Internat. J. Math. & Math. Sci.
Vol. 24, No. 4 (2000) 277–281
S0161171200003926

FANTASTIC FILTERS OF LATTICE IMPLICATION ALGEBRAS

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(Received 22 September 1999)

ABSTRACT. The notion of a fantastic filter in a lattice implication algebra is introduced, and the relations among filter, positive implicative filter, and fantastic filter are given. We investigate an equivalent condition for a filter to be fantastic, and state an extension property for fantastic filter.

Keywords and phrases. Lattice implication algebra, filter, implicative filter, positive implicative filter, fantastic filter.

2000 Mathematics Subject Classification. Primary 03G10, 06B10, 54E15.

1. Introduction. In order to research the logical system whose propositional value is given in a lattice, Xu [5] proposed the concept of lattice implication algebras, and discussed their some properties. Also, in [6], Xu and Qin discussed the properties of lattice H implication algebras, and gave some equivalent conditions about lattice H implication algebras. For the general development of lattice implication algebras, the filter theory plays an important role as well as ideal theory. Xu and Qin [7] introduced the notion of filters in a lattice implication algebra, and investigated their properties. In [2], we gave an equivalent condition of a filter, and provided some equivalent conditions that a filter is an implicative filter, and using this result an extension property for implicative filter is constructed. Jun et al. [4] introduced the concepts of a positive implicative filter and an associative filter in a lattice H implication algebra. They proved that (i) every positive implicative filter is an implicative filter, and (ii) every associative filter is a filter. They also provided equivalent conditions for both a positive implicative filter and an associative filter. In [3], Jun et al. defined an LI-ideal of a lattice implication algebra and showed that every LI-ideal is a lattice ideal. They gave an example that a lattice ideal may not be an LI-ideal, and showed that every lattice ideal is an LI-ideal in a lattice implication algebra. They discussed the relationship between filters and LI-ideals, and studied how to generate an LI-ideal by a set. Moreover they constructed the quotient structure by using an LI-ideal, and studied the properties of LI-ideals related to implication homomorphisms. In this paper, the notion of a fantastic filter in a lattice implication algebra is introduced, and then we give the relations among filter, positive implicative filter and fantastic filter. We investigate an equivalent condition for a filter to be fantastic, and state an extension property for fantastic filter.

2. Preliminaries. By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution "$'$" and a binary operation "$\to$" satisfying the following axioms:

(I1) $x \to (y \to z) = y \to (x \to z)$,
(I2) $x \to x = 1$,
(I3) $x \to y = y' \to x'$,
(I4) $x \to y = y \to x = 1 \Rightarrow x = y$,
(I5) $(x \to y) \to y = (y \to x) \to x$,
(L1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(L2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,

for all $x, y, z \in L$.

Note that the conditions (L1) and (L2) are equivalent to the conditions

(L3) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, and
(L4) $x \to (y \vee z) = (x \to y) \vee (x \to z)$, respectively.

Example 2.1. Let $L := \{0, a, b, c, 1\}$. Define the partially ordered relation on L as $0 < a < b < c < 1$, and define $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$ for all $x, y \in L$ and "$'$"and "$\to$" as follows:

Table 2.1.

x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra.

In the sequel the binary operation "$\to$" will be denoted by juxtaposition. We can define a partial ordering "$\leq$" on a lattice implication algebra L by $x \leq y$ if and only if $xy = 1$.

In a lattice implication algebra L, the following hold (see [5]):

(1) $0x = 1$, $1x = x$, and $x1 = 1$.
(2) $x' = x0$.
(3) $xy \leq (yz)(xz)$.
(4) $x \vee y = (xy)y$.
(5) $((yx)y')' = x \wedge y = ((xy)x')'$.
(6) $x \leq y$ implies $yz \leq xz$ and $zx \leq zy$.
(7) $x \leq (xy)y$.

In what follows, L denotes a lattice implication algebra unless otherwise specified.

Definition 2.2 (Xu et al. [7]). A subset F of L is called a *filter* of L if it satisfies:

(F1) $1 \in F$,
(F2) $x \in F$ and $xy \in F$ imply $y \in F$ for all $x, y \in L$.

A subset F of L is called an *implicative filter* of L if it satisfies (F1) and

(F3) $x(yz) \in F$ and $xy \in F$ imply $xz \in F$ for all $x, y, z \in L$.

Proposition 2.3 (Jun [2, Proposition 3.2]). *Every filter F of L has the following property:*

$$x \le y \text{ and } x \in F \text{ imply } y \in F. \tag{2.1}$$

Definition 2.4 (Jun et al. [4]). A subset F of L is called a *positive implicative filter* of L if it satisfies (F1) and
(F4) $x((yz)y) \in F$ and $x \in F$ imply $y \in F$ for all $x, y, z \in L$.

Proposition 2.5 (Jun [4, Theorem 3.1]). *Every positive implicative filter F of L is a filter.*

Proposition 2.6 (Jun [4, Theorem 3.3]). *Let F be a filter of L. Then F is a positive implicative filter of L if and only if*
(F5) $(xy)x \in F$ *implies* $x \in F$ *for all* $x, y \in L$.

Proposition 2.7 (Jun [2, Theorem 3.4]). *Let F be a non-empty subset of L. Then F is a filter of L if and only if it satisfies: for all $x, y \in F$ and $z \in L$,*
(F6) $x \le yz$ *implies* $z \in F$.

3. Fantastic filters

Definition 3.1. A non-empty subset F of L is called a *fantastic filter* of L if it satisfies (F1) and
(F7) $z(yx) \in F$ and $z \in F$ imply $((xy)y)x \in F$ for all $x, y, z \in L$.

Example 3.2. Let $L := \{0, a, b, c, d, 1\}$ be a set with Figure 3.1 as a partial ordering. Define a unary operation "$'$" and a binary operation denoted by juxtaposition on L as follows (Tables 3.2 and 3.3, respectively):

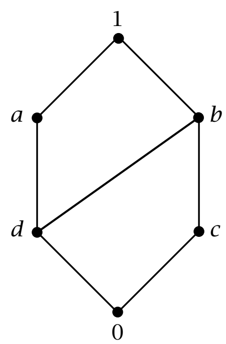

Figure 3.1.

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (xy)y, \qquad x \wedge y := ((x'y')y')', \tag{3.1}$$

TABLE 3.2.

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

TABLE 3.3.

	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

for all $x, y \in L$. Then L is a lattice implication algebra. One can see that $F := \{b, c, 1\}$ is a fantastic filter of L.

THEOREM 3.3. *Every fantastic filter of L is a filter.*

PROOF. Let F be a fantastic filter of L and let $zx \in F$ and $z \in F$. Then $z(1x) \in F$ and $z \in F$. It follows from (F7) that $x = ((x1)1)x \in F$ so that F is a filter. □

We now give an equivalent condition for a filter to be a fantastic filter.

THEOREM 3.4. *A filter F of L is fantastic if and only if it satisfies:*
(F8) $yx \in F$ *implies* $((xy)y)x \in F$ *for all* $x, y \in L$.

PROOF. Assume that F is a fantastic filter of L and let $x, y \in L$ be such that $yx \in F$. Then $1(yx) = yx \in F$ and $1 \in F$. It follows from (F7) that $((xy)y)x \in F$. Conversely let F be a filter of L satisfying (F8) and let $x, y, z \in L$ be such that $z(yx) \in F$ and $z \in F$. Then $yx \in F$ by (F2) and hence $((xy)y)x \in F$ by (F8). Therefore F is a fantastic filter of L. □

THEOREM 3.5. *Every positive implicative filter of L is fantastic.*

PROOF. Let F be a positive implicative filter of L. Then F is a filter of L (see Proposition 2.5). Let $x, y \in L$ be such that $yx \in F$. It is sufficient to show that $((xy)y)x \in F$. Since $x \leq ((xy)y)x$, we get $(((xy)y)x)y \leq xy$. Putting $a = ((xy)y)x$, we obtain

$$\begin{aligned}(ay)a &= ((((xy)y)x)y)(((xy)y)x) \\ &\geq (xy)(((xy)y)x) = ((xy)y)((xy)x) \geq yx.\end{aligned} \tag{3.2}$$

It follows from Proposition 2.3 that $(ay)a \in F$ so, from Proposition 2.6, that $a \in F$, i.e., $((xy)y)x \in F$. Hence F is a fantastic filter of L. □

Open problem. Does the converse of Theorem 3.5 hold?

Theorem 3.6 (extension property for fantastic filter). *Let F and G be filters of L such that $F \subseteq G$. If F is fantastic, then so is G.*

Proof. Let $x, y \in L$ be such that $yx \in G$. Then $y((yx)x) = (yx)(yx) = 1 \in F$. Since F is fantastic, it follows from Theorem 3.4 that

$$((((yx)x)y)y)((yx)x) \in F \tag{3.3}$$

so that $(yx)(((((yx)x)y)y)x) \in F \subseteq G$. Since $yx \in G$, therefore $((((yx)x)y)y)x \in G$. But

$$\begin{aligned}(((((yx)x)y)y)x)(((xy)y)x) &\geq ((xy)y)((((yx)x)y)y)\\ &\geq (((yx)x)y)(xy) \geq x((yx)x)\\ &= (yx)(xx) = (yx)1 = 1.\end{aligned} \tag{3.4}$$

Using Proposition 2.7, we get $((xy)y)x \in G$. Hence, by Theorem 3.4, G is a fantastic filter of L. □

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