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TOPOLOGICAL CONJUGACIES OF PIECEWISE MONOTONE INTERVAL MAPS

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Abstract. Our aim is to establish the topological conjugacy between piecewise monotone expansive interval maps and piecewise linear maps. First, we are concerned with maps satisfying a Markov condition and next with those admitting a certain countable partition. Finally, we compute the topological entropy in the Markov case.

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1. Introduction and preliminaries. Let I be a closed interval in $\mathbb{R}$, which is usually taken to be the interval $[0,1]$, and $f : I \to I$ a mapping. The iterates of f are the maps f^n defined inductively by $f^0 = \mathrm{id}_{\mathbb{R}}$, $f^1 = f$, $f^{n+1} = f^n \circ f$. The (forward or positive) *orbit* of a point $x \in I$ is the set $O(x) = \{f^n(x) : n \in \mathbb{N}\}$. The *ω-limit* set of x is the set of the limit points of $O(x)$ and is denoted by $\omega(x)$. Two maps $f : I \to I$ and $g : J \to J$ (J a closed interval in $\mathbb{R}$) are called *topologically conjugate* if there exists a homeomorphism $h : I \to J$ such that $h \circ f = g \circ h$.

The study of topological conjugacies has commenced with Poincaré in the 1880s. He considered homeomorphisms $f : S^1 \to S^1$ of the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ with no periodic points and showed that there exist a rotation $R : S^1 \to S^1$ and a continuous, surjective and monotone map $h : S^1 \to S^1$ such that $h \circ f = R \circ h$, that is, f and R are *topologically semiconjugate*. Similar results for piecewise monotone interval maps f were proved later by Parry [10] and Milnor and Thurston [9]. According to them, if $f : I \to I$ is continuous, piecewise monotone with positive topological entropy $h(f)$, then there exists a piecewise linear map $T : [0,1] \to [0,1]$ with slope $\pm\exp(h(f))$ such that f, T are topologically semiconjugate. f and T become topologically conjugate, if there are no attracting periodic points and no wandering intervals for f. The nonexistence of wandering intervals has been proved for a large class of functions satisfying some mild smoothness conditions (see [3, 6, 7, 8]).

In this paper, we consider the family $\mathcal{M}$ of functions which are piecewise monotone (but not necessarily continuous) and expansive. Particularly, $f : [0,1] \to [0,1]$ belongs to the family $\mathcal{M}$ if there exists a partition $0 = a_0 < a_1 < \cdots < a_r = 1$ ($r \geq 2$) of $[0,1]$ such that $f \mid [a_{i-1}, a_i]$ ($i = 1,2,\dots,r$) is a monotone C^1 function and satisfy the following Markov condition: for every $i = 1,2,\dots,r$, there exist $p(i), q(i) \in \{0,1,\dots,r\}$ with $p(i) < q(i)$ such that $f(a_{i-1}, a_i) = (a_{p(i)}, a_{q(i)})$. Furthermore, we assume that there is $\lambda > 1$ such that $|f'(x)| \geq \lambda$, for almost every $x \in [0,1]$, in which case, f is called *expansive*. Our aim is to show that every $f \in \mathcal{M}$ is topologically conjugate to a map T which is linear on each interval $[(i-1)/r, i/r]$ ($i = 1,2,\dots,r$). Next, we consider the class $\mathcal{M}_\infty$ where $[0,1]$ accepts a countable partition accumulating to 1. Finally, in the last section, we compute the topological entropy for continuous maps in $\mathcal{M}$.

Notation. If $J \subset [0,1]$ is an interval, we denote $|J|$ its length.

2. Topological conjugacies for maps in $\mathcal{M}$. In this section, we study the topological conjugacies for maps $f \in \mathcal{M}$. If $0 = a_0 < a_1 < \cdots < a_r = 1$ is the partition corresponding to f, we say that f is of *order* r. The points of the partition are called *critical points* of f. We denote by $I_1,\dots,I_r$ the intervals of the partition, that is, $I_j = (a_{j-1}, a_j)$. We assume that these intervals are maximal in the sense that if I is an interval which strictly contains one of them, then $f \mid I$ is neither continuous nor monotone. Also, we denote by f_j the restriction of f to I_j. Finally, we denote by $F_{j_1 j_2 \cdots j_k}$ the composition $f_{j_1}^{-1} \circ f_{j_2}^{-1} \circ \cdots \circ f_{j_k}^{-1}$. Note that $F_{j_1 j_2 \cdots j_k}$ is not necessarily defined for every (finite) sequence $j_1 j_2 \cdots j_k$. Moreover, $F_{j_1 j_2 \cdots j_k}(x)$ is the unique point $y \in I_{j_1}$ such that $f(y) \in I_{j_2}, \dots, f^{k-1}(y) \in I_{j_k}$ and $f^k(y) = x$.

An open interval $J \subset [0,1]$ is called a *branch* of f^n if $f^n \mid J$ is continuous, monotone and J is maximal with these properties. The set of branches of f^n is denoted by $B_n(f)$. Moreover, we define the sets

$$\begin{aligned} \mathcal{C}_n(f) &= \bigcup_{j=0}^{r} \bigcup_{i=0}^{n-1} f^{-i}(a_j), \quad n = 1,2,\dots, \\ \mathcal{C}(f) &= \bigcup_{j=0}^{r} \bigcup_{i=0}^{\infty} f^{-i}(a_j). \end{aligned} \tag{2.1}$$

Frequently, we write $\mathcal{C}_n$ and $\mathcal{C}$ instead of $\mathcal{C}_n(f)$ and $\mathcal{C}(f)$.

In what follows, we introduce some notions from symbolic dynamics. To each point x of $\mathcal{C}$, there corresponds a sequence of symbols which is related with the order of the points of $O(x)$.

Definition 2.1. The *itinerary* of $x \in \mathcal{C}$ with respect to $f \in \mathcal{M}$ is a sequence $\underline{i}_f(x) = \{i_n(x)\}_{n=0}^{\infty}$, where

$$i_n(x) = \begin{cases} j, & \text{if } f^n(x) \in I_j, \\ \dfrac{2j+1}{2}, & \text{if } f^n(x) = a_j. \end{cases} \tag{2.2}$$

An interesting notion in symbolic dynamics is the *shift map* σ: if $\underline{x} = \{x_n\}_{n=0}^{\infty}$, then $\sigma(\underline{x}) = \underline{y}$, where $\underline{y} = \{x_n\}_{n=1}^{\infty}$. Inductively, we have $\sigma^k(\underline{x}) = \{x_n\}_{n=k}^{\infty}$. To each $f \in \mathcal{M}$ of order r, we associate a subset of $\{1/2, 1, 3/2, \dots, r, (2r+1)/2\}^{\mathbb{N}}$. We describe this set in the following definition.

Definition 2.2. Let $f \in \mathcal{M}$ with partition $0 = a_0 < a_1 < \cdots < a_r = 1$. We define the set of sequences $\Sigma(f) = \{\underline{a} : \underline{a} = \{x_n\}_{n=0}^{\infty}\}$ with entries from the set $\{1/2, 1, 3/2, \dots, r, (2r+1)/2\}$, which satisfy the following conditions:

(i) Let $\underline{a} = \{x_n\} \in \Sigma(f)$. Then there exists an entry x_n of $\underline{a}$ of the form $(2k+1)/2$, where $k = 0,1,\dots,r$. Furthermore, if x_N is the first entry of $\underline{a}$ with this property, then $\sigma^N(\underline{a}) = \underline{i}_f(a_k)$.

(ii) If $n < N-1$ and $x_n = j$, then $p(j)+1 \le x_{n+1} \le q(j)$.

It is possible to define an order on the set $\underline{i}_f(\mathscr{C})$ which is consistent with the natural order of real numbers. Two sequences of symbols $\underline{x} = \{x_n\}_{n=0}^{\infty}$ and $\underline{y} = \{y_n\}_{n=0}^{\infty}$ belonging to $\{1/2, 1, 3/2, \dots, r, (2r+1)/2\}^{\mathbb{N}}$ are called to have *discrepancy* n if $x_i = y_i$, for $i = 0, 1, \dots, n-1$, and $x_n \neq y_n$. If the itineraries of two points of $\mathscr{C}$ have discrepancy n, then the first n points of their orbits are visiting simultaneously the same intervals of $B_1(f)$. Moreover, we define $1/2 \prec 1 \prec 3/2 \prec \cdots \prec r \prec (2r+1)/2$.

Definition 2.3. Let $f \in \mathcal{M}$ and $x, y \in \mathscr{C}$ with $x \neq y$. We assume that itineraries $\underline{i}_f(x)$ and $\underline{i}_f(y)$ have discrepancy n and that f is decreasing in k common intervals.

(i) When k is even, then $\underline{i}_f(x) \prec \underline{i}_f(y)$ if and only if $i_n(x) \prec i_n(y)$.

(ii) When k is odd, then $\underline{i}_f(x) \prec \underline{i}_f(y)$ if and only if $i_n(y) \prec i_n(x)$.

Lemma 2.4. *Let $f \in \mathcal{M}$ be of order r and let $x, y \in \mathscr{C}$ with $x \neq y$. Then $\underline{i}_f(x) \prec \underline{i}_f(y)$ if and only if $x < y$.*

Proof. We assume that itineraries $\underline{i}_f(x)$ and $\underline{i}_f(y)$ have discrepancy n. That is, $i_k(x) = i_k(y) = j_k$, for $k = 0, 1, \dots, n-1$, and $i_n(x) \neq i_n(y)$. We claim that $j_0, j_1, \dots, j_{n-1}$ are not of the form $(2s+1)/2$. To prove this, we assume the contrary, whence $\underline{i}_f(x) = \underline{i}_f(y)$, which is a contradiction, since $i_n(x) \neq i_n(y)$. From Definition 2.1, x, y belong to I_{j_0} and successively visit the intervals $I_{j_1}, \dots, I_{j_{n-1}}$. So, we can write $x = F_{j_0 j_1 \cdots j_{n-1}}(f^n(x))$ and $y = F_{j_0 j_1 \cdots j_{n-1}}(f^n(y))$. We assume that f is decreasing in k intervals among $I_{j_0}, I_{j_1}, \dots, I_{j_{n-1}}$. There are two cases.

(i) When k is even, then $F_{j_0 j_1 \cdots j_{n-1}}$ is increasing. Assume that $\underline{i}_f(x) \prec \underline{i}_f(y)$, then from Definition 2.3 we have $i_n(x) \prec i_n(y)$. This means that $f^n(x) < f^n(y)$ and, hence, $x = F_{j_0 j_1 \cdots j_{n-1}}(f^n(x)) < y = F_{j_0 j_1 \cdots j_{n-1}}(f^n(y))$.

(ii) When k is odd, then $F_{j_0 j_1 \cdots j_{n-1}}$ is decreasing. Assume that $\underline{i}_f(x) \prec \underline{i}_f(y)$, then from Definition 2.3 we have $i_n(y) \prec i_n(x)$. This means that $f^n(x) > f^n(y)$ and, hence, $x = F_{j_0 j_1 \cdots j_{n-1}}(f^n(x)) < y = F_{j_0 j_1 \cdots j_{n-1}}(f^n(y))$. □

Lemma 2.5. *Let $f \in \mathcal{M}$ be of order r. The map $\underline{i}_f : \mathscr{C} \to \Sigma(f)$ is a bijection.*

Proof. Let $x, y \in \mathscr{C}$ with $\underline{i}_f(x) = \underline{i}_f(y)$. Let k, m be the minimal integers for which $f^k(x)$, $f^m(y)$ are critical points of f. Assume that $k \neq m$ (let $k < m$). Since $f^k(x)$ is a critical point, then $f^{k+1}(x) = 0$ or 1, and, so, $i_{k+1}(x) = 1/2$ or $(2r+1)/2$. On the other hand, $i_k(y) = 1, 2, \dots, r$, and, hence, $i_{k+1}(y) \neq 1/2$ and $i_{k+1}(y) \neq (2r+1)/2$, which is a contradiction, since $i_{k+1}(x) = i_{k+1}(y)$. So, $k = m$. Furthermore, we observe that $f^k(x) = f^k(y)$, since $i_k(x) = i_k(y)$ and it is of the form $(2j+1)/2$. Consequently, $f^k(x) = f^k(y) = a_j$.

Assume that $i_n(x) = i_n(y) = j_n \in \mathbb{N}$, for $n = 0, 1, \dots, k-1$. From Definition 2.1, x, y belong to I_{j_0} and successively visit the intervals $I_{j_1}, \dots, I_{j_{k-1}}$. So, we can write $x = F_{j_0 j_1 \cdots j_{k-1}}(f^k(x))$ and $y = F_{j_0 j_1 \cdots j_{k-1}}(f^k(y))$. Since $f^k(x) = f^k(y)$, we have $x = y$. Thus, $\underline{i}_f$ is injective.

Let $\underline{a} = \{x_n\} \in \Sigma(f)$. We shall show that there exists an $x \in \mathscr{C}$ such that $\underline{i}_f(x) = \underline{a}$. From Definition 2.2, an entry of the sequence $\underline{a}$ is of the form $(2k+1)/2$. Let x_n be the first entry with this property. Then $x = F_{x_0 x_1 \cdots x_{n-1}}(a_k)$ satisfies the desired property. □

Proposition 2.6. *Let $f \in \mathcal{M}$ be of order r. Then $\mathcal{C}$ is dence in $[0,1]$.*

Proof. Let $\tilde{J} \subset [0,1]$ be an open interval such that $\tilde{J} \cap \mathcal{C} = \emptyset$. First, we show that $f^n(\tilde{J}) \cap \mathcal{C} = \emptyset$, for $n \in \mathbb{N}$. We assume, in the contrary, that there exists $x \in f^n(\tilde{J}) \cap \mathcal{C}$, then there is $y \in \tilde{J}$ such that $x = f^n(y)$. But, $f^m(x) = a_k$, for some $m \in \mathbb{N}$ and $k = 0,1,2,\dots,r$, since $x \in \mathcal{C}$. So, $f^{m+n}(y) = f^m(x) = a_k$, that is, $y \in \mathcal{C}$, which is a contradiction, since $\tilde{J} \cap \mathcal{C} = \emptyset$.

As $f^n(\tilde{J}) \cap \mathcal{C} = \emptyset$, for $n \in \mathbb{N}$, it turns out that f is monotone and C^1 on each interval $\tilde{J}, f(\tilde{J}), f^2(\tilde{J}), \dots$.

We prove by induction that $|f^n(\tilde{J})| \geq \lambda^n |\tilde{J}|$, for $n \geq 1$. From the mean value theorem and since $f|\tilde{J}$ is monotone, we have $|f(\tilde{J})|/|\tilde{J}| = |f'(a)|$, for some $a \in \tilde{J}$. But, $|f'(a)| \geq \lambda$ and, hence $|f(\tilde{J})| \geq \lambda |\tilde{J}|$. We assume that the claim is true for $k < n$. From the mean value theorem and since $f|f^{n-1}(\tilde{J})$ is monotone, we have $|f^n(\tilde{J})|/|f^{n-1}(\tilde{J})| = |f'(a_1)| \geq \lambda$, for some $a_1 \in f^{n-1}(\tilde{J})$. From the induction assumption, we have $|f^{n-1}(\tilde{J})| \geq \lambda^{n-1}|\tilde{J}|$. Combining the last two inequalities, we have $|f^n(\tilde{J})| \geq \lambda^n |\tilde{J}|$.

Thus, for some $n \in \mathbb{N}$, $\lambda^n |\tilde{J}| > 1$, which is a contradiction, since $|f^n(\tilde{J})| \leq 1$. □

Theorem 2.7. *Let $f \in \mathcal{M}$ be of order r with partition $0 = a_0 < a_1 < \cdots < a_r = 1$. We consider the map $T \in \mathcal{M}$ with partition $0 < 1/r < 2/r < \cdots < (r-1)/r < 1$ which is linear in each interval $[(i-1)/r, i/r]$ and $T((i-1)/r, i/r) = (p(i)/r, q(i)/r)$. Furthermore, $T|[(i-1)/r, i/r]$ is of the same monotonicity type with $f|[a_{i-1}, a_i]$ and it is continuous, from the right or from the left at i/r, when f is continuous, from the right or from the left at a_i, respectively. Then f and T are topologically conjugate. (Figure 2.1)*

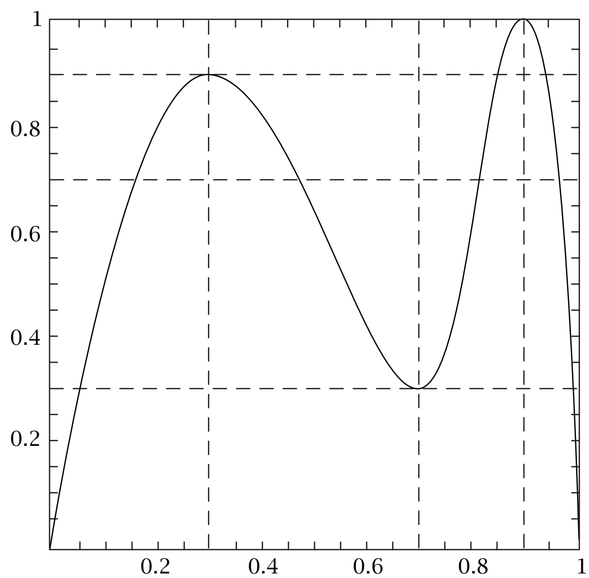

Figure 2.1.

Proof. From Definition 2.2, we have $\Sigma(f) = \Sigma(T)$. With this observation and since $\underline{i}_f$ and $\underline{i}_T$ are bijections (Lemma 2.5), we can define a correspondence $h : \mathcal{C}(f) \to$

$\mathscr{C}(T)$, which is an order preserving bijection and such that $h \circ f = T \circ h$. For $x \in \mathscr{C}(f)$, we define $h(x)$ to be the unique element of $\mathscr{C}(T)$, for which $\underline{i}_f(x) = \underline{i}_T(h(x))$. Equivalently, $h = \underline{i}_T^{-1} \circ \underline{i}_f$. But since $\underline{i}_f$ and $\underline{i}_T$ are bijections, we have that h is also a bijection. From Lemma 2.4, $\underline{i}_f$ and $\underline{i}_T$ are order preserving maps and, so, the same holds for h.

Let $x \in \mathscr{C}(f)$. We shall show that $h \circ f(x)$ and $T \circ h(x)$ have the same itinerary with respect to T. Indeed,

$$\underline{i}_T(h(f(x))) = \underline{i}_f(f(x)) = \sigma(\underline{i}_f(x)). \tag{2.3}$$

On the other hand,

$$\underline{i}_T(T(h(x))) = \sigma(\underline{i}_T(h(x))) = \sigma(\underline{i}_f(x)). \tag{2.4}$$

Since $\underline{i}_T$ is an injection, we have that $h \circ f(x) = T \circ h(x)$.

Since $\mathscr{C}(f)$ and $\mathscr{C}(T)$ are dense in [0,1] (Proposition 2.6), h can extend to a homeomorphism $\tilde{h} : [0,1] \to [0,1]$ such that $\tilde{h} \circ f = T \circ \tilde{h}$. □

3. Topological conjugacies for maps in $\mathcal{M}_\infty$. In the previous sections, we had studied functions with a finite partition. Here we study a special class of functions with countable partition. Some modifications are necessary.

Definition 3.1. A map $f : [0,1] \to [0,1]$ belongs to the class of functions $\mathcal{M}_\infty$ if there exists a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$ with $0 = a_0 < a_1 < a_2 < \cdots$ and $\lim_{n\to\infty} a_n = 1$ such that:

(i) f is C^1 and monotone on each interval $[a_{i-1}, a_i]$ of the partition.

(ii) For every $i \in \mathbb{N}^*$, there exist unique $p(i), q(i) \in \mathbb{N}$ such that $f(a_{i-1}, a_i) = (a_{p(i)}, a_{q(i)})$.

(iii) There exists $\lambda > 1$ such that $|f'(x)| \geq \lambda$, for every $x \neq a_i$.

In this case, $\mathscr{C}(f) = \cup_{j=0}^{\infty} \cup_{i=0}^{\infty} f^{-i}(a_j)$.

Definition 3.2. Let $f \in \mathcal{M}_\infty$ with partition $0 = a_0 < a_1 < a_2 < \cdots < 1$. We define the set of sequences $\Sigma_\infty(f) = \{\underline{a} : \underline{a} = \{x_n\}_{n=0}^{\infty}\}$ with entries from $\{1/2, 1, 3/2, \ldots\}$, which satisfy the following conditions:

(i) Let $\underline{a} = \{x_n\} \in \Sigma_\infty(f)$. Then there exists an entry x_n of $\underline{a}$, of the form $(2k+1)/2$, where $k = 0, 1, \ldots$. Furthermore, if x_N is the first entry of $\underline{a}$ with this property, then $\sigma^N(\underline{a}) = \underline{i}_f(a_k)$.

(ii) If $n < N-1$ and $x_n = j$, then $p(j) + 1 \leq x_{n+1} \leq q(j)$.

Theorem 3.3. *Let $f \in \mathcal{M}_\infty$ with partition $0 = a_0 < a_1 < a_2 < \cdots < 1$. We consider the map $T \in \mathcal{M}_\infty$ with partition $0 < 1/2 < 2/3 < 3/4 < \cdots < 1$ which is linear in each interval $[(i-1)/i, i/(i+1)]$ and $T((i-1)/i, i/(i+1)) = (p(i)/(p(i)+1), q(i)/(q(i)+1))$. Furthermore, $T \mid [(i-1)/i, i/(i+1)]$ is of the same monotonicity type with $f \mid [a_{i-1}, a_i]$ and it is continuous, from the right or from the left at $i/(i+1)$, when f is continuous, from the right or from the left at a_i, respectively. Then f and T are topologically conjugate.*

Proof. The proof of this theorem is the same as the proof of Theorem 2.7. □

4. Computation of topological entropy for continuous Markov maps. Topological entropy is a measure of the dynamical complexity of a map and it is a topological invariant. There is an important theorem connecting topological entropy with the number c_n of maximal intervals of monotonicity of the iterate f^n (see [1, 4]).

Theorem 4.1 (Misiurewicz-Szlenk). *Let $f : I \to I$ be a continuous, piecewise monotone map. Then the topological entropy of f is equal to the number*

$$\lim_{n\to\infty} \frac{1}{n} \ln c_n. \tag{4.1}$$

As a corollary of the above theorem, if f is a piecewise linear map with slope $\pm s$, then the topological entropy of f is equal to $\max\{0, \ln s\}$.

Let f be a continuous map in $\mathcal{M}$ and T as in Theorem 2.7. The slope of T is not necessarily constant. Observe that Theorem 2.7 still holds if we change the partition $0 < 1/r < 2/r < \cdots < (r-1)/r < 1$ with any other partition $0 = b_0 < b_1 < \cdots < b_r = 1$ of $[0,1]$. So, it is natural to ask the following question. Can we find a partition $0 = b_0 < b_1 < \cdots < b_r = 1$ of $[0,1]$ such that $|b_{q(i)} - b_{p(i)}|/(b_i - b_{i-1})$ is constant?

To answer this question, to each $f \in \mathcal{M}$, we associate an $r \times r$ matrix $A = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 0, & \text{if } (b_{i-1}, b_i) \cap f^{-1}(b_{j-1}, b_j) = \varnothing, \\ 1, & \text{if } (b_{i-1}, b_i) \cap f^{-1}(b_{j-1}, b_j) \neq \varnothing. \end{cases} \tag{4.2}$$

Observe that A is nonnegative. According to the *Perron-Frobenius theorem*, there exists a unique nonnegative eigenvalue $s \geq 0$, which is maximal in absolute value among all the other eigenvalues and corresponding to a nonnegative eigenvector (see Gantmacher [5]).

Proposition 4.2. *Assume that $f \in \mathcal{M}$ is a continuous map of order r, A is the corresponding matrix, and s is the "maximal" eigenvalue of A.*

(a) *If $s > 1$ and the corresponding eigenvector is positive, then the topological entropy of f is $\ln s$.*

(b) *If $s \leq 1$ or at least one component of the corresponding eigenvector is zero, then the topological entropy of f is zero.*

Proof. (a) Assume that there exist a partition $0 = b_0 < b_1 < \cdots < b_r = 1$ and a constant $s > 1$ such that $|T(b_{i-1}, b_i)| = s|(b_{i-1}, b_i)|$, for $i = 1, 2, \ldots, r$. If we let $x_i = b_i - b_{i-1} > 0$, the above relation gives

$$x_{p(i)+1} + x_{p(i)+2} + \cdots + x_{q(i)} = s x_i, \quad i = 1, 2, \ldots, r, \tag{4.3}$$

or, equivalently,

$$Ax = sx, \quad \text{where } x = (x_1, \ldots, x_r)^\tau. \tag{4.4}$$

Thus, there exist a partition $0 = b_0 < b_1 < \cdots < b_r = 1$ and a constant $s > 1$ such that $|T(b_{i-1}, b_i)| = s|(b_{i-1}, b_i)|$, for $i = 1, 2, \ldots, r$, if and only if (a) holds.

(b) Assume on the contrary that $h(f) > 0$. Then f is conjugate to a piecewise linear map with constant slope [9]. It follows that there exist a partition $0 = b_0 < b_1 < \cdots < b_r = 1$ and a constant $s > 1$ such that $|T(b_{i-1}, b_i)| = s|(b_{i-1}, b_i)|$, for $i = 1, 2, \dots, r$. This is equivalent to (a), which contradicts (b). □

Remark 4.3. There is a similar result in [2]. The proof we give here is more simple and is based heavily on Theorem 2.7.

The above proposition gives a method to construct the partition $0 = b_0 < b_1 < \cdots < b_r = 1$, when we are in case (a). Assume that $(u_1, u_2, \dots, u_r)^\tau$ is an eigenvector corresponding to the maximal eigenvalue. Then $b_0 = 0$ and

$$b_k = \frac{\sum_{i=1}^{k} u_i}{\sum_{i=1}^{r} u_i} \quad \text{for } k = 1, 2, \dots, r. \tag{4.5}$$

Consider the map $f \in \mathcal{M}$ whose graph is shown in Figure 2.1. According to Theorem 2.7, f is topologically conjugate with T which is piecewise linear (the graph of T is shown in Figure 4.1). The associated matrix to f is

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{4.6}$$

The maximal eigenvalue is $s = 2.8393$ and an eigenvector is

$$(0.6478, 0.4196, 0.7718, 1)^\tau. \tag{4.7}$$

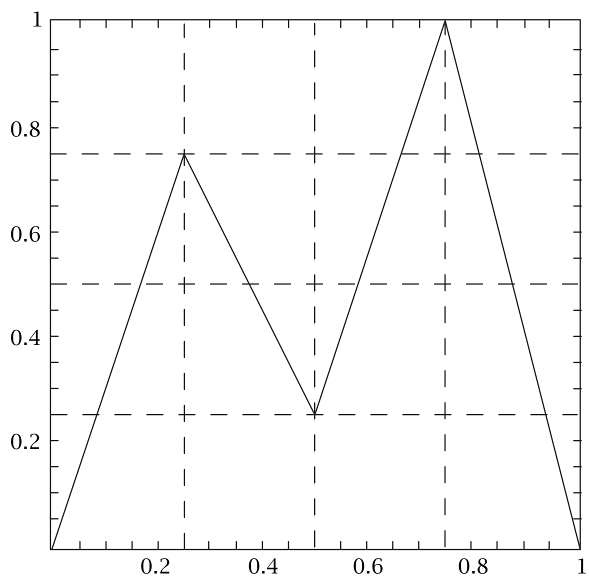

Figure 4.1.

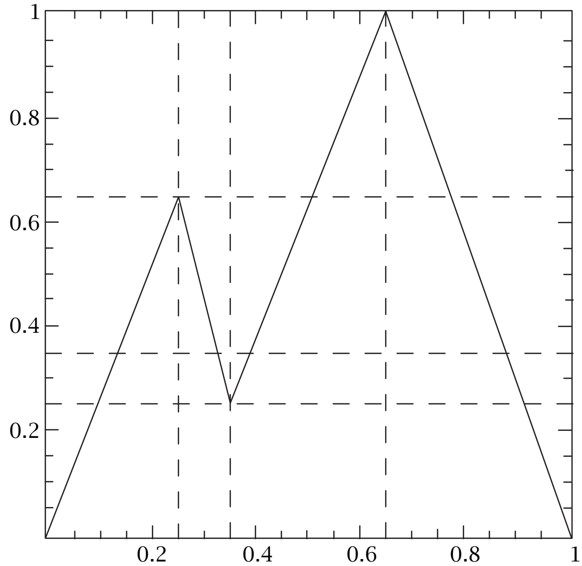

FIGURE 4.2.

Then from (4.5) we have $b_0 = 0$, $b_1 = 0.2282$, $b_2 = 0.3759$, $b_3 = 0.6478$, $b_4 = 1$. f is topologically conjugate to T' whose graph is shown in Figure 4.2. Since the slope of T' is constant in absolute value we have that $h(f) = \ln s = 1.0435$.

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