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A NOTE ON KAKUTANI TYPE FIXED POINT THEOREMS

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ABSTRACT. We present Kakutani type fixed point theorems for certain semigroups of self maps by relaxing conditions on the underlying set, family of self maps, and the mappings themselves in a locally convex space setting.

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1. Introduction. Using a technique of Tarafdar [9], we establish fixed point theorems by utilizing following semigroups under composition of self maps T on a subset M of a Hausdorff locally convex space

(i) $\mathcal{F} = C_T = \{f : M \to M \mid fT = Tf\}$,

(ii) $\mathcal{F} = \{T^n : n \in \mathbb{N} \cup \{0\}\}$,

(iii) $\mathcal{F} =$ identity map.

In the sequel (E,τ) will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is said to be an associated family of seminorms for τ if the family $\{rU : r > 0\}$, where $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighbourhoods of zero for τ. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is called an augmented associated family for τ if $\{p_\alpha : \alpha \in I\}$ is an associated family with the property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented associated families of seminorms shall be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space (E,τ), there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$ (see [7, page 203]).

The following construction will be crucial. Suppose that M is a τ-bounded subset of E. For this set M we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subset \lambda_\alpha U_\alpha$, where $U_\alpha = \{x : p_\alpha(x) \le 1\}$. Clearly, $B = \bigcap_\alpha \lambda_\alpha U_\alpha$ is τ-bounded, τ-closed, absolutely convex, and contains M. The linear span E_B of B in E is $\bigcup_{n=1}^{\infty} nB$. The Minkowski functional of B is a norm $\|\cdot\|_B$ on E_B. Thus $(E_B, \|\cdot\|_B)$ is a normed space with B as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$.

A self map T on M is said to be

(i) $A^*(\tau)$-nonexpansive if for all $x, y \in M$,

$$p_\alpha(Tx - Ty) \le p_\alpha(x - y) \quad \text{for each } p_\alpha \in A^*(\tau). \tag{1.1}$$

(ii) $A^*(\tau)$-asymptotically nonexpansive if for each $x, y \in M$,

$$p_\alpha(T^n x - T^n y) \le k_n p_\alpha(x-y), \quad n = 1,2,3,\dots, \text{ for each } p_\alpha \in A^*(\tau), \tag{1.2}$$

where $\{k_n\}$ is a fixed sequence of real numbers such that $k_n \to 1$ as $n \to \infty$.

In sequel, for simplicity, we shall call $A^*(\tau)$-nonexpansive ($A^*(\tau)$-asymptotically nonexpansive) maps to be nonexpansive (asymptotically nonexpansive).

Common fixed points of nonexpansive maps and best approximations have been considered in normed spaces (see [1, 3]). We prove common fixed point theorems for asymptotically nonexpansive maps in the setting of a locally convex space.

2. Results

Lemma 2.1. *Let M be a τ-bounded subset of a Hausdorff locally convex space (E,τ) and $T : M \to M$ be asymptotically nonexpansive map. Then T is asymptotically nonexpansive on M with respect to $\|\cdot\|_B$.*

Proof. By hypothesis for $x, y \in M$ and $n = 1,2,3,\dots,$

$$p_\alpha(T^n x - T^n y) \le k_n p_\alpha(x-y) \quad \text{for each } p_\alpha \in A^*(\tau), \tag{2.1}$$

where $\{k_n\}$ is a real sequence converging to 1,

$$\begin{gathered} \sup_\alpha p_\alpha\left(\frac{T^n x - T^n y}{\lambda_\alpha}\right) \le k_n \sup_\alpha p_\alpha\left(\frac{x-y}{\lambda_\alpha}\right), \\ \|T^n x - T^n y\|_B \le k_n \|x-y\|_B, \end{gathered} \tag{2.2}$$

where $\{k_n\} \to 1$ as $n \to \infty$ and is a fixed real sequence. This completes the proof. □

Note that $(E_B,\tau) \subset (E_B, \|\cdot\|_B)$ so a set compact in (E_B,τ) need not be compact in $(E_B, \|\cdot\|_B)$ (cf. [8, page 159, problem 3(c)]). To overcome this difficulty we use finite dimensionality to obtain following generalization of [9, Theorem 2.1].

Theorem 2.2. *Let M be a nonempty convex τ-bounded, τ-complete finite dimensional subset of a Hausdorff locally convex space (E,τ). Suppose $\mathcal{F}$ is a commutative semigroup of asymptotically nonexpansive self maps of M. Then there exists a point $a \in M$ such that*

$$T(a) = a \quad \text{for all } T \in \mathcal{F}. \tag{2.3}$$

Proof. Since M is τ-complete, it follows that $(E_B, \|\cdot\|_B)$ is a Banach space and M is complete in it. A closed, bounded and finite dimensional subset of a normed space is compact by [2, Theorem on page 10] so M is compact in $(E_B, \|\cdot\|_B)$. By Lemma 2.1, each $T \in \mathcal{F}$ is $\|\cdot\|_B$-asymptotically nonexpansive. Hence $\mathcal{F}$ is a commutative semigroup of asymptotically nonexpansive self maps of a compact convex subset M of the Banach space $(E_B, \|\cdot\|_B)$. The family $\mathcal{F}$ has a common fixed point by [4, Theorem 3.1]. □

We now prove another fixed point theorem for locally convex spaces by making use of Jungck and Sessa [6, Theorem 3]; see also [1, Corollary 2.3] and [5, Theorem 1].

Theorem 2.3. *Let M be a τ-bounded, τ-sequentially closed and finite dimensional subset of a Hausdorff locally convex space (E,τ). Suppose that M is starshaped with* *starcentre $q \in M$ and $T : M \to M$ is nonexpansive. Let $\mathcal{F}$ be a family of affine nonexpansive self maps of M commuting with T and leaving q fixed. Suppose for each pair $(x,y) \in M^2$, there exists $f = f(x,y)$ and $g = g(x,y)$ in $\mathcal{F}$ such that*

$$p_\alpha(Tx - Ty) \le p_\alpha(fx - gy) \quad \text{for all } p_\alpha \in A^*(\tau). \tag{2.4}$$

Then there exists $a \in M$ such that

$$a = T(a) = h(a) \quad \text{for all } h \in \mathcal{F}. \tag{2.5}$$

Proof. Since $\|\cdot\|_B$-topology is finer than the relative τ-topology on E_B, $\|\cdot\|_B$-cl$(M) \subset \tau$-sequential-cl$(M) = M$. Therefore, M is $\|\cdot\|_B$-closed in the normed space $(E_B, \|\cdot\|_B)$. As above, M is a compact subset of $(E_B, \|\cdot\|_B)$. Moreover, T and each $h \in \mathcal{F}$ is nonexpansive in (E,τ), which by Lemma 2.1 implies that T and each $h \in \mathcal{F}$ is $\|\cdot\|_B$-nonexpansive—so certainly $\|\cdot\|_B$-continuous. And from (2.4) we obtain for $x, y \in M$,

$$\sup_\alpha p_\alpha\left(\frac{Tx - Ty}{\lambda_\alpha}\right) \le \sup_\alpha p_\alpha\left(\frac{fx - gy}{\lambda_\alpha}\right). \tag{2.6}$$

Thus

$$\|Tx - Ty\|_B \le \|fx - gy\|_B \quad \text{for } x, y \in M. \tag{2.7}$$

A comparison of our hypothesis with that of [6, Theorem 3] tells us that we can now apply [6, Theorem 3] to M as a subset of $(E_B, \|\cdot\|_B)$ to conclude that there exists $a \in M$ such that $a = T(a) = h(a)$ for all $h \in \mathcal{F}$. □

Corollary 2.4. *Let M be a τ-bounded, τ-sequentially closed, and finite dimensional subset of a Hausdorff locally convex space (E,τ). Assume M is starshaped with starcentre $q \in M$. Suppose $T, I : M \to M$ are nonexpansive, I is affine and leaving q fixed and $TI = IT$. Suppose for $x, y \in M$, there exist $n = n(x,y)$, $m = m(x,y)$ in $\mathbb{N}_0 = \{0,1,2,\dots\}$ such that*

$$p_\alpha(Tx - Ty) \le p_\alpha(I^m x - I^n y) \quad \text{for each } p_\alpha \in A^*(\tau). \tag{2.8}$$

Then T and I have a common fixed point.

Proof. Let $\mathcal{F} = \{I^n : n \in \mathbb{N}_0\}$ $(I^0 x = x)$. For each n, I^n is affine, $TI^n = I^n T$ and $I^n : M \to M$ since I has these properties. Further (2.8) assures that $\mathcal{F}$ and its members satisfy (2.4) and the hypotheses of Theorem 2.3; consequently, the conclusion of the corollary follows. □

Corollary 2.5. *Let M be a τ-bounded, τ-closed finite dimensional starshaped subset of a Hausdorff locally convex space (E,τ) and T a nonexpansive self map of M. Then T has a fixed point.*

Finally, we consider an application of Corollary 2.4 to best approximation theory. A related result for normed spaces was given in [6, Theorem 4]. For any $\bar{x} \in E$, $C \subseteq E$

and $p_\alpha \in A^*(\tau)$, let

$$d_{p_\alpha}(\bar{x}, C) = \inf\{p_\alpha(y - \bar{x}) : y \in C\} \tag{2.9}$$

and let

$$D = \{y \in C : p_\alpha(y - \bar{x}) = d_{p_\alpha}(\bar{x}, C) \text{ for all } p_\alpha \in A^*(\tau)\}. \tag{2.10}$$

Theorem 2.6. *Let T and I be self maps of a Hausdorff locally convex space (E, τ) and let $C \subseteq E$ be such that $T : \partial C \to C$. Let T and I leave $\bar{x} \in E$ fixed and satisfy (2.8) for all $x, y \in D \cup \{\bar{x}\}$. Suppose I is nonexpansive and affine, T is nonexpansive on D, $IT = TI$ on D, and D is nonempty τ-bounded, τ-sequentially closed, finite dimensional and starshaped with respect to q. If I leaves q invariant and $I(D) \subseteq D$, then there exists $a \in D$ such that $a = I(a) = T(a)$.*

Proof. Let $y \in D$. Then $I^n y \in D$ for $n \in \mathbb{N}_0$ since $I(D) \subseteq D$. By definition of D, $y \in \partial C$ and since $T : \partial C \to C$, it follows that $Ty \in C$. By (2.8), for each $p_\alpha \in A^*(\tau)$,

$$p_\alpha(Ty - \bar{x}) = p_\alpha(Ty - T\bar{x}) \le p_\alpha(I^n y - I^m \bar{x}) \tag{2.11}$$

for some $n, m \in \mathbb{N}_0$. As $I^m \bar{x} = \bar{x}$, we get

$$p_\alpha(Ty - \bar{x}) \le p_\alpha(I^n y - \bar{x}) \quad \text{for all } p_\alpha \in A^*(\tau). \tag{2.12}$$

Again since $Ty \in C$ and $I^n y \in D$, the definition of D further implies that $Ty \in D$. Consequently, $T, I : D \to D$ and the conditions of Corollary 2.4 are satisfied. Hence there exists $a \in D$ such that $a = I(a) = T(a)$. □

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