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MATRIX-VARIATE BETA DISTRIBUTION

ARJUN K. GUPTA and DAYA K. NAGAR

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ABSTRACT. We propose matrix-variate beta type III distribution. Several properties of this distribution including Laplace transform, marginal distribution and its relationship with matrix-variate beta type I and type II distributions are also studied.

Keywords and phrases. Matrix variate, beta distribution, zonal polynomial, transformation, Laplace transform.

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1. Introduction. The random variable u with the probability density function (pdf)

$$\big(\beta(a,b)\big)^{-1}u^{a-1}(1-u)^{b-1}, \quad 0<u<1, \tag{1.1}$$

where $a>0$ and $b>0$, is said to have a beta type I distribution with parameters (a,b). The random variable v with pdf,

$$\big(\beta(a,b)\big)^{-1}v^{a-1}(1+v)^{-(a+b)}, \quad v>0, \tag{1.2}$$

where $a>0$ and $b>0$, is said to have beta type II distribution with parameters (a,b). Since (1.2) can be obtained from (1.1) by the transformation $v=u/(1-u)$, some authors call the distribution of v an "inverted beta distribution."

The matrix variate generalizations of (1.1) and (1.2) are given as follows (see [1, 3, 4, 6, 11]).

Definition 1.1. A $p\times p$ random symmetric positive definite matrix U is said to have a matrix-variate beta type I distribution with parameters (a,b), denoted as $U\sim B_p^I(a,b)$, if its pdf is given by

$$\big(\beta_p(a,b)\big)^{-1}\det(U)^{a-(p+1)/2}\det\left(I_p-U\right)^{b-(p+1)/2}, \quad 0<U<I_p, \tag{1.3}$$

where $a>(p-1)/2$, $b>(p-1)/2$, and $\beta_p(a,b)$ is the multivariate beta function given by

$$\beta_p(a,b)=\frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}, \tag{1.4}$$

where

$$\Gamma_p(a)=\pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma\left(a-\frac{j-1}{2}\right), \quad \operatorname{Re}(a)>\frac{p-1}{2}. \tag{1.5}$$

Definition 1.2. A $p \times p$ random symmetric positive definite matrix V is said to have a matrix-variate beta type II distribution with parameters (a,b), denoted as $V \sim B_p^{II}(a,b)$, if its pdf is given by

$$(\beta_p(a,b))^{-1}\det(V)^{a-(p+1)/2}\det\left(I_p+V\right)^{-(a+b)}, \quad V>0, \tag{1.6}$$

where $a > (p-1)/2$, $b > (p-1)/2$, and $\beta_p(a,b)$ is the multivariate beta function.

As in the univariate case, the density (1.6) can be obtained from (1.3) by transforming $U = (I_p+V)^{-1}V$, together with the Jacobian $J(U \to V) = \det(I_p+V)^{-(p+1)}$.

The matrix-variate beta type I and type II distributions have been studied by many authors, e.g., see [7, 9, 10, 13, 14].

In this paper, a new matrix-variate beta distribution has been defined. We call it "Matrix-variate beta type III" distribution, which is then derived by using matrix transformation. Several properties of this distribution and its relationship with matrix-variate beta type I and type II distributions have also been studied.

2. Density function.

First, we define the matrix-variate beta distribution of type III.

Definition 2.1. A $p \times p$ random symmetric positive definite matrix W is said to have a matrix-variate beta type III distribution with parameters (a,b), denoted as $W \sim B_p^{III}(a,b)$, if its pdf is given by

$$\begin{aligned}&2^{pb}(\beta_p(a,b))^{-1}\det(W)^{b-(p+1)/2}\\&\qquad\times\det\left(I_p-W\right)^{a-(p+1)/2}\det\left(I_p+W\right)^{-(a+b)}, \quad 0<W<I_p,\end{aligned} \tag{2.1}$$

where $a > (p-1)/2$, $b > (p-1)/2$, and $\beta_p(a,b)$ is the multivariate beta function.

For $p = 1$, the beta type III density is given by

$$2^b(\beta(a,b))^{-1}w^{b-1}(1-w)^{a-1}(1+w)^{-(a+b)}, \quad 0<w<1, \tag{2.2}$$

and in this case we write $w \sim B^{III}(a,b)$.

By means of a bilinear transformation of the random matrix U, the matrix-variate beta type III distribution is generated as in the following theorem.

Theorem 2.1. *Let $U \sim B_p^I(a,b)$. Define $W = (I_p+U)^{-1}(I_p-U)$. Then $W \sim B_p^{III}(a,b)$.*

Proof. Making the transformation $W = (I_p+U)^{-1}(I_p-U)$ with the Jacobian [12] in the pdf (1.3) of U, $J(U \to W) = 2^{p(p+1)/2}(I_p+W)^{-(p+1)}$, we get the desired result. □

From the density of beta type III matrix it is apparent that

$$\begin{aligned}\int_{0<W<I_p}&\det(W)^{b-(p+1)/2}\det\left(I_p-W\right)^{a-(p+1)/2}\\&\times\det\left(I_p+W\right)^{-(a+b)}dW = 2^{-pb}\beta_p(a,b).\end{aligned} \tag{2.3}$$

The cumulative distribution function (cdf) of W is obtained as

$$P(W<\Omega)=2^{pb}\left(\beta_p(a,b)\right)^{-1}\int_{0<W<\Omega}\det(W)^{b-(p+1)/2}\det\left(I_p-W\right)^{a-(p+1)/2}\times\det\left(I_p+W\right)^{-(a+b)}dW. \tag{2.4}$$

For evaluating the above integral, we use the following results involving zonal polynomials [2, 5, 8]:

$$\det\left(I_p-W\right)^{a-(p+1)/2}=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-a+(p+1)/2)_\kappa}{k!}C_\kappa(W), \tag{2.5}$$

$$\det\left(I_p+W\right)^{-(a+b)}=\sum_{t=0}^{\infty}\sum_{\tau}\frac{(a+b)_\tau}{t!}(-1)^tC_\tau(W), \tag{2.6}$$

$$C_\kappa(W)C_\tau(W)=\sum_{\delta}g_{\kappa,\tau}^{\delta}C_\delta(W), \tag{2.7}$$

$$\begin{aligned}\int_{0<W<I_p}&\det(W)^{a-(p+1)/2}\det\left(I_p-W\right)^{b-(p+1)/2}C_\kappa(RW)\,dW\\&=\beta_p(a,b)\frac{(a)_\kappa}{(a+b)_\kappa}C_\kappa(R),\quad \operatorname{Re}(a)>\frac{p-1}{2},\ \operatorname{Re}(b)>\frac{p-1}{2},\end{aligned} \tag{2.8}$$

where $\kappa=(k_1,\dots,k_p)$, $k_1\geq\cdots\geq k_p\geq 0$, $k_1+\cdots+k_p=k$, $\tau=(t_1,\dots,t_p)$, $t_1\geq\cdots\geq t_p\geq 0$, $t_1+\cdots+t_p=t$, $\delta=(d_1,\dots,d_p)$, $d_1\geq\cdots\geq d_p\geq 0$, $d_1+\cdots+d_p=d=k+t$, and $g_{\kappa,\tau}^{\delta}$ is the coefficient of $C_\delta(W)$ in $C_\kappa(W)C_\tau(W)$. Substituting (2.5) and (2.6) in (2.4) and subsequently using (2.7), we obtain

$$\begin{aligned}P(W<\Omega)=2^{pb}\left(\beta_p(a,b)\right)^{-1}&\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{\tau}\frac{(-a+(p+1)/2)_\kappa}{k!}\\&\times\frac{(a+b)_\tau}{t!}(-1)^t\sum_{\delta}g_{\kappa,\tau}^{\delta}\int_{0<W<\Omega}\det(W)^{b-(p+1)/2}C_\delta(W)\,dW.\end{aligned} \tag{2.9}$$

Now substituting $X=\Omega^{-1/2}W\Omega^{-1/2}$ with the Jacobian $J(W\to X)=\det(\Omega)^{(p+1)/2}$ in the above integral, we get

$$\begin{aligned}P(W<\Omega)&=2^{pb}\left(\beta_p(a,b)\right)^{-1}\det(\Omega)^b\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{\tau}\frac{(-a+(p+1)/2)_\kappa}{k!}\\&\quad\times\frac{(a+b)_\tau}{t!}(-1)^t\sum_{\delta}g_{\kappa,\tau}^{\delta}\int_{0<X<I_p}\det(X)^{b-(p+1)/2}C_\delta(\Omega X)\,dX\\&=2^{pb}\frac{\Gamma_p(a+b)\Gamma_p[(p+1)/2]}{\Gamma_p(a)\Gamma_p[b+(p+1)/2]}\det(\Omega)^b\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{\tau}\frac{(-a+(p+1)/2)_\kappa}{k!}\\&\quad\times\frac{(a+b)_\tau}{t!}(-1)^t\sum_{\delta}g_{\kappa,\tau}^{\delta}\frac{(b)_\delta}{(b+(p+1)/2)_\delta}C_\delta(\Omega),\end{aligned} \tag{2.10}$$

where the last step has been obtained by using (2.8).

The Laplace transform of the density of W is

$$L(Z) = 2^{pb}\left(\beta_p(a,b)\right)^{-1}\int_{0<W<I_p}\operatorname{etr}(-ZW)\det(W)^{b-(p+1)/2} \times\det\left(I_p-W\right)^{a-(p+1)/2}\det\left(I_p+W\right)^{-(a+b)}dW, \tag{2.11}$$

where $Z(p\times p) = ((1+\delta_{ij})z_{ij}/2)$. Now, using the expansions

$$\begin{aligned}\det\left(I_p+W\right)^{-(a+b)} &= 2^{-p(a+b)}\det\left(I_p-\frac{I_p-W}{2}\right)^{-(a+b)}\\ &= 2^{-p(a+b)}\sum_{t=0}^{\infty}\sum_{\tau}\frac{(a+b)_\tau}{t!}\left(\frac{1}{2}\right)^t C_\tau\left(I_p-W\right),\\ \operatorname{etr}(-ZW) = \operatorname{etr}(-Z)\operatorname{etr}\{Z(I_p-W)\} &= \operatorname{etr}(-Z)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}C_\kappa\left(Z\left(I_p-W\right)\right),\end{aligned} \tag{2.12}$$

in (2.11), we obtain

$$L(Z) = 2^{-pa}\left(\beta_p(a,b)\right)^{-1}\operatorname{etr}(-Z)\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{\tau}\frac{1}{k!}\frac{(a+b)_\tau}{2^t\,t!}\Phi(Z), \tag{2.13}$$

where

$$\begin{aligned}\Phi(Z) &= \int_{0<W<I_p}\det(W)^{b-(p+1)/2}\det\left(I_p-W\right)^{a-(p+1)/2}C_\kappa\left(Z\left(I_p-W\right)\right)C_\tau\left(I_p-W\right)dW\\ &= \int_{0<X<I_p}\det\left(I_p-X\right)^{b-(p+1)/2}\det(X)^{a-(p+1)/2}C_\kappa(ZX)C_\tau(X)\,dX.\end{aligned} \tag{2.14}$$

Since $\Phi(Z) = \Phi(H'ZH)$, $H\in O(p)$, integrating out H in $\Phi(H'ZH)$ using [5, equation 23], we have

$$\Phi(Z) = \frac{C_\kappa(Z)}{C_\kappa(I_p)}\int_{0<X<I_p}\det\left(I_p-X\right)^{b-(p+1)/2}\det(X)^{a-(p+1)/2}C_\kappa(X)C_\tau(X)\,dX. \tag{2.15}$$

Now using (2.7), and integrating X using (2.8), we obtain

$$\Phi(Z) = \frac{C_\kappa(Z)}{C_\kappa(I_p)}\sum_{\delta}g^{\delta}_{\kappa,\tau}\frac{(a)_\delta}{(a+b)_\delta}\frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}C_\delta(I_p). \tag{2.16}$$

Substituting (2.16) in (2.13), we finally obtain

$$L(Z) = 2^{-pa}\operatorname{etr}(-Z)\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{\tau}\frac{1}{k!}\frac{(a+b)_\tau}{2^t\,t!}\frac{C_\kappa(Z)}{C_\kappa(I_p)}\sum_{\delta}g^{\delta}_{\kappa,\tau}\frac{(a)_\delta}{(a+b)_\delta}C_\delta(I_p). \tag{2.17}$$

3. Properties. In this section, we study some properties of the random matrix distributed as matrix-variate beta type III.

THEOREM 3.1. *Let $W \sim B_p^{III}(a,b)$ and $A\,(p\times p)$ be a constant nonsingular matrix. Then the density of $X = AWA'$ is*

$$2^{pb}\left(\beta_p(a,b)\right)^{-1}\det(AA')^{(p+1)/2}\det(X)^{b-(p+1)/2} \times\det(AA'-X)^{a-(p+1)/2}\det(AA'+X)^{-(a+b)},\quad 0<X<AA'. \tag{3.1}$$

PROOF. Making the transformation $X = AWA'$ with the Jacobian $J(W\to X) = \det(A)^{-(p+1)}$ in the pdf (2.1) of W, the density of X is obtained as

$$2^{pb}\left(\beta_p(a,b)\right)^{-1}\det(AA')^{(p+1)/2}\det(X)^{b-(p+1)/2} \times\det(AA'-X)^{a-(p+1)/2}\det(AA'+X)^{-(a+b)},\quad 0<X<AA'. \tag{3.2}$$

which is the desired result. □

We will write $X \sim B_p^{III}(a,b;AA')$. In the next theorem, it is shown that the matrix-variate beta distribution of type III is orthogonally invariant.

THEOREM 3.2. *Let $W \sim B_p^{III}(a,b)$ and $H\,(p\times p)$ be an orthogonal matrix, whose elements are either constants or random variables distributed independently of W. Then, the distribution of W is invariant under the transformation $W \to HWH'$, and does not depend on H.*

PROOF. First, let H be a constant orthogonal matrix. Then, from Theorem 3.1, $HWH' \sim B_p^{III}(a,b)$ since $HH' = I_p$. If, however, H is a random orthogonal matrix, then the conditional distribution of $HUH' \mid H \sim B_p^{III}(a,b)$. Since this distribution does not depend on H, $HWH' \sim B_p^{III}(a,b)$. □

The relationship between beta type I, type II, and type III matrices is now exhibited. First, we derive the density of W^{-1}.

THEOREM 3.3. *Let $W \sim B_p^{III}(a,b)$. Then the density of $Y = W^{-1}$ is*

$$2^{pb}\left(\beta_p(a,b)\right)^{-1}\det\left(Y-I_p\right)^{a-(p+1)/2}\det\left(I_p+Y\right)^{-(a+b)},\quad Y>I_p. \tag{3.3}$$

PROOF. Making the transformation $Y = W^{-1}$ with the Jacobian $J(W\to Y) = \det(Y)^{-(p+1)}$, in the density of W the result follows. □

The density derived above may be called the inverse beta type III. From Theorem 3.3, it is clear that if $W \sim B_p^{III}(a,b)$, then W^{-1} does not follow beta type III distribution.

THEOREM 3.4. (i) *Let $U \sim B_p^{I}(a,b)$ and $W = (I_p+U)^{-1}(I_p-U)$, then $W \sim B_p^{III}(a,b)$.*
(ii) *Similarly, if $W \sim B_p^{III}(a,b)$ and $U = (I_p+W)^{-1}(I_p-W)$, then $U \sim B_p^{I}(a,b)$.*

PROOF. Part (i) is proved in Theorem 2.1. Since $W = (I_p+U)^{-1}(I_p-U)$, the Jacobian of transformation is $J(W\to U) = 2^{p(p+1)/2}\det(I_p+U)^{-(p+1)}$. Now, making the substitution in the density of W given by (2.1) the result (ii) follows. □

THEOREM 3.5. (i) *Let $V \sim B_p^{II}(a,b)$ and $W = (I_p+2V)^{-1}$, then $W \sim B_p^{III}(a,b)$.*
(ii) *Similarly, if $W \sim B_p^{III}(a,b)$ and $V = (I_p-W)W^{-1}/2$, then $V \sim B_p^{II}(a,b)$.*

The marginal distributions of W is given in the following.

Theorem 3.6. *Let* $W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$, $W_{11}\,(q \times q)$, *and* $W_{22\cdot 1} = W_{22} - W_{21}W_{11}^{-1}W_{12}$. *If* $W \sim B_p^{III}(a,b)$, *then* $W_{22\cdot 1} \sim B_{p-q}^{III}(a, b - q/2)$.

Proof. From the partition of W, we have

$$\det(W) = \det(W_{11})\det(W_{22\cdot 1}), \tag{3.4}$$

$$\det(I_p - W) = \det(I_q - W_{11})\det\left(I_{p-q} - W_{22\cdot 1} - W_{21}W_{11}^{-1}(I_q - W_{11})^{-1}W_{12}\right), \tag{3.5}$$

$$\det(I_p + W) = \det(I_q + W_{11})\det\left(I_{p-q} + W_{22\cdot 1} + W_{21}W_{11}^{-1}(I_q + W_{11})^{-1}W_{12}\right). \tag{3.6}$$

Now making the transformation $W_{11} = W_{11}$, $X = W_{21}W_{11}^{-1/2}$, and $W_{22\cdot 1} = W_{22} - W_{21}\,W_{11}^{-1}\,W_{12} = W_{22} - XX'$ with Jacobian $J(W_{11}, W_{22}, W_{21} \to W_{11}, W_{22\cdot 1}, X) = \det(W_{11})^{(p-q)/2}$ and substituting (3.4), (3.5), and (3.6) in the density of W, we get the joint density of W_{11}, $W_{22\cdot 1}$, and X as follows:

$$\begin{aligned}
&2^{pb}(\beta_p(a,b))^{-1}\det(W_{11})^{b-(q+1)/2}\det(I_q - W_{11})^{a-(p+1)/2}\det(I_q + W_{11})^{-(a+b)} \\
&\quad\times\det(W_{22\cdot 1})^{b-(p+1)/2}\det(I_{p-q} - W_{22\cdot 1})^{a-(p+1)/2}\det(I_{p-q} + W_{22\cdot 1})^{-(a+b)} \\
&\quad\times\det\left(I_{p-q} - (I_{p-q} - W_{22\cdot 1})^{-1}X(I_q - W_{11})^{-1}X'\right)^{a-(p+1)/2} \\
&\quad\times\det\left(I_{p-q} + (I_{p-q} + W_{22\cdot 1})^{-1}X(I_q + W_{11})^{-1}X'\right)^{-(a+b)}.
\end{aligned} \tag{3.7}$$

Substituting $Y = (I_{p-q} - W_{22\cdot 1})^{-1/2}X(I_q - W_{11})^{-1/2}$ with Jacobian $J(X \to Y) = \det(I_{p-q} - W_{22\cdot 1})^{q/2}\det(I_q - W_{11})^{(p-q)/2}$ and integrating Y, we get the joint density of W_{11} and $W_{22\cdot 1}$ as

$$\begin{aligned}
&2^{pb}(\beta_p(a,b))^{-1}\det(W_{11})^{b-(q+1)/2}\det(I_q - W_{11})^{a-(q+1)/2}\det(I_q + W_{11})^{-(a+b)} \\
&\quad\times\det(W_{22\cdot 1})^{b-(p+1)/2}\det(I_{p-q} - W_{22\cdot 1})^{a-(p-q+1)/2}\det(I_{p-q} + W_{22\cdot 1})^{-(a+b)}g(A,B),
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
g(A,B) &= \int_{I_{p-q} - YY' > 0} \det(I_{p-q} - YY')^{a-(p+1)/2}\det(I_{p-q} + AYBY')^{-(a+b)}\,dY \\
&= \int_{0 < Z < I_{p-q}} \int_{YY' = Z} \det(I_{p-q} - YY')^{a-(p+1)/2}\,{}_1F_0^{(q)}(a+b; -Y'AYB)\,dY\,dZ, \quad \text{for } p - q \le q, \\
&= \int_{0 < Z < I_q} \int_{Y'Y = Z} \det(I_q - Y'Y)^{a-(p+1)/2}\,{}_1F_0^{(p-q)}(a+b; -AYBY')\,dY\,dZ, \quad \text{for } p - q > q,
\end{aligned} \tag{3.9}$$

with $A = (I_{p-q} - W_{22\cdot 1})^{1/2}(I_{p-q} + W_{22\cdot 1})^{-1}(I_{p-q} - W_{22\cdot 1})^{1/2}$ and $B = (I_q - W_{11})^{1/2}\,(I_q + W_{11})^{-1}(I_q - W_{11})^{1/2}$. Since $g(A,B) = g(A, H'BH)$, $H \in O(q)$, by integrating H in $g(A, H'BH)$, we obtain

$$
\begin{aligned}
g(A,B) &= \int_{0<Z<I_{p-q}} \int_{YY'=Z} \det\left(I_{p-q}-YY'\right)^{a-(p+1)/2} {}_1F_0^{(q)}\left(a+b;-AYY',B\right) dY\, dZ \\
&= \frac{\pi^{q(p-q)/2}}{\Gamma_{p-q}(q/2)} \int_{0<Z<I_{p-q}} \det(Z)^{(q-p+q-1)/2} \det\left(I_{p-q}-Z\right)^{a-(p+1)/2} {}_1F_0^{(q)}(a+b;-AZ,B)\, dZ \\
&= \frac{\pi^{q(p-q)/2}}{\Gamma_{p-q}(q/2)} \cdot \frac{\Gamma_{p-q}(q/2)\Gamma_{p-q}(a-q/2)}{\Gamma_{p-q}(a)} {}_2F_1^{(q)}\left(\frac{q}{2},a+b;a;-A,B\right) \\
&= \frac{\pi^{q(p-q)/2}\Gamma_{p-q}(a-q/2)}{\Gamma_{p-q}(a)} {}_2F_1^{(q)}\left(\frac{q}{2},a+b;a;-A,B\right).
\end{aligned} \tag{3.10}
$$

Substituting $g(A,B)$ in (3.8), we get the joint density of W_{11} and $W_{22\cdot 1}$ as

$$
\begin{aligned}
&2^{qb}\left(\beta_q(a,b)\right)^{-1} \det(W_{11})^{b-(q+1)/2} \det\left(I_q-W_{11}\right)^{a-(q+1)/2} \\
&\quad\times \det\left(I_q+W_{11}\right)^{-(a+b)} 2^{(p-q)(b-q/2)} \left(\beta_{p-q}\left(a,b-\frac{q}{2}\right)\right)^{-1} \det\left(W_{22\cdot 1}\right)^{b-(p+1)/2} \\
&\quad\times \det\left(I_{p-q}-W_{22\cdot 1}\right)^{a-(p-q+1)/2} \det\left(I_{p-q}+W_{22\cdot 1}\right)^{-(a+b)} 2^{(p-q)q/2} \\
&\quad\times {}_2F_1^{(q)}\left(\frac{q}{2},a+b;a;-A,B\right).
\end{aligned} \tag{3.11}
$$

Clearly W_{11} and $W_{22\cdot 1}$ are not independent. Integrating W_{11}, using

$$
\begin{aligned}
&2^{qb} \int_{0<W_{11}<I_q} \det(W_{11})^{b-(q+1)/2} \det\left(I_q-W_{11}\right)^{a-(q+1)/2} \det\left(I_q+W_{11}\right)^{-(a+b)} \\
&\qquad\qquad \times {}_2F_1^{(q)}\left(\frac{q}{2},a+b;a;-A,B\right) dW_{11} \\
&\quad = \int_{0<B<I_q} \det(B)^{a-(q+1)/2} \det\left(I_q-B\right)^{b-(q+1)/2} {}_2F_1\left(\frac{q}{2},a+b;a;-A,B\right) dB \\
&\quad = \beta_q(a,b)\, {}_3F_2^{(q)}\left(a,\frac{q}{2},a+b;a,a+b;-A\right) \\
&\quad = \beta_q(a,b) \det\left(I_{p-q}+A\right)^{-q/2} \\
&\quad = 2^{-(p-q)q/2} \beta_q(a,b) \det\left(I_{p-q}+W_{22\cdot 1}\right)^{q/2},
\end{aligned} \tag{3.12}
$$

we get the density of $W_{22\cdot 1}$. For $q < p-q$, using (3.9) and following similar steps we get the same result. □

Alternately, Theorem 3.6 can be proved using the relationship between type II and type III beta matrices. Let $V \sim B_p^{II}(a,b)$ and $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$, V_{11} $(q\times q)$ and $V_{11\cdot 2} = V_{11} - V_{12}V_{22}^{-1}V_{21}$. It is well known that $V_{11\cdot 2}$ and V_{22} are distributed independently (see [4]), $V_{11\cdot 2} \sim B_q^{II}(a-(p-q)/2,b)$ and $V_{22} \sim B_{p-q}^{II}(a,b-q/2)$. According to Theorem 3.5(i), if $V \sim B_p^{II}(a,b)$, then $W = (I_p+2V)^{-1} \sim B_p^{III}(a,b)$. Furthermore,

$$
W^{-1} = \begin{pmatrix} W^{11} & W^{12} \\ W^{21} & W^{22} \end{pmatrix} = \begin{pmatrix} I_q+2V_{11} & 2V_{12} \\ 2V_{21} & I_{p-q}+2V_{22} \end{pmatrix}. \tag{3.13}
$$

That is, W^{22} $((p-q)\times(p-q)) = W_{22\cdot 1}^{-1} = I_{p-q}+2V_{22}$ and $W_{22\cdot 1} = (I_q+2V_{22})^{-1}$.

Now, since $V_{22} \sim B^{II}_{p-q}(a,b-q/2)$, then $W_{22\cdot 1} = (I_{p-q}+2V_{22})^{-1} \sim B^{III}_{p-q}(a,b-q/2)$.

The distribution of $(AW^{-1}A')^{-1}$ where $A\,(q\times p)$ is a constant matrix of rank $q\,(\le p)$, is now derived.

Theorem 3.7. *Let $A\,(q\times p)$ be a constant matrix of rank $q\,(\le p)$. If $W \sim B^{III}_p(a,b)$, then $(AW^{-1}A')^{-1} \sim B^{III}_q(a,b-(p-q)/2;(AA')^{-1})$.*

Proof. We write $A = M(I_q \quad 0)G$, where $M\,(q\times q)$ is nonsingular and $G\,(p\times p)$ is orthogonal. Now,

$$\begin{aligned}(AW^{-1}A')^{-1} &= \left(M(I_q \quad 0)GW^{-1}G'(I_q \quad 0)'M'\right)^{-1}\\ &= (M')^{-1}\left[(I_q \quad 0)Y^{-1}\begin{pmatrix} I_q\\ 0\end{pmatrix}\right]^{-1}M^{-1}\\ &= (M')^{-1}(Y^{11})^{-1}M^{-1},\end{aligned} \tag{3.14}$$

where $Y = \begin{pmatrix} Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{pmatrix} = GWG' \sim B^{III}_p(a,b)$, $Y_{11}\,(q\times q)$ and $Y^{11} = (Y_{11}-Y_{12}Y_{22}^{-1}Y_{21})^{-1} = Y_{11\cdot 2}^{-1}$. From Theorem 3.6, $Y_{11\cdot 2} \sim B^{III}_q(a,b-(p-q)/2)$ and from Theorem 3.1, $(M')^{-1}Y_{22\cdot 1}M^{-1} \sim B^{III}_q(a,b-(p-q)/2;(MM')^{-1})$. The proof is now completed by observing that $MM' = AA'$. □

Corollary 3.8. *Let $W \sim B^{III}_p(a,b)$ and $\boldsymbol{a} \in \Re^p$, $\boldsymbol{a} \ne \boldsymbol{0}$, then $\boldsymbol{a}'\boldsymbol{a}/\boldsymbol{a}'W^{-1}\boldsymbol{a} \sim B^{III}(a,b-(p-1)/2)$.*

In Corollary 3.8, the distribution of $\boldsymbol{a}'\boldsymbol{a}/\boldsymbol{a}'W^{-1}\boldsymbol{a}$ does not depend on $\boldsymbol{a}$. Thus, if $\boldsymbol{y}\,(p\times 1)$ is a random vector, independent of W, and $P(\boldsymbol{y}\ne \boldsymbol{0}) = 1$, then it follows that $\boldsymbol{y}'\boldsymbol{y}/\boldsymbol{y}'W^{-1}\boldsymbol{y} \sim B^{III}(a,b-(p-1)/2)$.

Theorem 3.9. *Let $W \sim B^{III}_p(a,b)$, then*

(i) $E\left[\dfrac{\det(W)^h}{\det(I_p+W)^h}\right] = \dfrac{\Gamma_p(a+b)\Gamma_p(b+h)}{2^{ph}\Gamma_p(b)(a+b+h)}, \quad \operatorname{Re}(h) > -b+\dfrac{p-1}{2}$,

(ii) $E[\det(W)^h]$
$$= \frac{2^{-pa}\Gamma_p(a+b)\Gamma_p(b+h)}{\Gamma_p(b)\Gamma_p(a+b+h)}\,{}_2F_1\left(a,a+b;a+b+h,\tfrac{1}{2}I_p\right), \quad \operatorname{Re}(h) > -b+\frac{p-1}{2},$$

(iii) $E[\det(I_p-W)^h]$
$$= \frac{2^{-pa}\Gamma_p(a+b)\Gamma_p(a+h)}{\Gamma_p(a)\Gamma_p(a+b+h)}\,{}_2F_1\left(a+h,a+b;a+b+h,\tfrac{1}{2}I_p\right), \quad \operatorname{Re}(h) > -a+\frac{p-1}{2},$$

where ${}_2F_1$ is the hypergeometric function of matrix argument.

Proof. (i) From the density of W, we have

$$\begin{aligned}E\left[\frac{\det(W)^h}{\det(I_p+W)^h}\right] &= 2^{pb}\left(\beta_p(a,b)\right)^{-1}\int_{0<W<I_p}\det(W)^{b+h-(p+1)/2}\det(I_p-W)^{a-(p+1)/2}\\ &\qquad\times\det(I_p+W)^{-(a+b+h)}\,dW\\ &= 2^{pb}\left(\beta_p(a,b)\right)^{-1}\frac{\beta_p(a,b+h)}{2^{p(b+h)}}, \quad \operatorname{Re}(h) > -b+\frac{p-1}{2}.\end{aligned} \tag{3.15}$$

Simplifying this last expression using (1.4), we get the desired result.

(ii) From the density of W, we have

$$E\Big[\det(W)^h\Big] = 2^{pb}\left(\beta_p(a,b)\right)^{-1} \times \int_{0<W<I_p} \det(W)^{b+h-(p+1)/2}\det\left(I_p-W\right)^{a-(p+1)/2}\det\left(I_p+W\right)^{-(a+b)}dW. \tag{3.16}$$

Writing $\det(I_p+W)^{-(a+b)}$ in series involving zonal polynomials using (2.12), we obtain

$$\begin{aligned}E\Big[\det(W)^h\Big] &= 2^{-pa}\left(\beta_p(a,b)\right)^{-1}\sum_{t=0}^{\infty}\sum_{\tau}\frac{(a+b)_\tau}{2^t t!} \\ &\quad\times\int_{0<W<I_p}\det(W)^{b+h-(p+1)/2}\det\left(I_p-W\right)^{a-(p+1)/2}C_\tau(I_p-W)\,dW \\ &= 2^{-pa}\frac{\beta_p(a,b+h)}{\beta_p(a,b)}\sum_{t=0}^{\infty}\sum_{\tau}\frac{(a+b)_\tau}{s^t t!}\cdot\frac{a_\tau}{(a+b+h)_\tau}C_\tau(I_p),\quad \operatorname{Re}(h)>-b+\frac{p-1}{2},\end{aligned} \tag{3.17}$$

where the integral has been evaluated using (2.8). Finally, simplifying the expression using results on hypergeometric functions [2, 5], we get the desired result.

Similarly $E[\det(I_p-W)^h]$ can be derived. □

From the density of W, we have

$$E\left[C_\kappa(W)\right] = 2^{pb}\left(\beta_p(a,b)\right)^{-1}\int_{0<W<I_p}C_\kappa(W)\det(W)^{b-(p+1)/2} \times\det\left(I_p-W\right)^{a-(p+1)/2}\det\left(I_p+W\right)^{-(a+b)}dW. \tag{3.18}$$

Writing $\det(I_p+W)^{-(a+b)}$ in series involving zonal polynomials using (2.6), we obtain

$$\begin{aligned}E\left[C_\kappa(W)\right] &= 2^{pb}\left(\beta_p(a,b)\right)^{-1}\sum_{t=0}^{\infty}\sum_{\tau}\frac{(a+b)_\tau}{t!}(-1)^t \\ &\quad\times\int_{0<W<I_p}C_\kappa(W)C_\tau(W)\det(W)^{b-(p+1)/2}\det\left(I_p-W\right)^{a-(p+1)/2}dW \\ &= 2^{pb}\left(\beta_p(a,b)\right)^{-1}\sum_{t=0}^{\infty}\sum_{\tau}\frac{(a+b)_\tau}{t!}(-1)^t\sum_{\delta}g_{\kappa,\tau}^{\delta} \\ &\quad\times\int_{0<W<I_p}C_\delta(W)\det(W)^{b-(p+1)/2}\det\left(I_p-W\right)^{a-(p+1)/2}dW \\ &= 2^{pb}\sum_{t=0}^{\infty}\sum_{\tau}\frac{(a+b)_\tau}{t!}(-1)^t\sum_{\delta}g_{\kappa,\tau}^{\delta}\frac{(b)_\delta}{(a+b)_\delta}C_\delta(I_p),\end{aligned} \tag{3.19}$$

where the last two steps have been obtained using (2.7) and (2.8).

Theorem 3.10. (i) *Let* $U \sim B_p^I(a,b)$ *and* $W \sim B_p^{III}(c,a+b)$ *be independent. Then* $Y = W^{1/2}UW^{1/2}$, $0 < Y < I_p$ *has the density*

$$2^{p(a+b)}(\beta_p(a,b+c))^{-1}\det(Y)^{a-(p+1)/2}\det(I_p-Y)^{b+c-(p+1)/2} \times \det(I_p+Y)^{-(a+b+c)}{}_2F_1(b,a+b+c;b+c;-(I_p+Y)^{-1}(I_p-Y)), \quad 0<Y<I_p. \tag{3.20}$$

Further, pdf of $X = (I_p+Y)^{-1}(I_p-Y) = (I_p+W^{1/2}UW^{1/2})^{-1}(I_p-W^{1/2}UW^{1/2})$ *is*

$$2^{pb}(\beta_p(a,b+c))^{-1}\det(X)^{b+c-(p+1)/2}\det(I_p-X)^{a-(p+1)/2} \times {}_2F_1(b,a+b+c;b+c;-X), \quad 0<X<I_p. \tag{3.21}$$

(ii) *Let* $U \sim B_p^I(a,b)$ *and* $V \sim B_p^I(a+b,c)$ *be independent. Then* $Y = (I_p+V^{1/2}UV^{1/2})^{-1} \times (I_p-V^{1/2}UV^{1/2}) \sim B_p^{III}(a,b+c)$.

Proof. The joint density of U and V is

$$2^{p(a+b)}(\beta_p(a,b)\beta_p(c,a+b))^{-1}\det(U)^{a-(p+1)/2}\det(I_p-U)^{b-(p+1)/2} \times\det(W)^{a+b-(p+1)/2}\det(I_p-W)^{c-(p+1)/2} \times\det(I_p+W)^{-(a+b+c)}, \quad 0<U<I_p,\ 0<W<I_p. \tag{3.22}$$

Making the transformation $Y = W^{1/2}UW^{1/2}$ with Jacobian $J(U\to Y) = \det(W)^{-(p+1)/2}$ in above, we get the joint density of Y and W as

$$2^{p(a+b)}(\beta_p(a,b)\beta_p(c,a+b))^{-1}\det(Y)^{a-(p+1)/2}\det(W-Y)^{b-(p+1)/2} \times\det(I_p-W)^{c-(p+1)/2}\det(I_p+W)^{-(a+b+c)}, \quad 0<Y<W<I_p. \tag{3.23}$$

Substituting $Z = (I_p-Y)^{-1/2}(W-Y)(I_p-Y)^{-1/2}$ with the Jacobian $J(W\to Z) = \det(I_p-Y)^{(p+1)/2}$ and integrating Z, we get the density of Y as

$$2^{p(a+b)}(\beta_p(a,b)\beta_p(c,a+b))^{-1}\det(Y)^{a-(p+1)/2} \times\det(I_p-Y)^{b+c-(p+1)/2}\det(I_p+Y)^{-(a+b+c)} \times\int_{0<Z<I_p}\det(Z)^{b-(p+1)/2}\det(I_p-Z)^{c-(p+1)/2} \times\det(I_p+(I_p+Y)^{-1}(I_p-Y)Z)^{-(a+b+c)}\,dZ. \tag{3.24}$$

Integration of Z using [5, equation 48] completes the proof.

(ii) From [6], we have $V^{1/2}UV^{1/2} \sim B_p^I(a,b+c)$ and from Theorem 2.1, we have $(I_p+V^{1/2}UV^{1/2})^{-1}(I_p-V^{1/2}UV^{1/2}) \sim B_p^{III}(a,b+c)$. □

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Arjun K. Gupta: Department of Mathematics and statistics, Bowling Green State University, Bowling Green, Ohio 43403-0221, USA

Daya K. Nagar: Departamento de Mathemáticas, Universidad de Antioquia, Medellín, A.A. 1226, Colombia