha f(u(x))u_{\beta\beta} + 2h_\beta^2 + 2hh_{\beta\beta}. \tag{2.4}$$

If we sum over all $\beta = x_k$, we get

$$\Delta L = 2\alpha f'(u(x))|\operatorname{grad} u|^2 + 2\alpha f(u(x))\Delta u + 2|\operatorname{grad} h|^2 + 2h\Delta h. \tag{2.5}$$

Substituting (2.2) into (2.5), we get

$$\Delta L = 2\alpha f'(u(x))|\operatorname{grad} u|^2 + 2|\operatorname{grad} h|^2. \tag{2.6}$$

Since $f'(u) \geq 0$ we see that $\Delta L \geq 0$. But $\Delta L \neq 0$ as u is nonconstant. Hence L is a nonconstant subharmonic function. And it follows from the maximum principle of subharmonic functions that $L(x)$ cannot attain its maximum at any interior point of Ω, that is,

$$L(x_0) > L(x) \tag{2.7}$$

for some $x_0 \in \partial\Omega$ and for all $x \in \Omega$.

It follows from (2.7) that

$$2\alpha \int_0^{u(x_0)} f(s)\,ds + (\Delta u(x_0))^2 > 2\alpha \int_0^{u(x)} f(s)\,ds + (\Delta u(x))^2. \tag{2.8}$$

However, since $(\Delta u(x_0)) = 0$, it yields

$$2\alpha \int_0^{u(x_0)} f(s)\,ds > 2\alpha \int_0^{u(x)} f(s)\,ds + (\Delta u(x_0))^2 > 2\alpha \int_0^{u(x)} f(s)\,ds \tag{2.9}$$

or, since $\alpha > 0$, we have

$$|u(x_0)| > |u(x)|, \quad \forall x \in \Omega. \tag{2.10}$$

Theorem 2.2. *Let u be a nonconstant solution of the partial differential equation*

$$\Delta\Delta u + \alpha\ (\Delta u)^k + g(x_1, x_2, \ldots, x_n)u = 0, \tag{2.11}$$

where α is a nonpositive constant, k is an odd integer and $g > 0$ is twice continuously differential and is such that

$$\Delta g \geq 8\left(g \operatorname{grad} \frac{1}{g^{1/2}}\right)^2. \tag{2.12}$$

Then

$$|u(x_0)| < |u(x)|. \tag{2.13}$$

For some $x_0 \in \partial\Omega$ and for all $x \in \partial\Omega$ provided

$$\Delta u = 0 \quad \text{on } \partial\Omega, \qquad g(x_0) < g(x). \tag{2.14}$$

Proof. A nonconstant solution u of (2.11) satisfies the equations

$$\Delta u = h, \qquad \Delta h = -\alpha h^k. \tag{2.15}$$

As in the proof of Theorem 2.1, we consider the function

$$L(x) = gu^2 + h^2. \tag{2.16}$$

□

Differentiating twice with respect to x_k and summing over all x_k, s, we get

$$\begin{aligned}\Delta L = {} & 2g\left(\frac{|\operatorname{grad} u| + u|\operatorname{grad} g|}{g}\right)^2 + \left(\Delta g - 8\left(\frac{g \operatorname{grad} 1}{g^{1/2}}\right)^2\right) \\ & + 2|\operatorname{grad} h|^2 + 2ug\Delta u + 2h\Delta h.\end{aligned} \tag{2.17}$$

Since $\alpha \le 0$ and $g > 0$, we can conclude with the help of (2.12) and (2.15) that L is subharmonic and it follows from the maximum principles of subharmonic functions that there exist a point $x_0 \in \partial\Omega$ such that

$$g(x_0)u^2(x_0) + (\Delta u(x_0))^2 > g(x)u^2(x) + (\Delta u(x))^2 \tag{2.18}$$

for all $x \in \bar{\Omega}$. But since $\Delta u = 0$ on $\partial\Omega$, the assertion is proved with the help of (2.14).

3. Concluding remarks.

(a) If $\alpha = 0$, then (2.11) reduces to

$$\Delta\Delta u + gu = 0. \tag{3.1}$$

Thus, Theorem 2 of Chow and Dunninger [1] regarding (3.1) becomes a particular case of Theorem 2.2.

(b) Theorem 1 of Dunninger [2] can be extended to

$$\Delta\Delta u + \alpha\Delta u + \beta u = 0, \tag{3.2}$$

where $\alpha \le 0$ and $\beta > 0$ are constants.

Clearly, if $k = 1$ and $g = \alpha$, equation (3.2) is obtained from (2.11).

(c) Theorem 2.1 can be extended to the solutions of the partial differential equation

$$\Delta^2 u + \alpha\Delta u + \beta f(u) = 0, \tag{3.3}$$

where $\alpha \le 0$, $\beta > 0$ are constants and $f(u)$ is a positive, nondecreasing, and a differentiable function.

(d) One may give extensions of the maximum principle to solutions of equations as $\Delta(h\Delta u) + \alpha(h\Delta u)^k + gu = 0$ under suitable assumptions.

References

[1] S. N. Chow and D. R. Dunninger, *A maximum principle for n-metaharmonic functions*, Proc. Amer. Math. Soc. **43** (1974), 79–83. MR 48#9090. Zbl 295.31004.

[2] D. R. Dunninger, *Maximum principles for solutions of some fourth-order elliptic equations*, J. Math. Anal. Appl. **37** (1972), 655–658. MR 47#602. Zbl 227.35039.

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