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THE INITIAL BOUNDARY VALUE PROBLEM OF A MIXED-TYPED HEMIVARIATIONAL INEQUALITY

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ABSTRACT. A mixed-typed differential inclusion with a weakly continuous nonlinear term and a nonmonotone discontinuous nonlinear multi-valued term is studied, and the existence and decay of solutions are established.

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1. Introduction. In the present paper, the following initial boundary value problem of a degenerate multi-valued hyperbolic-parabolic inequality will be considered:

$$\begin{gathered}\ddot{u}(t)+A(t)(\dot{u})(t)+B(u)(t)+\varphi(u(x,t))\ni f(t), \quad \text{a.e. } t\in[0,T],\\ u(x,t)=0, \quad \text{a.e. } (x,t)\in\Sigma=\partial\Omega\times[0,T],\\ u(0)=u_0, \qquad \dot{u}(0)=u_1,\end{gathered}\tag{1.1}$$

where A is a weakly continuous operator; B is a linear, continuous, and symmetric operator; φ is a nonmonotonous, discontinuous, and nonlinear set-valued mapping.

Physical motivations for studying inequality (1.1) come partly from problems of continuum mechanics and optimal control problems, where nonmonotone, nonlinear, discontinuous, and multi-valued constitutive laws and boundary or external constraints lead to various typed hemivariational inequalities, the mixed hyperbolic-parabolic hemivariational inequality is one of those [11, 12, 14].

For inequality (1.1), its stationary problems have been studied by many researchers (see [1, 2, 4, 13, 14, 15] and references therein). When φ degenerates into a class of single-valued mappings, inequality (1.1) becomes an equation, and when A and B were some special linear mappings and satisfy some conditions, equation (2.1) and some of its evolution equations have been investigated and applied intensively (see [5, 3, 6, 7, 8, 9, 10] and the references therein).

In this paper, we investigate the existence and decay of weak solution of the mixed hyperbolic-parabolic inequality (1.1) with φ, A, and B satisfying some conditions. We apply the Faedo-Galerkin method for the proof of existence of solutions.

2. Preliminaries. Let Ω be a bounded open set of $\mathbb{R}^n$ with regular boundary Γ. Let T denote a positive real number, $Q=\Omega\times[0,T]$. Suppose that $b\in L^\infty_{\text{loc}}(\mathbb{R})$, for every $\rho>0$, set

$$\underline{b}_\rho(\xi)=\operatorname*{ess\,inf}_{|\xi_1-\xi|<\rho} b(\xi_1), \qquad \bar{b}_\rho(\xi)=\operatorname*{ess\,sup}_{|\xi_1-\xi|<\rho} b(\xi_1),\tag{2.1}$$

they are all monotone for $\rho > 0$. Set

$$\underline{b}_\rho(\xi) = \lim_{\rho\to 0^+} \underline{b}_\rho(\xi), \qquad \bar{b}(\xi) = \lim_{\rho\to 0^+} \bar{b}_\rho(\xi), \qquad \varphi(\xi) = [\underline{b}(\xi), \bar{b}(\xi)]. \tag{2.2}$$

Let $J(\xi) = \int_0^\xi b(t)\,dt$, then $\partial^C J(\xi) \subseteq \varphi(\xi)$, where $\partial^C J(\xi)$ denotes the Clarke-subdifferential of J (see [2]). If $b(\xi_\pm)$ exists for every $\xi \in \mathbb{R}$, then $\varphi(\xi) = \partial^C J(\xi)$. If b is continuous at the point ξ, then $\varphi(\xi)$ is single-valued at ξ, if J is convex, $\varphi(\xi)$ is maximal monotone (see [2]).

Let $V = H_0^1(\Omega)$, $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$, $\langle\cdot,\cdot\rangle$ denotes the dual pair between V and $V' = H^{-1}(\Omega)$ which is compatible with the inner product of $L^2(\Omega)$. Let $|x|_X$ denote the norm of the element x of the Banach space X.

Considering the following initial boundary value problem of a hyperbolic-parabolic hemivariational inequality:

$$\begin{gathered} \ddot{u}(t) + A(t)\dot{u}(t) + Bu(t) + g(t) = f(t), \quad \text{a.e. } t \in [0,T], \\ u(x,t) = 0, \quad \text{a.e. } (x,t) \in \Sigma = \partial\Omega \times [0,T], \\ u(0) = u_0, \qquad \dot{u}(0) = u_1, \\ g(x,t) \in \varphi\big(u(x,t)\big), \quad \text{a.e. } (x,t) \in Q_T = \Omega \times [0,T], \end{gathered} \tag{2.3}$$

where f, u_0, and u_1 are given.

First we list some assumptions:

(1) $\exists c > 0$, $|b(\xi)| \le c(1+|\xi|)$, a.e. $\xi \in \mathbb{R}$.

(2) $A : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ is weakly continuous, and $A(t)$ is nonnegative, that is, $\langle A(t)v, v\rangle \ge 0$, for a.e. $t \ge 0$ and every $v \in L^2(\Omega)$.

(3) The function $t \to \langle A(t)u, v\rangle$ is measurable on $[0,T]$ for all $u, v \in L^2(\Omega)$.

(4) $B : H_0^1(\Omega) \to H^{-1}(\Omega)$ is linear, continuous, symmetric, and semicoercive, that is, $\exists c_1 > 0$, $c_2 > 0$, $c_3 > 0$

$$\begin{gathered} |Bv|_{H^{-1}(\Omega)} \le c_1 |v|_{H_0^1(\Omega)}, \quad \langle Bu, v\rangle = \langle Bv, u\rangle, \quad \forall u, v \in H_0^1(\Omega), \\ \langle Bv, v\rangle + c_3 |v|^2_{L^2(\Omega)} \ge c_2 |v|^2_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \end{gathered} \tag{2.4}$$

Let β be any mollifier satisfying $\beta \in C^\infty(\mathbb{R})$, $\beta \ge 0$, $\operatorname{supp}\beta \subset (-1,1)$, and $\int_R \beta(\xi)\, d\xi = 1$. Set

$$b_\varepsilon(\xi) = \frac{1}{\varepsilon}\int_{\mathbb{R}} \beta\left(\frac{\xi - z}{\varepsilon}\right) b(z)\,dz = \int_{|z|\le 1} \beta(z) b(\xi - \varepsilon z)\,dz, \quad \text{for every } \varepsilon > 0. \tag{2.5}$$

It is easy to see that b_ε is a smooth function, and also satisfies assumption (1) with possible different constant c if b is agreeable with assumption (1). For convenience, we denote $b_{1/n}$ by b_n for any positive integer n.

3. Existence of solution

Theorem 3.1. *Assume that* $f \in L^2(0,T;L^2(\Omega))$, $u_0 \in H_0^1(\Omega) \cap L^{p+1}(\Omega)$, $u_1 \in L^2(\Omega)$. *Then, under assumptions (1), (2), (3), and (4), there exists a function* u *defined in* $\Omega \times [0,T]$ *such that*

$$\begin{gathered} u \in L^\infty(0,T;H_0^1(\Omega)) \bigcap C([0,T];L^2(\Omega)), \\ \dot{u} \in L^\infty(0,T;L^2(\Omega)) \bigcap C([0,T];H^{-1}(\Omega)), \\ \ddot{u} \in L^2(0,T;H^{-1}(\Omega)), \end{gathered} \tag{3.1}$$

and

$$\begin{gathered} \ddot{u}(t) + A(t)\dot{u}(t) + Bu(t) + g(t) = f(t), \quad \text{in } L^2(0,T;H^{-1}(\Omega)), \\ g(t) \in \varphi(u(x,t)), \quad \text{a.e. } (x,t) \in \Omega \times [0,T], \\ u(0) = u_0, \qquad \dot{u}(0) = u_1. \end{gathered} \tag{3.2}$$

Proof. Let $\{e_n\}_{n=1}^\infty$ be a subset of $V = H_0^1(\Omega)$ satisfying $\overline{\text{span}\{e_n\}} = V$, $(e_i, e_j) = \delta_{ij}$. Let $x_n = \sum_1^n \omega_i^1 e_i \to u_0$ strongly in V and $L^{p+1}(\Omega)$, $y_n = \sum_1^n \omega_i^2 e_i \to u_1$ strongly in $L^2(\Omega)$.

Considering the following regularized equation of inequality (1.1)

$$\ddot{\xi}^n = M^n + N^n + h, \qquad \xi^n\big|_{t=0} = \omega^{1n}, \qquad \dot{\xi}^n\big|_{t=0} = \omega^{2n}, \tag{3.3}$$

where $\xi^n = \{\xi_i^n\}_{1\times n}$, $\omega^{1n} = \{\omega_i^1\}_{1\times n}$, $\omega^{2n} = \{\omega_i^2\}_{1\times n}$, $h = \{\langle f, e_i\rangle\}_{1\times n}$, $M^n = \{M_i^n\}_{1\times n}$, $M_i^n = -(A(t)(\sum_1^n \dot{\xi}_j^n e_j), e_i)$, $N^n = \{N_i^n\}_{1\times n}$, $N_i^n = -\langle B(\sum_1^n \xi_j^n e_j), e_i\rangle - \langle b_n(\sum_1^n \xi_j^n e_j), e_i\rangle$, where "$\cdot$" denotes time derivate.

Equation (3.3) is a vector-valued ordinary differential equation and its local solution ξ^n exists on $I_n = [0, T_n]$, $0 < T_n \le T$. Set $u_n(t) = \sum_1^n \xi_j^n e_j$ ($t \in I_n$). Equation (3.3) is equal to

$$\langle \ddot{u}_n, e_i\rangle = -(A(t)\dot{u}_n, e_i) - \langle Bu_n, e_i\rangle - \langle b_n(u_n), e_i\rangle + \langle f, e_i\rangle, \quad i = 1, 2, \dots, n. \tag{3.4}$$

Multiplying (3.4) by $\dot{\xi}_i^n$, summing over from $i = 1$ to $i = n$ and integrating over $[0,t]$ ($t \le I_n$), we get

$$\begin{aligned} |\dot{u}_n(t)|_{L^2(\Omega)}^2 + \langle Bu_n(t), u_n(t)\rangle + 2\int_0^t (A\dot{u}_n, \dot{u}_n)\,d\tau + 2\int_0^t \langle b_n(u_n), \dot{u}_n\rangle\,d\tau \\ = 2\int_0^t \langle f, \dot{u}_n\rangle\,d\tau + (y_n, y_n) + \langle Bx_n, x_n\rangle, \end{aligned} \tag{3.5}$$

but

$$\int_0^t \langle b_n(u_n),\dot{u}_n\rangle\, d\tau = \int_\Omega J(u_n(x,\tau))\big|_0^t\, dx$$
$$= \int_\Omega \left\{ \int_0^{u_n(x,t)} b_n(\lambda)\, d\lambda - \int_0^{u_n(x,0)} b_n(\lambda)\, d\lambda \right\} dx,$$
$$\left| \int_0^t \langle b_n(u_n),\dot{u}_n\rangle\, d\tau \right| \le c\int_\Omega \left\{ |u_n(x,t)| + |u_n(x,0)| + \left| \int_0^{u_n(x,t)} |\lambda|\, d\lambda \right| + \left| \int_0^{u_n(x,0)} |\lambda|\, d\lambda \right| \right\} dx, \tag{3.6}$$
$$\left| \int_0^{u_n(x,t)} |\lambda|\, d\lambda \right| = \frac{1}{2}|u_n(x,t)|^2,$$
$$\left| \int_0^{u_n(x,0)} |\lambda| d\lambda \right| = \frac{1}{2}|u_n(x,0)|^2,$$
$$\left| \int_0^t \langle b_n(u_n),\dot{u}_n\rangle\, d\tau \right| \le \frac{c}{2}(1+|\Omega|)\left\{ |u_n(t)|^2_{L^2(\Omega)} + |x_n|^2_{L^2(\Omega)} \right\},$$

where $|\Omega|$ denotes the Lebesgue measure of the domain Ω.

From (3.5), it follows that there exists $c_4 > 0$ such that

$$|\dot{u}_n(t)|^2_{L^2(\Omega)} + |u_n(t)|^2_{H_0^1(\Omega)} \le c_4 + \left\{ c_3 + \frac{c}{2}(1+|\Omega|) \right\} |u_n(t)|^2_{L^2(\Omega)} + 2\int_0^t \langle f,\dot{u}_n\rangle d\tau. \tag{3.7}$$

We note that

$$u_n(t) = u_n(0) + \int_0^t \dot{u}_n\, d\tau,$$
$$|u_n(t)|^2_{L^2(\Omega)} \le |u_n(0)|_{L^2(\Omega)} + \int_0^t |\dot{u}_n|_{L^2(\Omega)}\, d\tau, \tag{3.8}$$

using Hölder's inequality, we get that there exists $c_5, c_6 > 0$ such that

$$|u_n(t)|^2_{L^2(\Omega)} \le c_5 + c_6 \int_0^t |\dot{u}_n|^2_{L^2(\Omega)}\, d\tau, \tag{3.9}$$

$$\int_0^t \langle f,\dot{u}_n\rangle\, d\tau \le |f|_{L^2(0,T;L^2(\Omega))} \cdot |\dot{u}_n|_{L^2(0,t;L^2(\Omega))}$$
$$\le \frac{1}{2}\left(|f|^2_{L^2(0,T;L^2(\Omega))} + |\dot{u}_n|^2_{L^2(0,t;L^2(\Omega))} \right). \tag{3.10}$$

From (3.7), (3.9), and (3.10), we obtain that there exists $c_7, c_8 > 0$ such that

$$|\dot{u}_n(t)|^2_{L^2(\Omega)} + c_2 |u_n(t)|^2_{H_0^1(\Omega)} \le c_7 + c_8 \int_0^t |\dot{u}_n(\tau)|^2_{L^2(\Omega)}\, d\tau \quad (t \in I_n), \tag{3.11}$$

this implies that

$$|\dot{u}_n(t)|^2_{L^2(\Omega)} \le c_7 + c_8 \int_0^t |\dot{u}_n(\tau)|^2_{L^2(\Omega)}\, d\tau \quad (t \in I_n). \tag{3.12}$$

Using Gronwall's inequality it follows that

$$|\dot{u}_n(t)|^2_{L^2(\Omega)} \le c_7 \exp(c_8 t) \quad (t \in I_n). \tag{3.13}$$

Therefore, from (3.9), (3.11), and (3.13), we get that there exists $c_9 > 0$,

$$|\dot{u}_n(t)|_{L^2(\Omega)} \le c_9, \quad |u_n(t)|_{L^2(\Omega)} \le c_9, \quad |u_n(t)|_{H_0^1(\Omega)} \le c_9, \quad (t \in I_n), \tag{3.14}$$

where c_4, c_5, c_6, c_7, c_8, c_9 are positive constants independent of n and T_n, from which we can assert that $I_n = [0,T]$ $(\forall n)$.

For every $\eta \in \operatorname{span}\{e_1, e_2, \dots, e_n\}$, from (3.4)

$$\begin{aligned} |\langle \ddot{u}_n, \eta\rangle| \le{}& |A(t)(\dot{u}_n)|_{L^2(\Omega)} \cdot |\eta|_{L^2(\Omega)} + |f(t)|_{L^2(\Omega)} \cdot |\eta|_{L^2(\Omega)} \\ &+ |b_n(u_n)|_{L^2(\Omega)} \cdot |\eta|_{L^2(\Omega)} + |B| \cdot |u_n|_{H_0^1(\Omega)} \cdot |\eta|_{H_0^1(\Omega)}, \end{aligned} \tag{3.15}$$

where $|B|$ is the norm of linear continuous operator B

$$\begin{aligned} &|\ddot{u}_n(t)|_{H^{-1}(\Omega)} \\ &\quad = \sup_{|\eta|_V=1} |\langle \ddot{u}_n(t), \eta\rangle| = \sup_{\substack{\eta\in \operatorname{span}\{e_1,\dots,e_n\} \\ |\eta|_V=1}} |\langle \ddot{u}_n(t), \eta\rangle| \\ &\quad \le c_{10}\Big(|A(t)(\dot{u}_n)|_{L^2(\Omega)} + |f(t)|_{L^2(\Omega)} + |b_n(u_n)|_{L^2(\Omega)}\Big) + |B| \cdot |u_n(t)|_{H_0^1(\Omega)}, \end{aligned} \tag{3.16}$$

where c_{10} is the imbedding constant which $H_0^1(\Omega)$ imbeds in $L^2(\Omega)$

$$\begin{aligned} |b_n(u_n)(t)|_{L^2(\Omega)}^2 &= \int_\Omega |b_n(u_n)(t)|^2\,dx \le \int_\Omega c^2(1+|u_n(x,t)|)^2\,dx \\ &\le 2c^2 \int_\Omega \Big(1+|u_n(x,t)|^2\Big)\,dx = 2c^2\Big(|\Omega| + |u_n(t)|_{L^2(\Omega)}^2\Big), \end{aligned} \tag{3.17}$$

this shows that $\{b_n(u_n)\}$ is also a bounded subset of $L^\infty(0,T;L^2(\Omega))$. Since A is weakly continuous, it must be a bounded operator from $L^2(0,T;L^2(\Omega))$ to $L^2(0,T;L^2(\Omega))$. But $\{\dot{u}_n\}$ is a bounded subset of $L^2(0,T;L^2(\Omega))$, $\{A(t)(\dot{u}_n)\}$ must be a bounded subset of $L^2(0,T;L^2(\Omega))$. Inequality (3.16) implies that $\{\ddot{u}_n\}$ is a bounded subset of $L^2(0,T;H^{-1}(\Omega))$.

Therefore, there exist a subsequence of $\{u_n\}$, still denoted by itself, and a function u such that $u \in L^\infty(0,T;H_0^1(\Omega))$, $\dot{u} \in L^\infty(0,T;L^2(\Omega))$, $\ddot{u} \in L^2(0,T;H^{-1}(\Omega))$ satisfying

$$\begin{aligned} &u_n \longrightarrow u \quad \text{weakly-star in } L^\infty(0,T;H_0^1(\Omega)), \\ &\dot{u}_n \longrightarrow \dot{u} \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)), \\ &\ddot{u}_n \longrightarrow \ddot{u} \quad \text{weakly in } L^2(0,T;L^{-1}(\Omega)), \\ &b_n(u_n) \longrightarrow g \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)). \end{aligned} \tag{3.18}$$

Since, the space $W(V)$ defined by $W(V) = \{u \in L^2(0,T;V),\ \dot{u} \in L^2(0,T;V')\}$ forms a real Hilbert space with the norm $|u|_W = |u|_{L^2(0,T;V)} + |\dot{u}|_{L^2(0,T;V')}$ and is continuously imbedded in $C([0,T];L^2(\Omega))$, it is obvious that $u \in C(0,T;L^2(\Omega))$, $\dot{u} \in C(0,T;H^{-1}(\Omega))$. Hence, $u(0)$, $\dot{u}(0)$ make sense.

For $\lambda \in L^2(0,T)$, from (3.4) we have

$$\begin{aligned} \int_0^T \langle \ddot{u}_n, \lambda e_i\rangle\,dt ={}& -\int_0^T (A(t)(\dot{u}_n), \lambda e_i)\,dt - \int_0^T \langle B(u_n), \lambda e_i\rangle\,dt \\ &- \int_0^T \langle b_n(u_n), \lambda e_i\rangle\,dt + \int_0^T \langle f(t), \lambda e_i\rangle\,dt, \quad i = 1,2,\dots,n. \end{aligned} \tag{3.19}$$

For every given positive integer i, let $n \to \infty$ in (3.19), it follows that

$$\begin{aligned}\int_0^T \langle \ddot{u}, \lambda e_i\rangle\,dt = &-\int_0^T (A(t)(\dot{u}), \lambda e_i)\,dt - \int_0^T \langle B(u), \lambda e_i\rangle\,dt \\ &-\int_0^T \langle g, \lambda e_i\rangle\,dt + \int_0^T \langle f(t), \lambda e_i\rangle\,dt, \quad i = 1, 2, \ldots, n.\end{aligned} \tag{3.20}$$

Therefore, we have from (3.20)

$$\ddot{u}(t) + A(t)(\dot{u}) + B(u) + g(t) = f(t), \quad \text{in } L^2(0,T;H^{-1}(\Omega)). \tag{3.21}$$

Next, we demonstrate that

$$g(x,t) \in \varphi(u(x,t)) \quad \text{a.e. } (x,t) \in Q_T = \Omega \times [0,T]. \tag{3.22}$$

Since, $u_n(x,t) \to u(x,t)$ a.e. $(x,t) \in Q_T$, by Eropoß's theorem [9], for every $\delta > 0$, there exists a subset $Q_\delta \subseteq Q_T = \Omega \times [0,T]$, $|Q_\delta| \le \delta$,

$$u_n(x,t) \longrightarrow u(x,t) \quad \text{uniformly in } Q_T \backslash Q_\delta \tag{3.23}$$

that is, for every $\varepsilon > 0$, there exists a positive integer $\overline{N}$, when $n \ge \overline{N}$,

$$|u_n(x,t) - u(x,t)| \le \varepsilon \quad \forall (x,t) \in Q_T \backslash Q_\delta. \tag{3.24}$$

It is obvious that, when $1/n \le \varepsilon$ and $n \ge \overline{N}$, for almost everywhere $(x,t) \in Q_T \backslash Q_\delta$

$$\underline{b}_n(u_n(x,t)) = b_n(u_n(x,t)) = \bar{b}_n(u_n(x,t)) \le \bar{b}_\varepsilon(u_n(x,t)) \le \bar{b}_{2\varepsilon}(u(x,t)). \tag{3.25}$$

For every $\mu \in L^1(0,T;L^2(\Omega))$, $\mu \ge 0$

$$\begin{aligned}\int_{Q_T\backslash Q_\delta} g(x,t)\mu(x,t)\,dx\,dt &= \lim_{n\to\infty} \int_{Q_T\backslash Q_\delta} b_n(u_n(x,t))\mu(x,t)\,dx\,dt \\ &\le \int_{Q_T\backslash Q_\delta} \bar{b}_{2\varepsilon}(u(x,t))\mu(x,t)\,dx\,dt, \\ \int_{Q_T\backslash Q_\delta} g(x,t)\mu(x,t)\,dx\,dt &\le \limsup_{\varepsilon\to 0^+} \int_{Q_T\backslash Q_\delta} \bar{b}_{2\varepsilon}(u(x,t))\mu(x,t)\,dx\,dt \\ &\le \int_{Q_T\backslash Q_\delta} \bar{b}(u(x,t))\mu(x,t)\,dx\,dt.\end{aligned} \tag{3.26}$$

Analogously, we can obtain

$$\int_{Q_T\backslash Q_\delta} g(x,t)\mu(x,t)\,dx\,dt \ge \int_{Q_T\backslash Q_\delta} \underline{b}(u(x,t))\mu(x,t)\,dx\,dt. \tag{3.27}$$

Hence, (3.26) and (3.27) imply that

$$g(x,t) \in \varphi(u(x,t)) \quad \text{a.e. } (x,t) \in Q_T \backslash Q_\delta. \tag{3.28}$$

Finally, let $\delta \to 0^+$, we get

$$g(x,t) \in \varphi(u(x,t)) \quad \text{a.e. } (x,t) \in Q_T = \Omega \times [0,T]. \tag{3.29}$$

Let $\lambda \in C^1[0,T]$, $\lambda(T) = 0$, integrating by parts the left-hand side of equations (3.19) and (3.20) gives

$$\begin{aligned} -\langle \dot{u}_n(0), \lambda(0)e_i\rangle - \int_0^T \langle \dot{u}_n, \dot{\lambda} e_i\rangle\, dt = &-\int_0^T (A(t)(\dot{u}_n), \lambda e_i)\, dt - \int_0^t \langle B(u_n), \lambda e_i\rangle\, dt \\ &- \int_0^T \langle b_n(u_n), \lambda e_i\rangle\, dt - \int_0^T \langle f(t), \lambda e_i\rangle\, dt, \end{aligned} \tag{3.30}$$

$$\begin{aligned} -\langle \dot{u}(0), \lambda(0)e_i\rangle - \int_0^T \langle \dot{u}, \dot{\lambda} e_i\rangle\, dt = &-\int_0^T (A(t)(\dot{u}), \lambda e_i)\, dt - \int_0^t \langle B(u), \lambda e_i\rangle\, dt \\ &- \int_0^T \langle g, \lambda e_i\rangle\, dt - \int_0^T \langle f(t), \lambda e_i\rangle\, dt, \end{aligned} \tag{3.31}$$

making comparison between (3.30) and (3.31) we get that

$$\lim_{n\to\infty} \langle \dot{u}_n(0) - \dot{u}(0), e_i\rangle = 0, \quad i = 1, 2, \ldots, n \tag{3.32}$$

therefore, this implies that

$$\dot{u}_n(0) \longrightarrow \dot{u}(0) \quad \text{weakly in } H^{-1}(\Omega) \tag{3.33}$$

uniqueness of limit implies that $\dot{u}(0) = u_1$ (in $H^{-1}(\Omega)$).

Let $\lambda \in C^2[0,T]$, $\lambda(T) = 0$, $\dot{\lambda}(T) = 0$. Analogously, integrating by parts the left-hand side of equations (3.30) and (3.31), and making comparison with the obtained results again gives: $u(0) = u_0$ (in $L^2(\Omega)$). □

Theorem 3.2. *Let $f \in L^2(0,T; L^2(\Omega))$, $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that b satisfies*

$(1')$ $b(\xi)\xi \geq 0$ *for almost everywhere $\xi \in \mathbb{R}$, and $\exists \bar{c} > 0$,*

$$|b(\xi)| \leq \bar{c}(1 + |\xi|^p), \quad \text{a.e. } \xi \in \mathbb{R}, \text{ if } n > 2,\ 0 < p \leq \frac{2n}{n-2};\ \text{if } n \leq 2,\ 0 \leq p < \infty. \tag{3.34}$$

Then, under assumptions (2), (3), and (4), there exists a function v defined in $\Omega \times [0,T]$ satisfying

$$\begin{gathered} v \in L^\infty(0,T; H_0^1(\Omega)), \qquad \dot{v} \in L^\infty(0,T; L^2(\Omega)), \\ \ddot{v} + A(t)(\dot{v}) + B(v) + \bar{g}(t) = f(t) \quad \text{in } L^1(0,T; H^{-1}(\Omega) + L^1(\Omega)), \\ \bar{g}(x,t) \in \varphi(v(x,t)) \quad \text{a.e. } (x,t) \in Q_T = \Omega \times [0,T], \\ v(0) = u_0, \qquad \dot{v}(0) = u_1. \end{gathered} \tag{3.35}$$

Proof. It is also easy to see that b_ε satisfies assumption $(1)'$ with possible different constant $\bar{c}$. Analogously to Theorem 3.1, we still may get (3.5), where $\{e_n\}_{n=1}^\infty$ is a basis of $H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying $(e_i, e_j) = \delta_{ij}$. Set

$$J_n(\xi) = \int_0^\xi b_n(t)\, dt, \tag{3.36}$$

then $J_n(\xi) \geq 0$, $\forall \xi \in \mathbb{R}$, and

$$\begin{gathered} \int_0^t \langle b_n(u_n), \dot{u}_n\rangle\, d\tau = \int_\Omega J_n(u_n(x,t))\, dx - \int_\Omega J_n(u_n(x,0))\, dx \geq -\int_\Omega J_n(u_n(x,t))\, dx \\ |b_n(\xi)| \leq \left| \int_{|z|\leq 1} \beta(z) \left| b\left(\xi - \frac{z}{n}\right)\right|^p dz \right| \leq d_1 + d_2|\xi|^p, \end{gathered} \tag{3.37}$$

where d_1 and d_2 are positive constants independent of n.

$$\begin{aligned} |J_n(x_n)| &= \left|\int_0^{x_n} b_n(t)\,dt\right| \le (\operatorname{sgn} x_n)\cdot\int_0^{x_n} |b_n(t)|\,dt \\ &\le (\operatorname{sgn} x_n)\cdot\int_0^{x_n} (d_1+d_2|t|^p)\,dt = d_1|x_n|+\frac{d_2|x_n|^{p+1}}{(p+1)}, \end{aligned} \tag{3.38}$$

$$\left|\int_\Omega J_n(x_n)\,dt\right| \le \int_\Omega |J_n(x_n)|\,dx \le d_1|x_n|_{L^1(\Omega)}+\frac{d_2|x_n|^{p+1}_{L^{p+1}(\Omega)}}{(p+1)}.$$

Since $L^{p+1}(\Omega)\subset L^1(\Omega)$ and $u_n(0)=x_n\to u_0$ strongly in $L^{p+1}(\Omega)$, and $|x_n|_{L^1(\Omega)}$ are bounded, and so is $\int_\Omega J_n(x_n(x))\,dx$. From (3.5) we have

$$|\dot u_n(t)|^2_{L^2(\Omega)}+c_2|u_n(t)|^2_{H_0^1(\Omega)} \le c_4+c_3|u_n(t)|^2_{L^2(\Omega)}+2\int_0^t \langle f,\dot u_n\rangle\,d\tau. \tag{3.39}$$

It is easy to see that (3.9), (3.10), (3.11), (3.13), and (3.14) are still true and the solution of (3.3) can be extended to interval $[0,T]$. By Sobolev imbedding theorem, we have, for a.e. $t\in[0,T]$, if $n>2$, then $H_0^1(\Omega)\subset L^{p^*}(\Omega)\subset L^p(\Omega)$, $p^*=2n/(n-2)$, and $|u_n(t)|_{L^p(\Omega)}\le c_{10}|u_n(t)|_{H_0^1(\Omega)}\le c_{10}c_9$; if $n=2$, then $H_0^1(\Omega)\subset L^q(\Omega)$, when $1\le q<\infty$, so $|u_n(t)|_{L^p(\Omega)}\le c_{10}|u_n(t)|_{H_0^1(\Omega)}\le c_{10}c_9$; if $n=1$, then $H_0^1(\Omega)\subset C(\bar\Omega)$ and ditto, $|u_n(t)|_{C(\bar\Omega)}=\max_{x\in\bar\Omega}|u_n(x,t)|\le c_{10}c_9$, where $\bar\Omega$ denotes the closure of Ω and c_{10} is the imbedding constant which $H_0^1(\Omega)$ imbeds in $L^p(\Omega)$ or $C(\bar\Omega)$. Note that, we always have that $b_n(u_n)\in L^\infty(0,T;L^{p_0}(\Omega))$, where $p_0=(n+1)/(n-2)$ and $\{b_n(u_n)\}$ is a bounded subset of $L^\infty(0,T;L^{p_0}(\Omega))$. Therefore, there exist a subsequence of $\{u_n\}$, still denoted by itself, and a function v such that $v\in L^\infty(0,T;H_0^1(\Omega))$, $\dot v\in L^\infty(0,T;L^2(\Omega))$ satisfying

$$\begin{aligned} u_n &\longrightarrow v \quad \text{weakly-star in } L^\infty(0,T;H_0^1(\Omega)),\\ \dot u_n &\longrightarrow \dot v \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)),\\ b_n(u_n) &\longrightarrow g \quad \text{weakly-star in } L^\infty(0,T;L^{p_0}(\Omega)). \end{aligned} \tag{3.40}$$

Since, the dual of the space $H_0^1(\Omega)\cap L^\infty(\Omega)$ is the space $L^1(0,T;H^{-1}(\Omega)+L^1(\Omega))$, it is easy to obtain from (3.4) that

$$\ddot v(t)+A(t)\dot v+B(v)+\bar g(t)=f(t) \quad \text{in } L^1(0,T;H^{-1}(\Omega)+L^1(\Omega)). \tag{3.41}$$

Analogous to Theorem 3.1, we can complete the proof of this theorem. □

Remark 3.3. If $A(t)=A$ and A is linear, then the uniqueness of such solution will be obtained in the same way as in [3].

4. Decay of the solution

Theorem 4.1. *Let $T=+\infty$, $f\equiv 0$. Suppose that for every $t\ge 0$, the operator $A(t)$ satisfies*

$$(A(t)w,w)\ge \delta_0|w|^2_{L^2(\Omega)}, \quad \forall w\in L^2(\Omega). \tag{4.1}$$

Moreover, if $\langle Bw,w\rangle\ge 0$, $\forall w\in H_0^1(\Omega)$ or $c_3c_{10}^2\le c_2$, here c_{10} is an imbedding constant which $H_0^1(\Omega)$ imbeds in $L^2(\Omega)$. Then, under conditions of Theorem 3.2, the solution in

Theorem 3.2 obtained from the regularized equation (3.3) satisfies

$$|\dot{u}(t)|^2_{L^2(\Omega)} \le \mu_1 \exp(-\mu_2 t), \quad \text{a.e. } t \ge 0, \tag{4.2}$$

where δ_0, μ_1, μ_2 are positive constants.

Proof. Let u_n be a solution of (3.3), that is, satisfies (3.4) and (3.5). Since $J_n(u_n(x,t)) \ge 0$, by (3.5) we have

$$|\dot{u}(t)|^2_{L^2(\Omega)} + \langle Bu_n(t), u_n(t)\rangle \le c_{11} - 2\delta_0 \int_0^t |\dot{u}_n(\tau)|^2_{L^2(\Omega)}\, d\tau, \quad t \in [0,+\infty), \tag{4.3}$$

where c_{11} is a positive constant independent of n. If $\langle Bw, w\rangle \ge 0$, for every $w \in H^1_0(\Omega)$, $\langle Bu_n(t), u_n(t)\rangle \ge 0$. Analogously to [7, Theorem 4], we obtain:

$$|\dot{u}_n(t)|^2_{L^2(\Omega)} \le c_{11} \exp(-2\delta_0 t), \quad \text{a.e. } t \ge 0. \tag{4.4}$$

If $c_3 c_{10}^2 \le c_2$, we get from (4.3) that

$$\begin{aligned} |\dot{u}_n(t)|^2_{L^2(\Omega)} + c_2 |u_n(t)|^2_{H^1_0(\Omega)} &\le c_{11} + c_3 |u_n(t)|^2_{L^2(\Omega)} - 2\delta_0 \int_0^t |\dot{u}_n(\tau)|^2_{L^2(\Omega)}\, d\tau \\ &\le c_{11} + c_3 c_{10}^2 |u_n(t)|^2_{H^1_0(\Omega)} - 2\delta_0 \int_0^t |\dot{u}_n(\tau)|^2_{L^2(\Omega)}\, d\tau, \end{aligned} \tag{4.5}$$

from which it is permitted to get inequality (4.4).

Since $|\dot{u}_n(t)|_{L^2(\Omega)} \le c_9$, $\dot{u} \to \dot{u}$ weakly-star in $L^\infty(0,\infty; L^2(\Omega))$, it is easy to obtain that $\dot{u}(t) \to \dot{u}(t)$ weak in $L^2(\Omega)$ for a.e. $t \ge 0$. But $L^2(\Omega)$ is a real Hilbert space, therefore, $|\dot{u}(t)|_{L^2(\Omega)} \le \underline{\lim}_{n\to\infty} |\dot{u}_n(t)|_{L^2(\Omega)}$, a.e. $t \ge 0$. Finally, we get $|\dot{u}(t)|^2_{L^2(\Omega)} \le c_{11} \exp(-2\delta_0 t)$, (a.e. $t \ge 0$). □

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