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APPROXIMATING FIXED POINTS OF NONEXPANSIVE MAPPINGS

GUIMEI LIU, DENG LEI, and SHENGHONG LI

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ABSTRACT. We consider a mapping S of the form

$$S = \alpha_0 I + \alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_k T_k,$$

where $\alpha_i \geq 0$, $\alpha_0 > 0$, $\alpha_1 > 0$ and $\sum_{i=0}^{k} \alpha_i = 1$. We show that the Picard iterates of S converge to a common fixed point of T_i $(i = 1,2,\dots,k)$ in a Banach space when T_i $(i = 1,2,\dots,k)$ are nonexpansive.

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1. Introduction. Let X be a Banach space and C a convex subset of X. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all x, y in C.

Specifically, the iterative process studied by Kirk is given by

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T x_n + \alpha_2 T^2 x_n + \cdots + \alpha_k T^k x_n, \tag{1.1}$$

where $\alpha_i \geq 0$, $\alpha_1 > 0$ and $\sum_{i=0}^{k} \alpha_i = 1$.

Kirk [1] has investigated an iterative process for approximating fixed points of nonexpansive mapping on convex subset of a uniformly convex Banach space. Recently, Maiti and Saha [2] improved the result of Kirk.

Let $T_i : C \to C$ $(i = 1,2,\dots,k)$ be nonexpansive mappings, and let

$$S = \alpha_0 I + \alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_k T_k, \tag{1.2}$$

where $\alpha_i \geq 0$, $\alpha_0 > 0$, $\alpha_0 > 0$ and $\sum_{i=0}^{k} \alpha_i = 1$.

In this paper, we show that the Picard iterates (1.2) of S converge to a common fixed point of T_i $(i = 1,2,\dots,k)$ in a Banach space, without any assumption on convexity of Banach space. This result generalizes the corresponding result of Kirk [1], Maiti and Saha [2], Senter and Dotson [4].

2. Main results

LEMMA 2.1. *Let X be a normed space and $\{a_n\}$ and $\{b_n\}$ be two sequences in X satisfying*

(i) $\lim_{n\to\infty} \|a_n\| = d$,

(ii) $\limsup_{n\to\infty} \|b_n\| \leq d$ *and* $\{\sum_{i=1}^{n} b_i\}$ *is bounded,*

(iii) *there is a constant t with $0 < t < 1$ such that $a_{n+1} = (1-t)a_n + t b_n$ for all $n \geq 1$. Then $d = 0$.*

Proof. Suppose that $d > 0$ and it follows from (ii) that $\sum_{i=n}^{n+m-1} b_i$ is bounded for all n and m. Let

$$M = \sup\left\{\left\|\sum_{i=n}^{n+m-1} b_i\right\| : n, m = 1, 2, 3, \ldots\right\}. \tag{2.1}$$

Choose a number N such that

$$N > \max\left(\frac{2tM}{d}, 1\right). \tag{2.2}$$

We can choose a positive ε such that

$$1 - 2\varepsilon \exp\left(\frac{N+1}{1-t}\right) > \frac{1}{2}. \tag{2.3}$$

By $0 < t < 1$, there exists a natural k such that

$$N < kt \le N + 1. \tag{2.4}$$

Since $\lim_{n\to\infty} \|a_n\| = d$, $\limsup_{n\to\infty} \|b_n\| \le d$ and ε independent of n, without loss of generality we may assume that, for all $n \ge 1$,

$$d(1-\varepsilon) < \|a_n\| < d(1+\varepsilon) \qquad \text{and} \qquad \|b_n\| < d(1+\varepsilon). \tag{2.5}$$

Setting $s = 1 - t$ from (iii), we obtain by induction

$$a_{k+1} = s^k a_1 + ts^{k-1} b_1 + \cdots + tsb_{k-1} + tb_k, \qquad a_{k+1} \in B := \operatorname{co}\{a_1, b_1, b_2, \ldots, b_k\}. \tag{2.6}$$

Let $x = (1/k)\sum_{i=1}^{k} b_i$ and

$$y = \frac{s^k}{1-s^k}\{a_1 + t[s^{-1} - (kt)^{-1}]b_1 + t[s^{-2} - (kt)^{-1}]b_2 + \cdots + t[s^{-k} - (kt)^{-1}]b_k\}. \tag{2.7}$$

Then it is clear that $x, y \in B$ and $a_{k+1} = s^k x + (1-s^k)y$. Therefore,

$$d(1-\varepsilon) < \|a_{k+1}\| \le s^k\|x\| + (1-s^k)\|y\| \le s^k\|x\| + (1-s^k)d(1+\varepsilon). \tag{2.8}$$

Hence, we have

$$\begin{aligned} \|x\| &> d(1 - s^{-k}(2-s^k)\varepsilon) > d(1 - 2\varepsilon s^{-k}) \\ &= d\left\{1 - 2\varepsilon \exp\left[\sum_{i=1}^{k} \log\left(1 + \frac{t}{1-t}\right)\right]\right\} \ge d\left[1 - 2\varepsilon \exp\left(\sum_{i=1}^{k} \frac{t}{1-t}\right)\right] \\ &= d\left[1 - 2\varepsilon \exp\left(\frac{kt}{1-t}\right)\right] \ge d\left[1 - 2\varepsilon \exp\left(\frac{N+1}{1-t}\right)\right] > \frac{d}{2}, \end{aligned} \tag{2.9}$$

since $\log(1+u) \le u$ for $-1 < u < \infty$.

On the other hand, we have

$$\|x\| = \frac{1}{k}\left\|\sum_{i=1}^{k} b_i\right\| \le \frac{M}{k} \le \frac{d}{2M}M = \frac{d}{2}, \tag{2.10}$$

arriving at a contradiction. This completes the proof. □

Lemma 2.2. *Let C be a subset of a normed space X and $T_n : C \to C$ be a nonexpansive mapping for all $n = 1,2,\dots,k$. If for an arbitrary $x_0 \in C$ and $\{x_n\}$ is defined by (1.2), then*

$$\|x_{n+1} - p\| \le \|x_n - p\| \tag{2.11}$$

for all $n \ge 1$ and $p \in F(T)$, where $F(T)$ denotes the common fixed point set of T_i $(i = 1,2,\dots,k)$.

Proof. Since $p = Sp$ for all $p \in F(T)$ and T_i $(i = 1,2,\dots,k)$ is nonexpansive, we have

$$\|x_{n+1} - p\| = \|Sx_n - Sp\| \le \sum_{i=1}^{k} \alpha_i \|T_i x_n - T_i p\| = \|x_n - p\| \tag{2.12}$$

for all $n \ge 1$ and all $p \in F(T)$. This completes the proof. □

Theorem 2.3. *Let C be a nonempty closed convex and bounded subset of a Banach space X and $T_i : C \to C$ $(i = 1,2,\dots,k)$ be nonexpansive mappings. If for an arbitrary $x_0 \in C$ and $\{x_n\}$ is defined by (1.2), then $\|x_n - Sx_n\| \to 0$ as $n \to \infty$.*

Proof. By (1.2) and T_i is nonexpansive mapping, we have

$$\begin{aligned} \|x_{n+1} - Sx_{n+1}\| &\le \|Sx_n - Sx_{n+1}\| \\ &\le \alpha_0 \|x_n - x_{n+1}\| + \sum_{i=1}^{k} \alpha_i \|T_i x_n - T_i x_{n+1}\| \le \|x_n - Sx_n\|. \end{aligned} \tag{2.13}$$

Hence $\|x_n - Sx_n\| \to d$ as $n \to \infty$.

Set $a_n = x_n - Sx_n$, $b_n = 1/(1-\alpha_0)\sum_{i=1}^{k} \alpha_i (T_i x_n - T_i x_{n+1})$, we have $a_{n+1} = \alpha_0 a_n + (1-\alpha_0) b_n$ and

$$\|b_n\| \le \frac{1}{1-\alpha_0} \sum_{i=1}^{k} \alpha_i \|T_i x_n - T_i x_{n+1}\| \le \frac{1}{1-\alpha_0} \sum_{i=1}^{k} \alpha_i \|x_n - x_{n+1}\| = \|a_n\|. \tag{2.14}$$

Since $\lim_{n\to\infty} \|a_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = d$,

$$\limsup_{n \to \infty} \|b_n\| \le d. \tag{2.15}$$

Finally, we have

$$\begin{aligned} \left\| \sum_{j=1}^{n} b_j \right\| &= \left\| \sum_{j=1}^{n} \left[\frac{1}{1-\alpha_0} \sum_{i=1}^{k} \alpha_i (T_i x_j - T_i x_{j+1}) \right] \right\| \\ &= \frac{1}{1-\alpha_0} \left\| \sum_{i=1}^{k} \alpha_i \left[\sum_{j=1}^{n} (T_i x_j - T_i x_{j+1}) \right] \right\| \\ &= \frac{1}{1-\alpha_0} \left\| \sum_{i=1}^{k} \alpha_i (T_i x_1 - T_i x_{n+1}) \right\| \\ &\le \frac{1}{1-\alpha_0} \sum_{i=1}^{k} \alpha_i \|T_i x_1 - T_i x_{n+1}\| \le \|x_1 - x_{n+1}\|. \end{aligned} \tag{2.16}$$

Then $\|\sum_{j=1}^{n} b_j\|$ is bounded. Setting $t = 1 - \alpha_0$, then $a_{n+1} = (1-t)a_n + tb_n$ and $0 < t < 1$. It follows from Lemma 2.1 that $d = 0$, this completes the proof. □

Recall that a Banach space X is said to satisfy Opial's condition [3] if the condition $x_n \to x_0$ weakly implies

$$\limsup_{n \to \infty} ||x_n - x_0|| < \limsup_{n \to \infty} ||x_n - y|| \tag{2.17}$$

for all $y \neq x_0$.

Theorem 2.4. *Let X be a Banach space which satisfies Opial's condition, C be weakly compact and convex, and let T_i $(i = 1, 2, \ldots, k)$ and $\{x_n\}$ be as in Theorem 2.3. Then $\{x_n\}$ converges weakly to a fixed point of S.*

Proof. Due to weak compactness of C, there exists $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to a $p \in C$. With standard proof we show that $p = Sp$. We suppose that $\{x_n\}$ does not converge weakly to p; then there are $\{x_{n_l}\}$ and $q \neq p$ such that $x_{n_l} \to q$ weakly and $q = Sq$. By Theorem 2.3 and Opial's condition of X, we have

$$\begin{aligned} \lim_{n \to \infty} ||x_n - p|| &= \lim_{j \to \infty} ||x_{n_j} - p|| < \lim_{j \to \infty} ||x_{n_j} - q|| \\ &= \lim_{l \to \infty} ||x_{n_l} - q|| < \lim_{l \to \infty} ||x_{n_l} - p|| = \lim_{n \to \infty} ||x_n - p||, \end{aligned} \tag{2.18}$$

a contradiction. This completes the proof. □

Let D be a subset of a Banach space X. Mappings $T_i : D \to X$ $(i = 1, 2, \ldots, k)$ with a nonempty common fixed points set $F(T)$ in D will be said to satisfy condition A [2, 4] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$, such that $\|x - Sx\| \geq f(d(x, F(T)))$ for all $x \in D$, where S is defined by (1.2), $d(x, F(T)) = \inf\{\|x - z\| : z \in F(T)\}$.

Theorem 2.5. *Let X, C, and $\{x_n\}$ be as in Theorem 2.3. Let $T_i : C \to X$ $(i = 1, 2, \ldots, k)$ be nonexpansive mappings with a nonempty common fixed points set $F(T)$ in C. If T_i satisfies condition A, then $\{x_n\}$ converges to a member of $F(T)$.*

Proof. By condition A, we have

$$||x_n - Sx_n|| \geq f\{d[x_n, F(T)]\} \tag{2.19}$$

for all $n \geq 0$. Since $\{d[x_n, F(T)]\}$ is decreasing by Lemma 2.2, it follows from Theorem 2.3 that

$$\lim_{n \to \infty} \{d[x_n, F(T)]\} = 0. \tag{2.20}$$

We can thus choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$||x_{n_j} - p_j|| < 2^{-j} \tag{2.21}$$

for all integers $j \geq 1$ and some sequence $\{p_j\}$ in $F(T)$. Again by Lemma 2.2, we have $\|x_{n_j+1} - p_j\| \leq \|x_{n_j} - p_j\| < 2^{-j}$, and hence

$$||p_{j+1} - p_j|| \leq ||p_{j+1} - x_{n_j+1}|| + ||x_{n_j+1} - p_j|| \leq 2^{-(j+1)} + 2^{-j} < 2^{-j+1}, \tag{2.22}$$

which show that $\{p_j\}$ is Cauchy and therefore converges strongly to a point p in $F(T)$ since $F(T)$ is closed. Now it is readily seen that $\{x_{n_j}\}$ and hence $\{x_n\}$ itself, by Lemma 2.2, converges strongly to p. □

Remark 2.6. Theorem 2.5 generalizes [2, 4, Theorem 2.3] to a Banach space.

Theorem 2.7. *Let C be a closed convex subset of a Banach space X, and T_i $(i = 1, 2, \dots, k)$ be nonexpansive mappings from C into a compact subset of X. If $\{x_n\}$ is as in Theorem 2.3, then $\{x_n\}$ converges to a fixed point of S.*

Proof. By Theorem 2.3 and the precompactness of $S(C)$, we see that $\{x_n\}$ admits a strongly convergent subsequence $\{x_{n_j}\}$ whose limit we denote by z. Then, again by Theorem 2.3, we have $z = Sz$; namely, z is a fixed point of S. Since $\|x_n - z\|$ is decreasing by Lemma 2.1, z is actually the strong limit of the sequence $\{x_n\}$ itself. □

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Guimei Liu: Department of Mathematics, Zhejiang University, Zhejiang 310027, China

Deng Lei: Department of Mathematics, Southwest China Normal University, Beibei, Chongqing 400715, China

Shenghong Li: Department of Mathematics, Zhejiang University, Zhejiang 310027, China

E-mail address: lsh@math.zju.edu.cn