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a-MINIMAL SETS AND RELATED TOPICS IN TRANSFORMATION SEMIGROUPS (I)

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ABSTRACT. We deal with a-minimal sets instead of minimal right ideals of the enveloping semigroup and obtain a partition of disjoint isomorphic subgroups of some of its subsets. We also give some generalizations of almost periodicity and distality in the transformation semigroups and obtain similar results.

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1. Preliminaries. By a transformation semigroup (X,S,ρ) (or simply (X,S)) we mean a compact Hausdorff topological space X, a discrete topological semigroup S with identity e, and a continuous map $\rho : X\times S \to X$ ($\rho(x,s) = xs\ \forall x \in X, \forall s \in S$), such that

(1) $xe = x\,\forall x \in X$;

(2) $x(st) = (xs)t\,\forall x \in X,\ \forall s,t \in S$.

In the transformation semigroup (X,S), for each $s \in S$ define $\pi^s : X \to X$ by $\pi^s(x) = xs$ ($\forall x \in X$). We assume the semigroup S acts effectively on X, that is, for each $s,t \in S$, $s \neq t$ if and only if $\pi^s \neq \pi^t$. The closure of $\{\pi^s \mid s \in S\}$ in X^X (with pointwise convergence topology) is called the enveloping semigroup (or Ellis semigroup) of (X,S) and is denoted by $\mathrm{E}(X,S)$ (or simply $\mathrm{E}(X)$). The enveloping semigroup $\mathrm{E}(X)$ has a semigroup structure [1]. A nonempty subset I of $\mathrm{E}(X)$ is called a right ideal of $\mathrm{E}(X)$, if $I\mathrm{E}(X) \subseteq I$, moreover, if the right ideal I of $\mathrm{E}(X)$ does not have any proper subset which is a right ideal of $\mathrm{E}(X)$, then I is called a minimal right ideal of $\mathrm{E}(X)$, the set of all minimal right ideals of $\mathrm{E}(X)$ is denoted by $\mathrm{Min}(\mathrm{E}(X))$. An element u of $\mathrm{E}(X)$ is called idempotent if $u^2 = u$. For $p \in \mathrm{E}(X)$ and $a \in X$, the maps $\mathrm{L}_p : \mathrm{E}(X) \to \mathrm{E}(X)$ and $\theta_a : \mathrm{E}(X) \to X$ defined by $\mathrm{L}_p(q) = pq$ and $\theta_a(q) = aq$ ($q \in \mathrm{E}(X)$), respectively, are continuous [2, Proposition 3.2]

Dealing with a-minimal sets (see Definition 1.1) where $a \in X$, it turns out that if K is an a-minimal set functions $\mathrm{L}_p : K \to K$, $\mathrm{L}_p(q) = pq$ ($p,q \in K$) that are bijective, play an important role in this area. In fact Ellis [2, Proposition 3.5] showed that for minimal right ideal I of $\mathrm{E}(X)$, $\{Iv \mid v \in \mathrm{J}(I)\}$ is a partition of subgroups of I and $\mathrm{L}_v = id_I$ ($v \in \mathrm{J}(I)$). Now if we want to have similar results for some of the subsets of a-minimal set K, we need to deal with elements $p \in K$ such that L_p is bijective. Let I be a right ideal in $\mathrm{E}(X)$, $B \subseteq \mathrm{E}(X)$, $C \subseteq X$ ($B,C \neq \emptyset$) and $a \in X$. Standing notations:

$$\mathrm{S}(I) = \{p \in I \mid \mathrm{L}_p : I \longrightarrow I \text{ is surjective}\}, \qquad \mathrm{F}(a,B) = \{p \in B \mid ap = a\},$$
$$\mathrm{I}(I) = \{p \in I \mid \mathrm{L}_p : I \longrightarrow I \text{ is injective}\}, \qquad \mathrm{F}(C,B) = \bigcap_{c\in C} \mathrm{F}(c,B),$$

$$\mathrm{B}(I) = \{p \in I \mid \mathrm{L}_p : I \longrightarrow I \text{ is bijective}\}, \qquad \overline{\mathrm{F}}(C,B) = \{p \in B \mid Cp = C\},$$
$$\mathrm{J}(B) = \{u \in B \mid u^2 = u\}. \tag{1.1}$$

A nonempty subset Z of X is called invariant if $ZS \subseteq Z$, moreover, a closed invariant subset Z of X is called minimal if it does not have any proper closed invariant subset. An element $a \in X$ is called almost periodic if $\overline{aS} = a\mathrm{E}(X)$ is a minimal subset of X [3, Theorems 1.15 and 1.17], and (X,S) is called distal if for each $x,y \in X$ and each $p \in \mathrm{E}(X)$, $xp = yp$ implies $x = y$. For an arbitrary map g, the restriction of g to A is denoted by $g|_A$.

For the remainder of this paper (X,S) is a fixed transformation semigroup, with e as the identity element of S.

Definition 1.1. Let A be a nonempty subset of X, $a_0 \in X$, and let K be a closed right ideal of $\mathrm{E}(X)$.

(a) K is called an a_0-minimal set if
  (i) $a_0K = a_0\mathrm{E}(X)$,
  (ii) K is minimal among all closed right ideals of $\mathrm{E}(X)$ with property (i).

The set of all a_0-minimal sets is denoted by $\mathrm{M}_{(X,S)}(a_0)$ (or simply $\mathrm{M}(a_0)$).

(b) K is called an A-$\overline{\text{minimal}}$ set if
  (i) $\forall a \in A,\ aK = a\mathrm{E}(X)$,
  (ii) K is minimal among all closed right ideals of $\mathrm{E}(X)$ with property (i).

The set of all A-$\overline{\text{minimal}}$ sets is denoted by $\overline{\mathrm{M}}_{(X,S)}(A)$ (or simply $\overline{\mathrm{M}}(A)$).

(c) K is called an A-$\overline{\overline{\text{minimal}}}$ set if
  (i) $AK = A\mathrm{E}(X)$,
  (ii) K is minimal among all closed right ideals of $\mathrm{E}(X)$ with property (i).

The set of all A-$\overline{\overline{\text{minimal}}}$ sets is denoted by $\overline{\overline{\mathrm{M}}}_{(X,S)}(A)$ (or simply $\overline{\overline{\mathrm{M}}}(A)$).

For more information about a-minimal sets we refer the reader to [5].

Theorem 1.2. *Let $a_0 \in X$ and let A be a nonempty subset of X, we have*

(a) $\mathrm{M}(a_0) = \overline{\mathrm{M}}(\{a_0\}) = \overline{\overline{\mathrm{M}}}(\{a_0\})$,

(b) $\overline{\mathrm{M}}(A) \neq \emptyset$,

(c) *if for each $b \in A\mathrm{E}(X)$, $\bigcup_{a\in A}\theta_a^{-1}(b)$ is a closed subset of $\mathrm{E}(X)$, then $\overline{\overline{\mathrm{M}}}(A) \neq \emptyset$.*

Proof. (b) Let

$$\mathcal{A} = \{K \mid K \text{ is a closed right ideal of } \mathrm{E}(X) \text{ and for each } a \in A,\ aK = a\mathrm{E}(X)\}, \tag{1.2}$$

then $\mathrm{E}(X) \in \mathcal{A}$ and for each chain such as $(K_\alpha)_{\alpha\in\Gamma}$ in the ordered set $(\mathcal{A},\subseteq)$, $\bigcap_{\alpha\in\Gamma}K_\alpha$ is a closed right ideal of $\mathrm{E}(X)$, moreover, for each $a \in A$, $b \in a\mathrm{E}(X)$, and $\alpha \in \Gamma$ define

$$K_\alpha(a,b) = \{p \in K_\alpha \mid ap = b\}\left(= K_\alpha \cap \theta_a^{-1}(b)\right), \tag{1.3}$$

by continuity of θ_a, $K_\alpha(a,b)$ is closed and by compactness of $\mathrm{E}(X)$, $\bigcap_{\alpha\in\Gamma}K_\alpha(a,b)(= \bigcap_{\alpha\in\Gamma}K_\alpha \cap \theta_a^{-1}(b))$ is nonempty, thus $b \in a(\bigcap_{\alpha\in\Gamma}K_\alpha)$ and $a(\bigcap_{\alpha\in\Gamma}K_\alpha) = a\mathrm{E}(X)$ (for each $a \in A$) thus $\bigcap_{\alpha\in\Gamma}K_\alpha \in \mathcal{A}$. Using Zorn's Lemma $(\mathcal{A},\subseteq)$ has a minimal element K, which is an A-$\overline{\text{minimal}}$ set.

We introduce the following sets:

$$\begin{aligned}\overline{\mathcal{M}}(X,S) &= \{B \subseteq X \mid B \neq \varnothing,\ \forall K \in \overline{M}(B),\ J(F(B,K)) \neq \varnothing\},\\ \overline{\overline{\mathcal{M}}}(X,S) &= \{B \subseteq X \mid B \neq \varnothing,\ \overline{\overline{M}}(B) \neq \varnothing,\ \forall K \in \overline{\overline{M}}(B),\ J(F(B,K)) \neq \varnothing\}.\end{aligned} \tag{1.4}$$

(c) Let

$$\mathcal{A} = \{K \mid K \text{ is a closed right ideal of } \mathrm{E}(X) \text{ and } AK = A\mathrm{E}(X)\}, \tag{1.5}$$

then $\mathrm{E}(X) \in \mathcal{A}$ and for each chain such as $(K_\alpha)_{\alpha\in\Gamma}$ in the ordered set $(\mathcal{A},\subseteq)$, $\bigcap_{\alpha\in\Gamma} K_\alpha$ is a closed right ideal of $\mathrm{E}(X)$, moreover, for each $b \in A\mathrm{E}(X)$ and $\alpha \in \Gamma$ define

$$K_\alpha(b) = \{p \in K_\alpha \mid \exists a \in A\ ap = b\}\left(= K_\alpha \cap \left(\bigcup_{a\in A} \theta_a^{-1}(b)\right)\right), \tag{1.6}$$

using an argument similar to the one given for (b) we have $b \in A(\bigcap_{\alpha\in\Gamma} K_\alpha)$, thus $\bigcap_{\alpha\in\Gamma} K_\alpha \in \mathcal{A}$. So $(\mathcal{A},\subseteq)$ has a minimal element like K, which is an A-minimal set. □

Corollary 1.3. *Let $a_0 \in X$, $\varnothing \neq A \subseteq X$ and let K be a right ideal of $\mathrm{E}(X)$, we have*

(a) *$a_0K = a_0\mathrm{E}(X)$ if and only if there exists $L \in \mathrm{M}(a_0)$, such that $L \subseteq K$,*

(b) *for each $a \in A$, $aK = a\mathrm{E}(X)$ if and only if there exists $L \in \overline{\mathrm{M}}(A)$ such that $L \subseteq K$,*

(c) *if for each $b \in A\mathrm{E}(X)$, $\bigcup_{a\in A} \theta_a^{-1}(b)$ is a closed subset of $\mathrm{E}(X)$, then $AK = A\mathrm{E}(X)$ if and only if there exists $L \in \overline{\overline{\mathrm{M}}}(A)$ such that $L \subseteq K$. Moreover, if A is finite, then for each $b \in A\mathrm{E}(X)$, $\bigcup_{a\in A} \theta_a^{-1}(b)$ is a closed subset of $\mathrm{E}(X)$ and $\overline{\overline{\mathrm{M}}}(A) \neq \varnothing$.*

Proof. The proof follows immediately by Theorem 1.2. □

Theorem 1.4. *Let $\varnothing \neq A \subseteq X$, K be a closed right ideal of $\mathrm{E}(X)$, $I \in \overline{\mathrm{M}}(A)$ and $J \in \overline{\overline{\mathrm{M}}}(A)$ ($\overline{\overline{\mathrm{M}}}(A)$ may be empty in which case the last item will be disregarded) we have Table 1.1.*

Proof. Second row. For each $u \in \mathrm{J}(\mathrm{S}(K))$ we have

$$\begin{aligned} u \in \mathrm{S}(K) &\Longrightarrow uK = K\\ &\Longrightarrow \forall p \in K,\ \exists q \in K,\ p = uq\\ &\Longrightarrow \forall p \in K,\ \exists q \in K,\ p = uq = u^2q = up = \mathrm{L}_u(p)\ (\text{since } u^2 = u)\\ &\Longrightarrow \mathrm{L}_u|_K = id_K,\ u \text{ is a left identity of } K\\ &\Longrightarrow u \text{ is the identity of the semigroup } Ku. \end{aligned} \tag{1.7}$$

For each $u,v \in \mathrm{J}(\mathrm{S}(K))$, define $\varphi_{u,v} : Ku \to Kv$ by $\varphi_{u,v}(p) = pv$ ($p \in Ku$). $\varphi_{u,v}$ is a semigroup isomorphism and $\varphi_{u,v}^{-1} = \varphi_{v,u}$. On the other hand, for each $u \in \mathrm{J}(\mathrm{I}(K))$, we have $u^2K = uK$, now since $u \in \mathrm{I}(K)$, so $uK = K$, thus $u \in \mathrm{J}(\mathrm{S}(K))$. Using the above facts we get $\mathrm{J}(\mathrm{S}(K)) = \mathrm{J}(\mathrm{I}(K)) = \mathrm{J}(\mathrm{B}(K))\ (= \{u \in K \mid \mathrm{L}_u|_K = id_K\})$.

Third row. For each $p \in \mathrm{F}(A,I)$, pI is a closed right ideal of $\mathrm{E}(X)$ and a subset of I, moreover, for each $a \in A$, $a(pI)(= (ap)I = aI = a\mathrm{E}(X))$, since $I \in \overline{\mathrm{M}}(A)$, so $pI = I$ and $p \in \mathrm{S}(I)$. For each $p \in \overline{\mathrm{F}}(A,J)$, pJ is a closed right ideal of $\mathrm{E}(X)$ and a subset of

TABLE 1.1. The mark $\surd$ indicates that for the corresponding case $\pi(\mathrm{Q})$ is true, where α is: (if $\mathrm{Q} \neq \varnothing$ then Q is a subsemigroup of C), β is: ($\forall u, v \in \mathrm{J}(\mathrm{C})$ (u is a left identity of C) $\wedge$ (the identity of Cu) $\wedge$ ($\mathrm{C}u \cong \mathrm{C}v$)), and γ is: (($\forall u \in \mathrm{J}(\mathrm{Q})$ (Qu is a group with identity u))) $\wedge$ ($\{\mathrm{Q}v \mid v \in \mathrm{J}(\mathrm{Q})\}$ is a partition of Q into some of its disjoint isomorphic subgroups) $\wedge\,\mathrm{card}(\{\mathrm{Q}v \mid v \in \mathrm{J}(\mathrm{Q})\}) = \mathrm{card}(\mathrm{J}(Q))$.

	$\pi(\mathrm{Q})$	Q / C	$\mathrm{F}(A,C)$	$\overline{\mathrm{F}}(A,C)$	$\mathrm{B}(C)\cap\overline{\mathrm{F}}(A,C)$	$\mathrm{B}(C)\cap\mathrm{F}(A,C)$	$\mathrm{B}(C)$	$\mathrm{S}(C)\cap\overline{\mathrm{F}}(A,C)$	$\mathrm{S}(C)\cap\mathrm{F}(A,C)$	$\mathrm{S}(C)$	$\mathrm{I}(C)$
First row	α	K or I or J	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
	β	K			$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
Third row	β	I or J	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
	γ	K			$\surd$	$\surd$	$\surd$				
Fifth row	γ	I	$\surd$		$\surd$	$\surd$	$\surd$		$\surd$		
	γ	J	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$		

J, moreover, $A(pJ) = (Ap)J = AJ = A\mathrm{E}(X)$, since $J \in \overline{\overline{\mathrm{M}}}(A)$, so $pJ = J$ and $p \in \mathrm{S}(J)$. Also note that $\mathrm{J}(\mathrm{F}(A,K)) = \mathrm{J}(\overline{\mathrm{F}}(A,K))$. Using this fact and the second row, we get the third row.

Fourth row. Let $u \in \mathrm{J}(\mathrm{B}(K))$, by the first and second rows, $\mathrm{B}(K)u$ is a semigroup with identity u. Also we have

$$\begin{aligned} &\forall p \in \mathrm{B}(K),\ pK = K \\ &\quad \Longrightarrow \forall p \in \mathrm{B}(K),\ \exists q \in K,\ pq = u \\ &\quad \Longrightarrow \forall p \in \mathrm{B}(K),\ \exists q \in \mathrm{B}(K),\ pq = u \ (\text{since } p,\ u \in \mathrm{B}(K)) \\ &\quad \Longrightarrow \forall p \in \mathrm{B}(K)u(\subseteq \mathrm{B}(K))\ \exists q \in \mathrm{B}(K),\ pq = u = u^2 = p(qu) \\ &\quad \Longrightarrow \forall p \in \mathrm{B}(K)u,\ \exists q \in \mathrm{B}(K)u,\ pq = u \end{aligned} \tag{1.8}$$

thus $\mathrm{B}(K)u$ is a group with identity u.

Let $p \in \mathrm{B}(K)$, then $pK = K$ and $\{q \in K \mid pq = p\}$ is a nonempty closed subsemigroup of $\mathrm{E}(X)$ and has an idempotent element u [2, Lemma 2.9], since $pu = p$ and $p \in \mathrm{B}(K)$ so $u \in \mathrm{J}(\mathrm{B}(K))$ and $p = pu \in \mathrm{B}(K)u$. Thus $\mathrm{B}(K) = \bigcup_{u \in \mathrm{J}(\mathrm{B}(K))} \mathrm{B}(K)u$. Moreover, let $u,v \in \mathrm{J}(\mathrm{B}(K))$, if $\mathrm{B}(K)u \cap \mathrm{B}(K)v \neq \emptyset$ and $p \in \mathrm{B}(K)u \cap \mathrm{B}(K)v$, then there exist $q \in \mathrm{B}(K)u$ and $r \in \mathrm{B}(K)v$ such that $pq = qp = u$ and $pr = rp = v$, thus $u = pq = (vp)q = v(pq) = (rp)u = r(pu) = rp = v$, therefore $u = v$ if and only if $\mathrm{B}(K)u \cap \mathrm{B}(K)v \neq \emptyset$. Similar methods described above, and the second row will complete the proof of the fourth row.

The proofs of the third and fourth rows conclude the fifth and sixth rows. □

Corollary 1.5. *Let $\emptyset \neq A \subseteq X$, K be a right ideal of $\mathrm{E}(X)$, $I \in \overline{\mathrm{M}}(A)$ and if $\overline{\overline{\mathrm{M}}}(A)$ is nonempty let $J \in \overline{\overline{\mathrm{M}}}(A)$, we have Tables 1.2 and 1.3.*

Proof. Use an argument similar to the one given in the proof of Theorem 1.4. □

Theorem 1.6. *Let $A \in \overline{\mathcal{M}}(X,S)$ and K be a closed right ideal of $\mathrm{E}(X)$ such that for each $a \in A$, $aK = a\mathrm{E}(X)$, then the following statements are equivalent:*
- (a) $K \in \overline{\mathrm{M}}(A)$,
- (b) $\mathrm{J}(\mathrm{F}(A,K)) \subseteq \mathrm{S}(K)$,
- (c) $u\mathrm{E}(X) = K\ \forall u \in \mathrm{J}(\mathrm{F}(A,K))$.

Proof. (a)⇒(b). Use Corollary 1.5 and Table 1.2.

(b)⇒(c). For $p \in \mathrm{S}(K)$, we have $K = pK \subseteq p\mathrm{E}(X) \subseteq K\mathrm{E}(X) \subseteq K$ so $p\mathrm{E}(X) = K$.

(c)⇒(a). By Corollary 1.3(b), there exists $L \in \overline{\mathrm{M}}(A)$ and $L \subseteq K$ and $u \in \mathrm{J}(\mathrm{F}(A,L)) \subseteq \mathrm{J}(\mathrm{F}(A,K))$, thus $K = u\mathrm{E}(X) \subseteq L$ and $K = L \in \overline{\mathrm{M}}(A)$. □

Theorem 1.7. *Let A be a nonempty subset of X then*
- (a) *for each $K,L \in \overline{\mathrm{M}}(A)$, we have*
  - (i) $\forall p \in \mathrm{F}(A,K)$, $pL = K$,
  - (ii) $\forall u \in \mathrm{J}(\mathrm{F}(A,K))$, $\exists! v \in \mathrm{J}(\mathrm{F}(A,L))$, $uv = u \wedge vu = v$,
  - (iii) $\forall u \in \mathrm{J}(\mathrm{F}(A,K))$, $\exists! v \in \mathrm{J}(\mathrm{F}(A,L))$, $uv = u$,
  - (iv) $\forall u \in \mathrm{J}(\mathrm{F}(A,K))$, $\mathrm{card}(\mathrm{J}((\mathrm{L}_u|_L)^{-1}(u))) = 1$,
  - (v) $\mathrm{card}(\mathrm{J}(\mathrm{F}(A,K))) = \mathrm{card}(\mathrm{J}(\mathrm{F}(A,L)))$,
  - (vi) $\mathrm{card}(\overline{\mathrm{M}}(A))\,\mathrm{card}(\mathrm{J}(\mathrm{F}(A,K))) = \mathrm{card}(\bigcup_{N \in \overline{\mathrm{M}}(A)} \mathrm{J}(\mathrm{F}(A,N)))$,

TABLE 1.2. The mark √ indicates that for the corresponding case $\mathrm{D} \subseteq \mathrm{G}$.

D	C \ G	$\mathrm{F}(A,C)\cap\mathrm{B}(C)$	$\mathrm{F}(A,C)\cap\mathrm{S}(C)$	$\mathrm{F}(A,C)$	$\overline{\mathrm{F}}(A,C)\cap\mathrm{B}(C)$	$\overline{\mathrm{F}}(A,C)\cap\mathrm{S}(C)$	$\overline{\mathrm{F}}(A,C)$	$\mathrm{B}(C)$	$\mathrm{S}(C)$	$\mathrm{I}(C)$
$\mathrm{F}(A,C)\cap\mathrm{B}(C)$	K or I or J	√	√	√	√	√	√	√	√	√
$\mathrm{F}(A,C)\cap\mathrm{S}(C)$	K		√	√		√	√		√	
	I or J	√	√	√	√	√	√	√	√	√
$\mathrm{F}(A,C)$	K			√			√			
	I or J	√	√	√	√	√	√	√	√	√
$\overline{\mathrm{F}}(A,C)\cap\mathrm{B}(C)$	K or I or J				√	√	√	√	√	√
$\overline{\mathrm{F}}(A,C)\cap\mathrm{S}(C)$	K or I					√	√		√	
	J				√	√	√	√	√	√
$\overline{\mathrm{F}}(A,C)$	K or I						√			
	J				√	√	√	√	√	√
$\mathrm{B}(C)$	K or I or J							√	√	√
$\mathrm{S}(C)$	K or I or J								√	
$\mathrm{I}(C)$	K or I or J									√

TABLE 1.3. The mark √ indicates that for the corresponding case $\mathrm{J}(\mathrm{D}) \subseteq \mathrm{J}(\mathrm{G})$.

D	G / C	$\mathrm{F}(A,C)\cap\mathrm{B}(C)$ or $\mathrm{F}(A,C)\cap\mathrm{S}(C)$ or $\overline{\mathrm{F}}(A,C)\cap\mathrm{B}(C)$ or $\overline{\mathrm{F}}(A,C)\cap\mathrm{S}(C)$	$\mathrm{F}(A,C)$ or $\overline{\mathrm{F}}(A,C)$	$\mathrm{B}(C)$ or $\mathrm{S}(C)$ or $\mathrm{I}(C)$
$\mathrm{F}(A,C)\cap\mathrm{B}(C)$ or $\mathrm{F}(A,C)\cap\mathrm{S}(C)$ or $\overline{\mathrm{F}}(A,C)\cap\mathrm{B}(C)$ or $\overline{\mathrm{F}}(A,C)\cap\mathrm{S}(C)$	K or I or J	√	√	√
$\mathrm{F}(A,C)$ or $\overline{\mathrm{F}}(A,C)$	K		√	
	I or J	√	√	√
$\mathrm{B}(C)$ or $\mathrm{S}(C)$ or $\mathrm{I}(C)$	K or I or J			√

(b) *for each* $K,L \in \overline{\overline{\mathrm{M}}}(A)$*, we have*
 (i) $\forall p \in \overline{\mathrm{F}}(A,K),\ pL = K$,
 (ii) $\forall u \in \mathrm{J}(\mathrm{F}(A,K)),\ \exists! v \in \mathrm{J}(\mathrm{F}(A,L)),\ uv = u \wedge vu = v$,
 (iii) $\forall u \in \mathrm{J}(\mathrm{F}(A,K)),\ \exists! v \in \mathrm{J}(\mathrm{F}(A,L)),\ uv = u$,
 (iv) $\forall u \in \mathrm{J}(\mathrm{F}(A,K)),\ \mathrm{card}(\mathrm{J}((\mathrm{L}_u|_L)^{-1}(u))) = 1$,
 (v) $\mathrm{card}(\mathrm{J}(\mathrm{F}(A,K))) = \mathrm{card}(\mathrm{J}(\mathrm{F}(A,L)))$,
 (vi) $\mathrm{card}(\overline{\overline{\mathrm{M}}}(A))\,\mathrm{card}(\mathrm{J}(\mathrm{F}(A,K))) = \mathrm{card}(\bigcup_{N\in\overline{\overline{\mathrm{M}}}(A)} \mathrm{J}(\mathrm{F}(A,N)))$.

PROOF. (a)(i). For each $p \in \mathrm{F}(A,K)$, pL is a closed right ideal of $\mathrm{E}(X)$ and a subset of K, moreover, for each $a \in A$, $a(pL) = (ap)L = aL = a\mathrm{E}(X)$, thus $pL = K$.

(ii), (iii), and (iv). For each $u \in \mathrm{J}(\mathrm{F}(A,K))$ we have $uL = K$ (by (i)), thus $\{q \in L \mid uq = u\}$ $(= (\mathrm{L}_u|_L)^{-1}(u))$ is a nonempty closed subsemigroup of $\mathrm{E}(X)$ and has an idempotent like v [2, Lemma 2.9], as $uv = u$ and for each $a \in A$, $a = au = a(uv) = (au)v = av$, we have $v \in \mathrm{J}(\mathrm{F}(A,L))$, moreover, $(vu)^2 = v(uv)u = vu^2 = vu \in vK = L$, thus $vu \in \mathrm{J}(\mathrm{F}(A,L))$ and by Theorem 1.4 (Table 1.1 (third row)) $\mathrm{L}_{vu}(v) = v$, that is, $v = (vu)v = v(uv) = vu$. Now let $v' \in \mathrm{J}(\mathrm{F}(A,L))$ be such that $uv' = u$, by an argument similar to the one given for v we have $v'u = v'$ and $v' = v'u = v'(vu) = (v'v)u = vu = v$, this gives the desired result.

(v) and (vi). By (ii), (iii), and (iv) there exists a unique map $\phi_{K,L} : \mathrm{J}(\mathrm{F}(A,K)) \to \mathrm{J}(\mathrm{F}(A,L))$ such that for each $u \in \mathrm{J}(\mathrm{F}(A,K))$, $\phi_{K,L}(u) \in \mathrm{J}((\mathrm{L}_u|_L)^{-1}(u))$, moreover, $\phi_{K,L}^{-1} = \phi_{L,K}$.

(b) Use a similar argument like (a). □

LEMMA 1.8. *Let A be a nonempty subset of X and let K be a closed right ideal of* $\mathrm{E}(X)$*. Then* $\mathrm{J}(\mathrm{F}(A,K)) \neq \varnothing$ *if and only if* $\mathrm{F}(A,K) \neq \varnothing$.

PROOF. $\mathrm{F}(A,K) = \{p \in K \mid \forall a \in A\ ap = a\}$ $(= \bigcap_{a\in A}\theta_a^{-1}(a) \cap K)$ is a closed subsemigroup of $\mathrm{E}(X)$, by [2, Lemma 2.9], it is nonempty if and only if it has an idempotent. □

Corollary 1.9. *Let A be a nonempty subset of X. We have*

(a) *the following statements are equivalent:*
  (i) $\forall K \in \overline{\mathrm{M}}(A)$, $\mathrm{J}(\mathrm{F}(A,K)) \neq \varnothing$ *(or* $A \in \overline{\mathcal{M}}(X,S)$*),*
  (ii) $\forall K \in \overline{\mathrm{M}}(A)$, $\mathrm{F}(A,K) \neq \varnothing$,
  (iii) $\exists K \in \overline{\mathrm{M}}(A)$, $\mathrm{J}(\mathrm{F}(A,K)) \neq \varnothing$,
  (iv) $\exists K \in \overline{\mathrm{M}}(A)$, $\mathrm{F}(A,K) \neq \varnothing$,

(b) *the following statements are equivalent:*
  (i) $\overline{\overline{\mathrm{M}}}(A) \neq \varnothing \wedge (\forall K \in \overline{\mathrm{M}}(A),\ \mathrm{J}(\mathrm{F}(A,K)) \neq \varnothing)$ *(or* $A \in \overline{\overline{\mathcal{M}}}(X,S)$*),*
  (ii) $\overline{\overline{\mathrm{M}}}(A) \neq \varnothing \wedge (\forall K \in \overline{\mathrm{M}}(A),\ \mathrm{F}(A,K) \neq \varnothing)$,
  (iii) $\exists K \in \overline{\overline{\mathrm{M}}}(A)$, $\mathrm{J}(\mathrm{F}(A,K)) \neq \varnothing$,
  (iv) $\exists K \in \overline{\overline{\mathrm{M}}}(A)$, $\mathrm{F}(A,K) \neq \varnothing$.

Proof. Use Theorem 1.7 and Lemma 1.8. □

Theorem 1.10. *For $i \in \{1,\dots,n\}$, let (X_i,S_i) be a transformation semigroup and let A_i be a nonempty subset of X_i. If $\prod_{i=1}^n A_i \in \overline{\mathcal{M}}(\prod_{i=1}^n X_i, \prod_{i=1}^n S_i)$, then $\overline{\mathrm{M}}_{(\prod_{i=1}^n X_i,\prod_{i=1}^n S_i)}(\prod_{i=1}^n A_i) = \prod_{i=1}^n \overline{\mathrm{M}}_{(X_i,S_i)}(A_i)$ and for each $i \in \{1,\dots,n\}$ we have $A_i \in \overline{\mathcal{M}}(X_i,S_i)$.*

Proof. Let $K \in \overline{\mathrm{M}}_{(\prod_{i=1}^n X_i,\prod_{i=1}^n S_i)}(\prod_{i=1}^n A_i)$, since $\prod_{i=1}^n A_i \in \overline{\mathcal{M}}(\prod_{i=1}^n X_i, \prod_{i=1}^n S_i)$, there exists $u = (u_1,\dots,u_n) \in \mathrm{J}(\mathrm{F}(\prod_{i=1}^n A_i, K))$ (for each $i \in \{1,\dots,n\}$, $u_i \in \mathrm{J}(\mathrm{F}(A_i,\mathrm{E}(X_i,S_i))))$ so $K = (u_1,\dots,u_n)\mathrm{E}(\prod_{i=1}^n X_i, \prod_{i=1}^n S_i) = \prod_{i=1}^n u_i \mathrm{E}(X_i,S_i)$, moreover, for each $i \in \{1,\dots,n\}$, and $a \in A_i$, we have $a(u_i\mathrm{E}(X_i,S_i)) = a\mathrm{E}(X_i,S_i)$. Since $K \in \overline{\mathrm{M}}_{(\prod_{i=1}^n X_i,\prod_{i=1}^n S_i)}(\prod_{i=1}^n A_i)$, it is easy to see that for each $i \in \{1,\dots,n\}$, $u_i\mathrm{E}(X_i,S_i) \in \overline{\mathrm{M}}_{(X_i,S_i)}(A_i)$, thus $\overline{\mathrm{M}}_{(\prod_{i=1}^n X_i,\prod_{i=1}^n S_i)}(\prod_{i=1}^n A_i) \subseteq \prod_{i=1}^n \overline{\mathrm{M}}_{(X_i,S_i)}(A_i)$ and for each $i \in \{1,\dots,n\}$, $A_i \in \overline{\mathcal{M}}(X_i,S_i)$. On the other hand, for each $i \in \{1,\dots,n\}$ let $K_i \in \overline{\mathrm{M}}_{(X_i,S_i)}(A_i)$, then for each $(a_1,\dots,a_n) \in \prod_{i=1}^n A_i$, we have

$$\begin{aligned}(a_1,\dots,a_n)\prod_{i=1}^n K_i &= \prod_{i=1}^n a_iK_i = \prod_{i=1}^n a_i\mathrm{E}(X_i,S_i)\\ &= (a_1,\dots,a_n)\prod_{i=1}^n \mathrm{E}(X_i,S_i) = (a_1,\dots,a_n)\mathrm{E}\left(\prod_{i=1}^n X_i, \prod_{i=1}^n S_i\right).\end{aligned} \tag{1.9}$$

Thus by Corollary 1.3(b), there exists $K \in \overline{\mathrm{M}}_{(\prod_{i=1}^n X_i,\prod_{i=1}^n S_i)}(\prod_{i=1}^n A_i)$ such that $K \subseteq \prod_{i=1}^n K_i$. Since $\overline{\mathrm{M}}_{(\prod_{i=1}^n X_i,\prod_{i=1}^n S_i)}(\prod_{i=1}^n A_i) \subseteq \prod_{i=1}^n \overline{\mathrm{M}}_{(X_i,S_i)}(A_i)$, for each $i \in \{1,\dots,n\}$ there exists $K_i' \in \overline{\mathrm{M}}_{(X_i,S_i)}(A_i)$, such that $\prod_{i=1}^n K_i' = K \subseteq \prod_{i=1}^n K_i$. Thus for each $i \in \{1,\dots,n\}$, $K_i' = K_i$ and $K = \prod_{i=1}^n K_i$, therefore $\overline{\mathrm{M}}_{(\prod_{i=1}^n X_i,\prod_{i=1}^n S_i)}(\prod_{i=1}^n A_i) \supseteq \prod_{i=1}^n \overline{\mathrm{M}}_{(X_i,S_i)}(A_i)$. □

Note 1.11. Let $\varnothing \neq A \subseteq X$, and K, L be right ideals of $\mathrm{E}(X)$, then from the following table we have

Table 1.4.

	Part 1	Part 2	Part 3
P	$\mathrm{S}(K)$, $\mathrm{J}(\mathrm{S}(K)) = \mathrm{J}(\mathrm{B}(K)) = \mathrm{J}(\mathrm{I}(K))$, $\mathrm{B}(K)$	$\mathrm{F}(A,K)$, $\mathrm{J}(\mathrm{F}(A,K))$	$\overline{\mathrm{F}}(A,K)$
Q	$\mathrm{S}(L)$, $\mathrm{J}(\mathrm{S}(L)) = \mathrm{J}(\mathrm{B}(L)) = \mathrm{J}(\mathrm{I}(L))$, $\mathrm{B}(L)$	$\mathrm{F}(A,L)$, $\mathrm{J}(\mathrm{F}(A,L))$	$\overline{\mathrm{F}}(A,L)$

(a) in part 1, if $\mathrm{P}\cap\mathrm{Q}\neq\emptyset$, then $K=L$,
(b) in parts 1 and 2, if $K,L\in\overline{\mathrm{M}}(A)$ and $\mathrm{P}\cap\mathrm{Q}\neq\emptyset$, then $K=L$,
(c) in parts 1 and 2, if $K,L\in\overline{\mathrm{M}}(A)$ and $A\in\overline{\mathcal{M}}(X,S)$, then $\mathrm{P}\cap\mathrm{Q}\neq\emptyset$ if and only if $K=L$,
(d) in parts 1, 2, and 3, if $K,L\in\overline{\overline{\mathrm{M}}}(A)$ and $\mathrm{P}\cap\mathrm{Q}\neq\emptyset$, then $K=L$,
(e) in parts 1, 2, and 3, if $K,L\in\overline{\overline{\mathrm{M}}}(A)$ and $A\in\overline{\overline{\mathcal{M}}}(X,S)$, then $\mathrm{P}\cap\mathrm{Q}\neq\emptyset$ if and only if $K=L$.

Use the fact that for each $p\in\mathrm{S}(K)$ we have $p\mathrm{E}(X)=K$ and use Corollary 1.5 (Table 1.2).

Note 1.12. Let A be a nonempty subset of X. Then the following statements are equivalent:
(a) for each $a\in A$, a is almost periodic,
(b) $\overline{\mathrm{M}}(A)=\mathrm{Min}(\mathrm{E}(X))$,
(c) $\overline{\mathrm{M}}(A)\cap\mathrm{Min}(\mathrm{E}(X))\neq\emptyset$,
(d) $\overline{\overline{\mathrm{M}}}(A)=\mathrm{Min}(\mathrm{E}(X))$,
(e) $\overline{\overline{\mathrm{M}}}(A)\cap\mathrm{Min}(\mathrm{E}(X))\neq\emptyset$.

Proof. Let $K\in\mathrm{Min}(\mathrm{E}(X))$, then each $a\in A$ is almost periodic if and only if for each $a\in A$, $aK=a\mathrm{E}(X)$ if and only if $K\in\overline{\mathrm{M}}(A)$, moreover, $K\in\overline{\overline{\mathrm{M}}}(A)$ if and only if $AK=A\mathrm{E}(X)$ if and only if for each $a\in A$ there exists $b\in A$ such that $a\in bK$, if and only if each $a\in A$ is almost periodic. □

Definition 1.13. Let $\mathrm{Q},\mathrm{R}\in\{\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}}\}$ and A, B be nonempty subsets of X, such that whenever $\mathrm{R}=\overline{\overline{\mathrm{M}}}$, then $\overline{\overline{\mathrm{M}}}(A)\neq\emptyset$. We say
(a) (X,S) is $A\frac{(-)}{}$distal (or simply A-distal) if for each $a\in A$, $\mathrm{E}(X)\in\mathrm{M}(a)$,
(b) (X,S) is $A\frac{(\mathrm{Q})}{}$distal if $\mathrm{E}(X)\in\mathrm{Q}(a)$,
(c) B is $A\frac{(-,-)}{}$almost periodic (or simply A-almost periodic) if

$$\forall b\in B,\ \forall a\in A,\ \text{and}\ \forall K\in\mathrm{M}(a),\ \exists L\in\mathrm{M}(b)\ \text{such that}\ L\subseteq K, \tag{1.10}$$

(d) B is $A\frac{(-,\mathrm{R})}{}$almost periodic if

$$\forall b\in B\ \text{and}\ \forall K\in\mathrm{R}(A),\ \exists L\in\mathrm{M}(b)\ \text{such that}\ L\subseteq K, \tag{1.11}$$

(e) B is $A\frac{(\mathrm{Q},-)}{}$almost periodic if

$$\forall a\in A\ \text{and}\ \forall K\in\mathrm{M}(a),\ \exists L\in\mathrm{Q}(B)\ \text{such that}\ L\subseteq K, \tag{1.12}$$

(f) B is $A\frac{(\mathrm{Q},\mathrm{R})}{}$almost periodic if

$$\forall K\in\mathrm{R}(A)\ \exists L\in\mathrm{Q}(B)\ \text{such that}\ L\subseteq K, \tag{1.13}$$

(g) whenever A or B is singleton, instead of the symbol of the corresponding set we will use the symbol of its element.

Theorem 1.14. *Let $a \in X$ and let Z be a closed nonempty invariant subset of X, then the following statements are equivalent:*

(a) *a is almost periodic,*
(b) $\mathrm{Min}(\mathrm{E}(X)) = \mathrm{M}(a)$,
(c) $\mathrm{Min}(\mathrm{E}(X)) \cap \mathrm{M}(a) \neq \emptyset$,
(d) $\forall x \in \overline{aS}\quad \mathrm{M}(x) = \mathrm{M}(a)$,
(e) $\forall x \in \overline{aS}\quad \mathrm{M}(x) \cap \mathrm{M}(a) \neq \emptyset$,
(f) *for each $x \in Z$, a is x-almost periodic,*
(g) *there exists an almost periodic point $x \in X$ such that a is x-almost periodic.*

Proof. (a), (b), and (c) are equivalent by Note 1.12.

(a), (b)⇒(d). a is almost periodic if and only if $\overline{aS}$ is minimal, if and only if for each $x \in \overline{aS}$, $\overline{xS} = \overline{aS}$ is minimal, if and only if for each $x \in \overline{aS}$, x is almost periodic [3, Theorem 1.15 and 1.17]. Thus for each $x \in \overline{aS}$, $\mathrm{M}(x) = \mathrm{M}(a) = \mathrm{Min}(\mathrm{E}(X))$.

(d)⇒(e). It is clear.

(e)⇒(c). $\overline{aS}$ has an almost periodic point say x. By Note 1.12, $\mathrm{M}(x) = \mathrm{Min}(\mathrm{E}(X))$, thus $\mathrm{Min}(\mathrm{E}(X)) \cap \mathrm{M}(a) \neq \emptyset$.

(b)⇒(f). Use the fact that each closed right ideal of $\mathrm{E}(X)$ contains a minimal right ideal.

(f)⇒(g). Since Z is a closed invariant subset of X, it has an almost periodic point say x, and a is x-almost periodic.

(g)⇒(c). Let $x \in X$ be an almost periodic point such that a is x-almost periodic. By Note 1.12, $\mathrm{M}(x) = \mathrm{Min}(\mathrm{E}(X))$, let $K \in \mathrm{M}(x) = \mathrm{Min}(\mathrm{E}(X))$, there exists $L \in \mathrm{M}(a)$, such that $L \subseteq K$, thus $L = K \in \mathrm{Min}(\mathrm{E}(X))$, that is, $\mathrm{Min}(\mathrm{E}(X)) \subseteq \mathrm{M}(a)$ and $\mathrm{Min}(\mathrm{E}(X)) \cap \mathrm{M}(a) \neq \emptyset$. □

Lemma 1.15. *Let A, B, and C be nonempty subsets of X, then*

(a) *the following statements are equivalent:*
  (i) *B is $A^{\underline{(-,-)}}$ almost periodic,*
  (ii) *B is $A^{\underline{(\overline{\mathrm{M}},-)}}$ almost periodic,*
  (iii) $\forall b \in B,\ \forall a \in A,\ \forall K \in \mathrm{M}(a),\ bK = b\mathrm{E}(X)$,
(b) *the following statements are equivalent:*
  (i) *B is $A^{\underline{(-,\overline{\mathrm{M}})}}$ almost periodic,*
  (ii) *B is $A^{\underline{(\overline{\mathrm{M}},\overline{\mathrm{M}})}}$ almost periodic,*
  (iii) $\forall b \in B,\ \forall K \in \overline{\mathrm{M}}(A),\ bK = b\mathrm{E}(X)$,
(c) *the following statements are equivalent:*
  (i) *B is $A^{\underline{(-,\overline{\overline{\mathrm{M}}})}}$ almost periodic,*
  (ii) *B is $A^{\underline{(\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}})}}$ almost periodic,*
  (iii) $\forall b \in B,\ \forall K \in \overline{\overline{\mathrm{M}}}(A),\ bK = b\mathrm{E}(X)$,
(d) *let $\mathrm{P},\mathrm{Q},\mathrm{R} \in \{-,\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}}\}$, if C is $B^{\underline{(\mathrm{P},\mathrm{Q})}}$ almost periodic and B is $A^{\underline{(\mathrm{Q},\mathrm{R})}}$ almost periodic, then C is $A^{\underline{(\mathrm{P},\mathrm{R})}}$ almost periodic,*
(e) *the following statements are valid:*
  (i) *B is $A^{\underline{(\overline{\overline{\mathrm{M}}},-)}}$ almost periodic* $\Rightarrow \forall a \in A,\ \forall L \in \mathrm{M}(a),\ BL = B\mathrm{E}(X)$,
  (ii) *B is $A^{\underline{(\overline{\overline{\mathrm{M}}},\overline{\mathrm{M}})}}$ almost periodic* $\Rightarrow \forall L \in \overline{\mathrm{M}}(A),\ BL = B\mathrm{E}(X)$,
  (iii) *B is $A^{\underline{(\overline{\overline{\mathrm{M}}},\overline{\overline{\mathrm{M}}})}}$ almost periodic* $\Rightarrow \forall L \in \overline{\overline{\mathrm{M}}}(A),\ BL = B\mathrm{E}(X)$.

Proof. (a) We have

$$
\begin{aligned}
&B \text{ is } A\,\frac{(-,-)}{}\,\text{almost periodic} \\
&\quad\Longleftrightarrow \forall b \in B,\ \forall a \in A,\ \forall K \in \mathrm{M}(a),\ \exists L \in \mathrm{M}(b),\ L \subseteq K \\
&\quad\Longleftrightarrow \forall b \in B,\ \forall a \in A,\ \forall K \in \mathrm{M}(a),\ bK = b\mathrm{E}(X) \ (\text{by Corollary 1.3(a)}) \\
&\quad\Longleftrightarrow \forall a \in A,\ \forall K \in \mathrm{M}(a),\ \forall b \in B,\ bK = b\mathrm{E}(X) \\
&\quad\Longleftrightarrow \forall a \in A,\ \forall K \in \mathrm{M}(a),\ \exists L \in \overline{\mathrm{M}}(B),\ L \subseteq K,\ (\text{by Corollary 1.3(b)}) \\
&\quad\Longleftrightarrow B \text{ is } A\,\frac{(\overline{\mathrm{M}},-)}{}\,\text{almost periodic.}
\end{aligned}
\tag{1.14}
$$

(b) We have

$$
\begin{aligned}
&B \text{ is } A\,\frac{(-,\overline{\mathrm{M}})}{}\,\text{almost periodic} \\
&\quad\Longleftrightarrow\ \forall b \in B,\ \forall K \in \overline{\mathrm{M}}(A),\ \exists L \in \mathrm{M}(b),\ L \subseteq K \\
&\quad\Longleftrightarrow\ \forall b \in B,\ \forall K \in \overline{\mathrm{M}}(A),\ bK = b\mathrm{E}(X) \ (\text{by Corollary 1.3(a)}) \\
&\quad\Longleftrightarrow\ \forall K \in \overline{\mathrm{M}}(A),\ \forall b \in B,\ bK = b\mathrm{E}(X) \\
&\quad\Longleftrightarrow\ \forall K \in \overline{\mathrm{M}}(A),\ \exists L \in \overline{\mathrm{M}}(B),\ L \subseteq K,\ (\text{by Corollary 1.3(b)}) \\
&\quad\Longleftrightarrow B \text{ is } A\,\frac{(\overline{\mathrm{M}},\overline{\mathrm{M}})}{}\,\text{almost periodic.}
\end{aligned}
\tag{1.15}
$$

(c) Use an argument similar to (a) and (b).

(d) Each case should be checked, for example, we check the cases $\mathrm{P},\mathrm{Q},\mathrm{R} \in \{\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}}\}$ (thus $\mathrm{Q}(B)$ and $\mathrm{R}(A)$ are nonempty), we have

$$
\begin{aligned}
&\Big(\big(C \text{ is } B\,\frac{(\mathrm{P},\mathrm{Q})}{}\,\text{almost periodic}\big) \wedge \big(B \text{ is } A\,\frac{(\mathrm{Q},\mathrm{R})}{}\,\text{almost periodic}\big)\Big) \\
&\quad\Longrightarrow \big(\big(\forall K \in \mathrm{Q}(B),\ \exists L \in \mathrm{P}(C),\ L \subseteq K\big) \wedge \big(\forall I \in \mathrm{R}(A),\ \exists K \in \mathrm{Q}(B),\ K \subseteq I\big)\big) \\
&\quad\Longrightarrow \forall I \in \mathrm{R}(A),\ \exists L \in \mathrm{P}(C),\ L \subseteq I \\
&\quad\Longrightarrow C \text{ is } A\,\frac{(\mathrm{P},\mathrm{R})}{}\,\text{almost periodic.}
\end{aligned}
\tag{1.16}
$$

(e) (i) B is $A\,\frac{(\overline{\overline{\mathrm{M}}},-)}{}$ almost periodic

$$
\begin{aligned}
&\Longrightarrow \forall a \in A,\ \forall L \in \mathrm{M}(a),\ \exists K \in \overline{\overline{\mathrm{M}}}(B),\ K \subseteq L \\
&\Longrightarrow \forall a \in A,\ \forall L \in \mathrm{M}(a),\ \exists K \in \overline{\overline{\mathrm{M}}}(B),\ B\mathrm{E}(X) = BK \subseteq BL \subseteq B\mathrm{E}(X) \\
&\Longrightarrow \forall a \in A,\ \forall L \in \mathrm{M}(a),\ BL = B\mathrm{E}(X).
\end{aligned}
\tag{1.17}
$$

For (ii) and (iii), use a similar argument like (i). □

Note 1.16. If A is a nonempty subset of X, then by Corollary 1.3(a), A is $A\,\frac{(-,\overline{\mathrm{M}})}{}$ almost periodic.

Theorem 1.17. *Let A and B be nonempty subsets of X, then we have Table 1.5.*

TABLE 1.5. The mark $\surd$ indicates that for the corresponding case if B is $A^{\underline{\alpha}}$almost periodic, then B is $A^{\underline{\beta}}$almost periodic, and The mark $\underline{\underline{\surd}}$ indicates that for the corresponding case if B is $A^{\underline{\alpha}}$almost periodic and A is $A^{\underline{(\overline{\overline{\mathrm{M}}},\overline{\mathrm{M}})}}$almost periodic and B is $B^{\underline{(\overline{\overline{\mathrm{M}}},\overline{\mathrm{M}})}}$almost periodic, then B is $A^{\underline{\beta}}$almost periodic.

β / α	$(-,-)$ *or* $(\overline{\mathrm{M}},-)$	$(-,\overline{\mathrm{M}})$ *or* $(\overline{\mathrm{M}},\overline{\mathrm{M}})$	$(-,\overline{\overline{\mathrm{M}}})$ *or* $(\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}})$	$(\overline{\overline{\mathrm{M}}},-)$	$(\overline{\overline{\mathrm{M}}},\overline{\mathrm{M}})$	$(\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}})$
$(-,-)$ *or* $(\overline{\mathrm{M}},-)$	$\surd$	$\surd$		$\underline{\underline{\surd}}$	$\underline{\underline{\surd}}$	
$(-,\overline{\mathrm{M}})$ *or* $(\overline{\mathrm{M}},\overline{\mathrm{M}})$		$\surd$			$\underline{\underline{\surd}}$	
$(-,\overline{\overline{\mathrm{M}}})$ *or* $(\overline{\mathrm{M}},\overline{\overline{\mathrm{M}}})$		$\underline{\underline{\surd}}$	$\surd$		$\underline{\underline{\surd}}$	$\underline{\underline{\surd}}$
$(\overline{\overline{\mathrm{M}}},-)$				$\surd$	$\surd$	
$(\overline{\overline{\mathrm{M}}},\overline{\mathrm{M}})$					$\surd$	
$(\overline{\overline{\mathrm{M}}},\overline{\overline{\mathrm{M}}})$					$\underline{\underline{\surd}}$	$\surd$

PROOF. For most of the cases use Lemma 1.15 and Note 1.16. By Lemma 1.15((a) and (b)) we have the main diagonal.

For example in the first row the following statements are valid:

- (B is $A^{\underline{(\overline{\mathrm{M}},-)}}$almost periodic) by using Note 1.16 $\Rightarrow$ ((B is $A^{\underline{(\overline{\mathrm{M}},-)}}$almost periodic)) $\wedge$ A is $A^{\underline{(-,\overline{\mathrm{M}})}}$almost periodic by using Lemma 1.15(d) $\Rightarrow$ (B is $A^{\underline{(\overline{\mathrm{M}},\overline{\mathrm{M}})}}$almost periodic)
- ((B is $B^{\underline{(\overline{\overline{\mathrm{M}}},\overline{\mathrm{M}})}}$almost periodic) $\wedge$ (B is $A^{\underline{(\overline{\mathrm{M}},-)}}$almost periodic)) by using Lemma 1.15(d) $\Rightarrow$ (B is $A^{\underline{(\overline{\overline{\mathrm{M}}},-)}}$almost periodic).
- ((B is $B^{\underline{(\overline{\overline{\mathrm{M}}},\overline{\mathrm{M}})}}$almost periodic) $\wedge$ (B is $A^{\underline{(\overline{\mathrm{M}},-)}}$almost periodic)) by using Lemma 1.15(d) and Note 1.16 $\Rightarrow$ ((B is $A^{\underline{(\overline{\overline{\mathrm{M}}},-)}}$almost periodic) $\wedge$ (A is $A^{\underline{(-,\overline{\mathrm{M}})}}$almost periodic)) Lemma 1.15(d) $\Rightarrow$ (B is $A^{\underline{(\overline{\overline{\mathrm{M}}},\overline{\mathrm{M}})}}$almost periodic). □

THEOREM 1.18. *Let $n \in \mathbb{N}$ and A be a nonempty subset of X, then*

(a) *the following statements are equivalent:*

(i) (X,S) *is distal,*

(ii) $\mathrm{Min}(\mathrm{E}(X)) = \{\mathrm{E}(X)\}$,

(iii) $\forall x \in X$, (X,S) *is x-distal,*

(iv) $\exists x \in X$ *(x is almost periodic)* $\wedge$*((X,S) is x-distal),*

(in these cases $\mathrm{E}(X)$ *is a group),*

(b) *the following statements are equivalent:*

(i) (X,S) *is A-distal,*

(ii) $\forall a \in A$, (X,S) *is a-distal,*

(iii) $\forall a \in A$, $\mathrm{M}(a) = \{\mathrm{E}(X)\}$,

(iv) $\forall a \in A$, $\mathrm{F}(a,\mathrm{E}(X))$ *is a subgroup of* $\mathrm{E}(X)$,

(v) $\forall a \in A$, $\mathrm{J}(\mathrm{F}(a,\mathrm{E}(X)))$ *is a subgroup of* $\mathrm{E}(X)$,

(vi) $\forall a \in A$, $\mathrm{J}(\mathrm{F}(a,\mathrm{E}(X))) = \{e\}$,

(vii) (X^n,S^n) *is* A^n*-distal,*

(in these cases for each $a \in A$, $\mathrm{F}(a,\mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X)) \cap \mathrm{F}(a,\mathrm{E}(X))$, $\mathrm{F}(A,\mathrm{E}(X))$ *and* $\mathrm{B}(\mathrm{E}(X)) \cap \mathrm{F}(A,\mathrm{E}(X))$ *are subgroups of* $\mathrm{E}(X)$*),*

(c) *if* $A^n \in \overline{\mathcal{M}}(X^n,S^n)$, *then the following statements are equivalent:*

(i) (X,S) *is* $A^{\underline{(\overline{\mathrm{M}})}}$ *distal,*

(ii) $\overline{\mathrm{M}}(A) = \{\mathrm{E}(X)\}$,

(iii) $\mathrm{F}(A,\mathrm{E}(X))$ *is a subgroup of* $\mathrm{E}(X)$,

(iv) $\mathrm{J}(\mathrm{F}(A,\mathrm{E}(X)))$ *is a subgroup of* $\mathrm{E}(X)$,

(v) $\mathrm{J}(\mathrm{F}(A,\mathrm{E}(X))) = \{e\}$,

(vi) (X^n,S^n) *is* $A^{n\,\underline{(\overline{\mathrm{M}})}}$ *distal,*

(in these cases $\mathrm{F}(A,\mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X)) \cap \mathrm{F}(A,\mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X)) \cap \overline{\mathrm{F}}(A,\mathrm{E}(X))$ *and* $\mathrm{S}(\mathrm{E}(X)) \cap \mathrm{F}(A,\mathrm{E}(X))$ *are subgroups of* $\mathrm{E}(X)$*),*

(d) *if* $A \in \overline{\overline{\mathcal{M}}}(X,S)$, *then the following statements are equivalent:*

(i) (X,S) *is* $A^{\underline{(\overline{\overline{\mathrm{M}}})}}$ *distal,*

(ii) $\overline{\overline{\mathrm{M}}}(A) = \{\mathrm{E}(X)\}$,

(iii) $\overline{\mathrm{F}}(A,\mathrm{E}(X))$ *is a subgroup of* $\mathrm{E}(X)$,

(iv) $\mathrm{F}(A,\mathrm{E}(X))$ *is a subgroup of* $\mathrm{E}(X)$,

(v) $\mathrm{J}(\mathrm{F}(A,\mathrm{E}(X)))$ $(= \mathrm{J}(\overline{\mathrm{F}}(A,\mathrm{E}(X))))$ *is a subgroup of* $\mathrm{E}(X)$,

(vi) $\mathrm{J}(\mathrm{F}(A,\mathrm{E}(X))) = \{e\}$,

(in these cases $\mathrm{F}(A,\mathrm{E}(X))$, $\overline{\mathrm{F}}(A,\mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X)) \cap \mathrm{F}(A,\mathrm{E}(X))$, $\mathrm{B}(\mathrm{E}(X)) \cap \overline{\mathrm{F}}(A,\mathrm{E}(X))$, $\mathrm{S}(\mathrm{E}(X)) \cap \mathrm{F}(A,\mathrm{E}(X))$ *and* $\mathrm{S}(\mathrm{E}(X)) \cap \overline{\mathrm{F}}(A,\mathrm{E}(X))$ *are subgroups of* $\mathrm{E}(X)$*).*

Proof. (a) (i) and (ii) are equivalent by [2, Proposition 5.3]. Moreover:

$$\begin{aligned} \text{(ii)} &\Longrightarrow \mathrm{E}(X) \text{ is the unique closed right ideal of } \mathrm{E}(X) \\ &\Longrightarrow \forall x \in X,\ \forall K \in \mathrm{M}(x),\ K = \mathrm{E}(X) \\ &\Longrightarrow \forall x \in X,\ \mathrm{E}(X) \in \mathrm{M}(x) \Longrightarrow \text{(iii)} \end{aligned} \tag{1.18}$$

in addition, by Theorem 1.14, (ii) is a corollary of (iv)

(b) (i) and (ii) are equivalent by Definition 1.13. And

$$\begin{aligned} \text{(ii)} &\Longrightarrow \forall a \in A,\ \mathrm{E}(X) \in \mathrm{M}(a) \\ &\Longrightarrow \forall a \in A,\ \forall K \in \mathrm{M}(a),\ K \subseteq \mathrm{E}(X) \wedge \mathrm{E}(X) \in \mathrm{M}(a) \\ &\Longrightarrow \forall a \in A,\ \forall K \in \mathrm{M}(a),\ K = \mathrm{E}(X) \Longrightarrow \text{(iii)}, \\ \text{(iii)} &\Longrightarrow \forall a \in A,\ \mathrm{E}(X) \in \mathrm{M}(a) \wedge e \in \mathrm{J}(\mathrm{F}(a,\mathrm{E}(X))) \\ &\Longrightarrow \forall a \in A,\ \mathrm{F}(a,\mathrm{E}(X))e \text{ is a subgroup of } \mathrm{F}(a,\mathrm{E}(X)) \text{ (Theorem 1.4 (Table 1.1))} \\ &\Longrightarrow \forall a \in A,\ \mathrm{F}(a,\mathrm{E}(X)) \text{ is a subgroup of } \mathrm{E}(X) \Longrightarrow \text{(iv)}, \end{aligned} \tag{1.19}$$

since the set of idempotents of each group is a subgroup of that group, and the unique idempotent of each group is its identity element, (v) follows from (iv) and (vi) follows from (v), in addition:

$$\begin{aligned}
\text{(vi)} &\Longrightarrow \forall a \in A,\ \mathrm{J}(\mathrm{F}(a,\mathrm{E}(X))) = \{e\} \\
&\Longrightarrow \forall a \in A,\ \forall K \in \mathrm{M}(a),\ \mathrm{J}(\mathrm{F}(a,K)) \subseteq \mathrm{J}(\mathrm{F}(a,\mathrm{E}(X))) = \{e\} \\
&\Longrightarrow \forall a \in A,\ \forall K \in \mathrm{M}(a),\ \mathrm{J}(\mathrm{F}(a,K)) = \{e\} \\
&\qquad (\text{since for each } a \in A \text{ and } K \in \mathrm{M}(a),\ \mathrm{J}(\mathrm{F}(a,K)) \neq \emptyset) \\
&\Longrightarrow \forall a \in A,\ \forall K \in \mathrm{M}(a),\ e \in K \\
&\Longrightarrow \forall a \in A,\ \forall K \in \mathrm{M}(a),\ K = \mathrm{E}(X) \\
&\Longrightarrow \forall a \in A,\ \mathrm{E}(X) \in \mathrm{M}(a) \Longrightarrow \text{(ii)}.
\end{aligned} \tag{1.20}$$

On the other hand, since $\mathrm{E}(X^n,S^n) = (\mathrm{E}(X))^n$, and for each $(a_1,\dots,a_n) \in A^n$, $\mathrm{F}((a_1,\dots,a_n),\mathrm{E}(X^n,S^n)) = \prod_{i=1}^n \mathrm{F}(a_i,\mathrm{E}(X))$, thus for each $b \in A^n$, $\mathrm{F}(b,\mathrm{E}(X^n,S^n))$ is a group if and only if for each $a \in A$, $\mathrm{F}(a,\mathrm{E}(X))$ is a group, using this fact and the equivalence of (i) and (v), we have the equivalence of (vii) and (i).

(c) By Theorem 1.10, we have $A \in \overline{\mathcal{M}}(X,S)$ and $\overline{\mathrm{M}}_{(X^n,S^n)}(A^n) = \prod_{i=1}^n \overline{\mathrm{M}}_{(X,S)}(A)$ $(= \{\prod_{i=1}^n K_i \mid \forall i \in \{1,\dots,n\}\ K_i \in \overline{\mathrm{M}}_{(X,S)}(A)\})$, moreover,

$$\begin{aligned}
\text{(i)} &\Longrightarrow \mathrm{E}(X) \in \overline{\mathrm{M}}(A) \\
&\Longrightarrow \forall K \in \overline{\mathrm{M}}(A),\ K \subseteq \mathrm{E}(X) \wedge \mathrm{E}(X) \in \overline{\mathrm{M}}(A) \\
&\Longrightarrow \forall K \in \overline{\mathrm{M}}(A),\ K = \mathrm{E}(X) \Longrightarrow \text{(ii)}, \\
\text{(ii)} &\Longrightarrow \mathrm{E}(X) \in \overline{\mathrm{M}}(A) \wedge e \in \mathrm{J}(\mathrm{F}(A,\mathrm{E}(X))) \\
&\Longrightarrow \mathrm{F}(a,\mathrm{E}(X))e \text{ is a subgroup of } \mathrm{F}(a,\mathrm{E}(X)) \text{ (Theorem 1.4 (Table 1.1))} \\
&\Longrightarrow \mathrm{F}(A,\mathrm{E}(X)) \text{ is a subgroup of } \mathrm{E}(X) \Longrightarrow \text{(iii)},
\end{aligned} \tag{1.21}$$

since the set of idempotents of each group is a subgroup of that group, and the unique idempotent of each group is its identity element, (iv) follows from (iii) and (v) follows from (iv), in addition

$$\begin{aligned}
\text{(v)} &\Longrightarrow \forall K \in \overline{\mathrm{M}}(A) \quad \mathrm{J}(\mathrm{F}(A,K)) \subseteq \mathrm{J}(\mathrm{F}(A,\mathrm{E}(X))) = \{e\} \\
&\Longrightarrow \forall K \in \overline{\mathrm{M}}(A) \quad \mathrm{J}(\mathrm{F}(A,K)) = \{e\} \\
&\qquad (\text{since } A \in \overline{\mathcal{M}}(X,S),\ \text{for each } K \in \overline{\mathrm{M}}(A), \mathrm{J}(\mathrm{F}(A,K)) \neq \emptyset) \\
&\Longrightarrow \forall K \in \overline{\mathrm{M}}(A) \quad e \in K \\
&\Longrightarrow \forall K \in \overline{\mathrm{M}}(A) \quad K = \mathrm{E}(X) \Longrightarrow \text{(i)}.
\end{aligned} \tag{1.22}$$

On the other hand, since $\mathrm{E}(X^n,S^n) = (\mathrm{E}(X))^n$ and $\mathrm{F}(A^n,\mathrm{E}(X^n,S^n)) = (\mathrm{F}(A,\mathrm{E}(X)))^n$, thus $\mathrm{F}(A^n,\mathrm{E}(X^n,S^n))$ is a group if and only if $\mathrm{F}(A,\mathrm{E}(X))$ is a group, using this fact and the equivalence of (i) and (iv), we have the equivalence of (vi) and (i).

(d) Use a similar argument described for (c).

Each part ((b), (c), and (d)) may be extended by using Theorem 1.4 (Table 1.1). □

Note 1.19. Let $A \in \overline{\mathcal{M}}(X,S) \cap \overline{\overline{\mathcal{M}}}(X,S)$, then by Theorem 1.18, (X,S) is $A\underline{\overset{(\overline{\mathrm{M}})}{\quad}}$distal if and only if (X,S) is $A\underline{\overset{(\overline{\overline{\mathrm{M}}})}{\quad}}$distal (you can verify (as an exercise) that $\overline{\mathcal{M}}(X,S) \subseteq \overline{\overline{\mathcal{M}}}(X,S)$!).

Theorem 1.20. *Let A be a nonempty subset of X, then we have the following table:*

TABLE 1.6. The mark $\surd$ indicates that for the corresponding case if (X,S) is $A^{\underline{(\alpha)}}$distal, then (X,S) is $A^{\underline{(\beta)}}$distal.

β / α	$-$	$\overline{\mathrm{M}}$	$\overline{\overline{\mathrm{M}}}$
$-$	$\surd$	$\surd$	
$\overline{\mathrm{M}}$		$\surd$	
$\overline{\overline{\mathrm{M}}}$		$\surd$	$\surd$

PROOF. Let (X,S) be A-distal, $a \in A$ and $K \in \overline{\mathrm{M}}(A)$, then $aK = a\mathrm{E}(X)$ and by Corollary 1.3(a), there exists $L \in \mathrm{M}(a)$, such that $L \subseteq K$. By Theorem 1.18, the only choice for L is $\mathrm{E}(X)$, so $\mathrm{E}(X) = K \in \overline{\mathrm{M}}(A)$ and (X,S) is $A^{\underline{(\overline{\mathrm{M}})}}$distal.

Let (X,S) be $A^{\underline{(\overline{\overline{\mathrm{M}}})}}$distal and $K \in \overline{\mathrm{M}}(A)$, then for each $a \in A$, $aK = a\mathrm{E}(X)$ and $AK = A\mathrm{E}(X)$. Since $\mathrm{E}(X) \in \overline{\overline{\mathrm{M}}}(A)$, we have $\mathrm{E}(X) = K \in \overline{\mathrm{M}}(A)$ so (X,S) is $A^{\underline{(\overline{\mathrm{M}})}}$distal. □

THEOREM 1.21. *Let $\{(X_\alpha,S)\}_{\alpha\in\Gamma}$ be a nonempty collection of transformation semigroups and for each $\alpha\in\Gamma$, let A_α be a nonempty subset of X_α, then we have*

(a) *if for each $\alpha\in\Gamma$, (X_α,S) is distal, then $(\prod_{\alpha\in\Gamma}X_\alpha,S)$ is distal,*

(b) *if for each $\alpha\in\Gamma$, (X_α,S) is A_α-distal, then $(\prod_{\alpha\in\Gamma}X_\alpha,S)$ is $\prod_{\alpha\in\Gamma}A_\alpha$-distal,*

(c) *if for each $\alpha\in\Gamma$, (X_α,S) is $A_\alpha{}^{\underline{(\overline{\mathrm{M}})}}$distal, and $\prod_{\alpha\in\Gamma}A_\alpha \in \overline{\mathcal{M}}(\prod_{\alpha\in\Gamma}X_\alpha,S)$, then $(\prod_{\alpha\in\Gamma}X_\alpha,S)$ is $\prod_{\alpha\in\Gamma}A_\alpha{}^{\underline{(\overline{\mathrm{M}})}}$distal,*

(d) *if for each $\alpha\in\Gamma$, (X_α,S) is $A_\alpha{}^{\underline{(\overline{\overline{\mathrm{M}}})}}$distal, and $\prod_{\alpha\in\Gamma}A_\alpha \in \overline{\overline{\mathcal{M}}}(\prod_{\alpha\in\Gamma}X_\alpha,S)$, then $(\prod_{\alpha\in\Gamma}X_\alpha,S)$ is $\prod_{\alpha\in\Gamma}A_\alpha{}^{\underline{(\overline{\mathrm{M}})}}$distal.*

PROOF. (b) Let $(a_\alpha)_{\alpha\in\Gamma} \in \prod_{\alpha\in\Gamma}A_\alpha$, $u \in \mathrm{J}(\mathrm{F}((a_\alpha)_{\alpha\in\Gamma},\mathrm{E}(\prod_{\alpha\in\Gamma}X_\alpha)))$, and $(s_\omega)_{\omega\in\Omega}$ be a net in S converging to u in $\mathrm{E}(\prod_{\alpha\in\Gamma}X_\alpha)$, then for each $\alpha\in\Gamma$, $(s_\omega)_{\omega\in\Omega}$ is a convergent net in $\mathrm{E}(X_\alpha)$ and $\lim_{\omega\in\Omega}s_\omega \in \mathrm{J}(\mathrm{F}(a_\alpha,\mathrm{E}(X_\alpha)))$. Since (X_α,S) is A_α-distal, by Theorem 1.18, $\lim_{\omega\in\Omega}s_\omega = e$ (in $\mathrm{E}(X_\alpha)$), thus for each $x_\alpha \in X_\alpha$, $\lim_{\omega\in\Omega}x_\alpha s_\omega = x_\alpha e = x_\alpha$ and for each $(x_\alpha)_{\alpha\in\Gamma} \in \prod_{\alpha\in\Gamma}X_\alpha$, $\lim_{\omega\in\Omega}(x_\alpha)_{\alpha\in\Gamma}s_\omega = \lim_{\omega\in\Omega}(x_\alpha s_\omega)_{\alpha\in\Gamma} = (x_\alpha)_{\alpha\in\Gamma}$, that is, $u = \lim_{\omega\in\Omega}s_\omega = e$ and $\mathrm{J}(\mathrm{F}((a_\alpha)_{\alpha\in\Gamma},\mathrm{E}(\prod_{\alpha\in\Gamma}X_\alpha))) = \{e\}$. So by Theorem 1.18, $(\prod_{\alpha\in\Gamma}X_\alpha,S)$ is $\prod_{\alpha\in\Gamma}A_\alpha$-distal.

To prove (c) and (d), use an argument similar to the one given for (b). □

NOTE 1.22. Let (X_i,S_i) be a transformation semigroup for each $i\in\{1,\dots,n\}$ and A_i, B_i be nonempty subsets of X_i such that $\prod_{i=1}^n A_i$, $\prod_{i=1}^n B_i \in \overline{\mathcal{M}}(\prod_{i=1}^n X_i,\prod_{i=1}^n S_i)$, then

(a) $(\prod_{i=1}^n X_i,\prod_{i=1}^n S_i)$ is $\prod_{i=1}^n A_i{}^{\underline{(\overline{\mathrm{M}})}}$distal if and only if for each $i\in\{1,\dots,n\}$ (X_i,S_i) is $A_i{}^{\underline{(\overline{\mathrm{M}})}}$distal,

(b) $\prod_{i=1}^n B_i$ is $\prod_{i=1}^n A_i{}^{\underline{(\overline{\mathrm{M}},\overline{\mathrm{M}})}}$ almost periodic if and only if for each $i\in\{1,\dots,n\}$ B_i is $A_i{}^{\underline{(\overline{\mathrm{M}},\overline{\mathrm{M}})}}$ almost periodic.

PROOF. By [4, Lemma 7], we have $\mathrm{E}(\prod_{i=1}^n X_i,\prod_{i=1}^n S_i) = \prod_{i=1}^n \mathrm{E}(X_i,S_i)$, by Theorem 1.10, $\overline{\mathrm{M}}_{(\prod_{i=1}^n X_i,\prod_{i=1}^n S_i)}(\prod_{i=1}^n A_i) = \prod_{i=1}^n \overline{\mathrm{M}}_{(X_i,S_i)}(A_i)$ and $\overline{\mathrm{M}}_{(\prod_{i=1}^n X_i,\prod_{i=1}^n S_i)}(\prod_{i=1}^n B_i) = \prod_{i=1}^n \overline{\mathrm{M}}_{(X_i,S_i)}(B_i)$. Now Theorem 1.18, leads to the desired result. □

Theorem 1.23. *Let Z be a closed invariant subset of X and $\emptyset \neq A \subseteq Z$, then*

(a) $\mathrm{E}(Z,S) = \{p\,|_Z : p \in \mathrm{E}(X,S)\}$,

(b) $\mathrm{F}(A,\mathrm{E}(Z)) = \{p\,|_Z : p \in \mathrm{F}(A,\mathrm{E}(X))\}$,

(c) *if (X,S) is A-distal, then (Z,S) is A-distal,*

(d) *if (X,S) is $A^{\underline{(\overline{\mathrm{M}})}}$ distal and $A \in \overline{\mathcal{M}}(Z,S)$, then (Z,S) is $A^{\underline{(\overline{\mathrm{M}})}}$ distal,*

(e) *if (X,S) is $A^{\underline{(\overline{\overline{\mathrm{M}}})}}$ distal and $A \in \overline{\overline{\mathcal{M}}}(Z,S)$, then (Z,S) is $A^{\underline{(\overline{\overline{\mathrm{M}}})}}$ distal.*

Proof. (a) and (b) are clear.

(c) Let (X,S) be A-distal, by Theorem 1.18, for each $a \in A$, $\mathrm{F}(a,\mathrm{E}(X))$ is a group and by (b), $\mathrm{F}(a,\mathrm{E}(Z)) = \{p\,|_Z : p \in \mathrm{F}(a,\mathrm{E}(X))\}$ is a group. By Theorem 1.18, (Z,S) is A-distal.

To prove (d) and (e), use an argument similar to the one given for (c). □

Note 1.24. Let Z be a closed invariant subset of X and $\emptyset \neq A \subseteq Z$, then

(a) for each $K \in \overline{\mathrm{M}}_{(X,S)}(A)$, there exists $L \in \overline{\mathrm{M}}_{(Z,S)}(A)$, such that $L \subseteq K|_Z (= \{p\,|_Z : p \in K\})$,

(b) let $A \in \overline{\mathcal{M}}(Z,S)$, then $A \in \overline{\mathcal{M}}(X,S)$ and $\overline{\mathrm{M}}_{(Z,S)}(A) = \{K\,|_Z : K \in \overline{\mathrm{M}}_{(X,S)}(A)\}$.

Proof. (a) If $K \in \overline{\mathrm{M}}_{(X,S)}(A)$, then for each $a \in A$ we have $aK = a\mathrm{E}(X)$ and $a\mathrm{E}(Z) = a(\mathrm{E}(X)|_Z) = a\mathrm{E}(X) = aK = a(K|_Z)$ ($K|_Z$ is a closed right ideal of $\mathrm{E}(Z)$), so by Corollary 1.3(b), there exists $L \in \overline{\mathrm{M}}_{(Z,S)}(A)$ such that $L \subseteq K|_Z$.

(b) Let $K \in \overline{\mathrm{M}}_{(X,S)}(A)$, by (a), there exists $L \in \overline{\mathrm{M}}_{(Z,S)}(A)$, such that $L \subseteq K|_Z$, let $p \in \mathrm{F}(A,L)$, and choose $q \in \mathrm{F}(A,K)$ such that $p = q|_Z$, by Corollary 1.5 Table 1.2 $p \in \mathrm{S}(L)$ and $q \in \mathrm{S}(K)$, thus $L = p\mathrm{E}(Z) = (q|_Z)(\mathrm{E}(X)|_Z) = (q\mathrm{E}(X))|_Z = K|_Z$. Thus $\{K|_Z : K \in \overline{\mathrm{M}}_{(X,S)}(A)\} \subseteq \overline{\mathrm{M}}_{(Z,S)}(A)$. On the other hand, if $L \in \overline{\mathrm{M}}_{(Z,S)}(A)$ and $p \in \mathrm{F}(A,L)$, choose $q \in \mathrm{F}(A,\mathrm{E}(X))$ such that $q|_Z = p$, moreover, for each $a \in A$, $a(q\mathrm{E}(X)) = a\mathrm{E}(X)$, by Corollary 1.3(b), there exists $K \in \overline{\mathrm{M}}_{(X,S)}(A)$ such that $K \subseteq q\mathrm{E}(X)$, by the last argument $K|_Z \in \overline{\mathrm{M}}_{(Z,S)}(A)$, since $L = p\mathrm{E}(Z) = (q|_Z)(\mathrm{E}(X)|_Z) = (q\mathrm{E}(X))|_Z \supseteq K|_Z$ and $L, K|_Z \in \overline{\mathrm{M}}_{(Z,S)}(A)$ we have $L = K|_Z$. Thus $\overline{\mathrm{M}}_{(Z,S)}(A) \subseteq \{K\,|_Z : K \in \overline{\mathrm{M}}_{(X,S)}(A)\}$. □

Corollary 1.25. *Let A and B be nonempty subsets of X, then we have Table 1.7.*

Table 1.7. In the corresponding case we have: "B is $A^{\underline{(\mathrm{Q},\mathrm{R})}}$ almost periodic and A is $B^{\underline{(\mathrm{R},\mathrm{Q})}}$ almost periodic if and only if $\pi(\mathrm{Q},\mathrm{R},B,A)$" and "if $\pi(\mathrm{Q},\mathrm{R},B,A)$, then $\vartheta(\mathrm{Q},\mathrm{R},B,A)$."

Q	R	$\pi(\mathrm{Q},\mathrm{R},B,A)$	$\vartheta(\mathrm{Q},\mathrm{R},B,A)$
–	–	$\forall b \in B,\ \forall a \in A,\ \mathrm{M}(b) = \mathrm{M}(a)$	$(\forall b \in B,\ \mathrm{M}(b) = \overline{\mathrm{M}}(A)) \wedge (\forall a \in A,\ \mathrm{M}(a) = \overline{\mathrm{M}}(B))$ $\wedge \overline{\mathrm{M}}(B) = \overline{\mathrm{M}}(A)$
–	$\overline{\mathrm{M}}$	$\forall b \in B,\ \mathrm{M}(b) = \overline{\mathrm{M}}(A)$	$\overline{\mathrm{M}}(B) = \overline{\mathrm{M}}(A)$
–	$\overline{\overline{\mathrm{M}}}$	$\forall b \in B,\ \mathrm{M}(b) = \overline{\overline{\mathrm{M}}}(A)$	$\overline{\mathrm{M}}(B) = \overline{\overline{\mathrm{M}}}(A)$
$\overline{\mathrm{M}}$	$\overline{\mathrm{M}}$	$\overline{\mathrm{M}}(B) = \overline{\mathrm{M}}(A)$	–
$\overline{\mathrm{M}}$	$\overline{\overline{\mathrm{M}}}$	$\overline{\mathrm{M}}(B) = \overline{\overline{\mathrm{M}}}(A)$	–
$\overline{\overline{\mathrm{M}}}$	$\overline{\overline{\mathrm{M}}}$	$\overline{\overline{\mathrm{M}}}(B) = \overline{\overline{\mathrm{M}}}(A)$	–

Proof. First row. We have ((B is $A\frac{(-,-)}{}$almost periodic) $\wedge$ (A is $B\frac{(-,-)}{}$almost periodic))

$$\begin{aligned}
&\Longleftrightarrow \forall a \in A,\ \forall b \in B,\\
&\qquad ((\forall K \in \mathrm{M}(a),\ \exists L \in \mathrm{M}(b),\ L \subseteq K) \wedge (\forall L \in \mathrm{M}(b),\ \exists K \in \mathrm{M}(a),\ K \subseteq L))\\
&\Longleftrightarrow \forall a \in A,\ \forall b \in B,\ \forall K \in \mathrm{M}(a),\ \forall L \in \mathrm{M}(b),\\
&\qquad ((\exists L' \in \mathrm{M}(b),\ \exists K' \in \mathrm{M}(a),\ K' \subseteq L' \subseteq K) \wedge (\exists K' \in \mathrm{M}(a),\ \exists L' \in \mathrm{M}(b),\ L \subseteq K' \subseteq L))\\
&\Longleftrightarrow \forall a \in A,\ \forall b \in B,\ \forall K \in \mathrm{M}(a),\ \forall L \in \mathrm{M}(b),\\
&\qquad\qquad ((\exists L' \in \mathrm{M}(b)\ L' = K) \wedge (\exists K' \in \mathrm{M}(a)\ K' = L))\\
&\Longleftrightarrow \forall a \in A,\ \forall b \in B,\ \mathrm{M}(a) \subseteq \mathrm{M}(b) \wedge \mathrm{M}(b) \subseteq \mathrm{M}(a)\\
&\Longleftrightarrow \forall a \in A,\ \forall b \in B,\ \mathrm{M}(a) = \mathrm{M}(b),
\end{aligned} \tag{1.23}$$

moreover, suppose for each $a \in A$ and $b \in B$ we have $\mathrm{M}(b) = \mathrm{M}(a)$, then if $a \in A$ and $K \in \mathrm{M}(a)$, for each $a' \in A$ we have $K \in \mathrm{M}(a')$. So for each $a' \in A$ we have $a'K = a'\mathrm{E}(X)$ and $K \in \overline{\mathrm{M}}(A)$ (K does not have any proper subset like L, such that L is a closed right ideal of $\mathrm{E}(X)$ and $aL = a\mathrm{E}(X)$). Therefore, $\mathrm{M}(a) \subseteq \overline{\mathrm{M}}(A)$, on the other hand, if $K \in \overline{\mathrm{M}}(A)$, by Corollary 1.3, there exists $L \in \mathrm{M}(a)$, such that $L \subseteq K$, moreover, by the above argument we have $L \in \overline{\mathrm{M}}(A)$ and $L = K$, thus $\overline{\mathrm{M}}(A) \subseteq \mathrm{M}(a)$. Therefore $\overline{\mathrm{M}}(A) = \mathrm{M}(a)$, now let $b \in B$, a similar argument will show $\overline{\mathrm{M}}(B) = \mathrm{M}(b)$ and $\overline{\mathrm{M}}(A) = \overline{\mathrm{M}}(B)$.

For the other rows use similar methods. □

Remark 1.26. Let $\Gamma_1 = \{\overline{\mathrm{M}}(A) \mid A \subseteq X,\ A \neq \varnothing\}$ and $\Gamma_2 = \{\overline{\overline{\mathrm{M}}}(A) \mid A \subseteq X,\ A \neq \varnothing, \overline{\overline{\mathrm{M}}}(A) \neq \varnothing\}$.

For each nonempty subsets A and B of X define

$$\overline{\mathrm{M}}(A) \leq_1 \overline{\mathrm{M}}(B) \quad \text{if and only if} \quad A \text{ is } B\frac{(\overline{\mathrm{M}},\overline{\mathrm{M}})}{}\text{almost periodic}, \tag{1.24}$$

for each nonempty subsets A and B of X, such that $\overline{\overline{\mathrm{M}}}(A)$ and $\overline{\overline{\mathrm{M}}}(B)$ are nonempty define

$$\overline{\overline{\mathrm{M}}}(A) \leq_2 \overline{\overline{\mathrm{M}}}(B) \quad \text{if and only if} \quad A \text{ is } B\frac{(\overline{\overline{\mathrm{M}}},\overline{\overline{\mathrm{M}}})}{}\text{almost periodic}, \tag{1.25}$$

then

(a) $(\Gamma_1, \leq_1)$ and $(\Gamma_2, \leq_2)$ are partially ordered sets,

(b) for each nonempty subset A of X:

  (i) if (X,S) is $A\frac{(\overline{\mathrm{M}})}{}$ distal, then $\overline{\mathrm{M}}(A) = \{\mathrm{E}(X)\}$ is the maximum element in $(\Gamma_1, \leq_1)$,

  (ii) if (X,S) is $A\frac{(\overline{\overline{\mathrm{M}}})}{}$ distal, then $\overline{\overline{\mathrm{M}}}(A) = \{\mathrm{E}(X)\}$ is the maximum element in $(\Gamma_2, \leq_2)$,

(c) $\mathrm{Min}(\mathrm{E}(X))$ is the minimum element in $(\Gamma_1, \leq_1)$ and $(\Gamma_2, \leq_2)$ (if A is a nonempty subset of X such that all of its elements are almost periodic, then by Note 1.12, $\overline{\overline{\mathrm{M}}}(A) = \overline{\mathrm{M}}(A) = \mathrm{Min}(\mathrm{E}(X))$).

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References

[1] R. Ellis, *A semigroup associated with a transformation group*, Trans. Amer. Math. Soc. **94** (1960), 272–281. MR 23#A961. Zbl 094.17402.

[2] ______, *Lectures on Topological Dynamics*, W. A. Benjamin, New York, 1969. MR 42#2463. Zbl 193.51502.

[3] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, New Jersey, 1981, M. B. Porter Lectures. MR 82j:28010. Zbl 459.28023.

[4] M. Sabbaghan and F. A. Z. Shirazi, *Idempotents and finite product associated with a-minimal sets*, J. Sci. Univ. Tehran Int. Ed. **3** (1998), no. 2, 115–121. MR 2000c:54031.

[5] M. Sabbaghan, F. A. Z. Shirazi, and Ta-Sun Wu, *a-minimal sets*, J. Sci. Univ. Tehran Int. Ed. **2** (1997), no. 1, 1–12. MR 99a:54027.

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