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GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTION OF SECOND-ORDER NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider the global existence and asymptotic behavior of solution of second-order nonlinear impulsive differential equations.

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1. Introduction. In recent years, there have been many papers considering the impulsive differential equations, see, for example, [2, 3, 1, 4, 5] and the references cited in [3]. In this paper, our results extend those in [6]. We consider the second-order nonlinear impulsive differential equation

$$(p(t)y'(t))' = f(t, y(t), y'(t)), \quad t \neq t_k,\ t \geq 0, \tag{1.1}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k)), \tag{1.2}$$

$$y'(t_k^+) - y'(t_k) = J_k(y'(t_k)), \quad k = 1, 2, \ldots, \tag{1.3}$$

under the following standing assumptions on p, f, I_k and J_k:

(A_1) $p : [0,\infty) \to (0,\infty)$ is continuous and

$$P(t) = \int_0^t \frac{1}{p(s)}\,ds \longrightarrow \infty \quad \text{as } t \longrightarrow \infty; \tag{1.4}$$

(A_2) $f : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to (0,\infty)$ is continuous and $f(t,u,v)$ is nondecreasing in u and v;

(A_3) I_k and $J_k : \mathbb{R} \to (0,\infty)$ are continuous and I_k and J_k are nondecreasing for $k = 1, 2, \ldots$;

(A_4) $0 \leq t_0 < t_1 < t_2 < \cdots < t_n < \cdots$ with $\lim_{n\to\infty} t_n = \infty$;

(A_5) Every Cauchy problem for (1.1), (1.2), and (1.3) has a unique solution.

Let $y(t)$ be a solution of (1.1), (1.2), and (1.3) with the maximal interval of existence $[0, T_y)$. From (1.1), (1.2), and (1.3) we have $(p(t)y'(t))' > 0$ for $t \neq t_k$, so that $p(t)y'(t)$ is increasing on $[0, T_y)$. It happens that either $T_y < \infty$ and $\lim_{t\to T_y} p(t)y'(t) = \infty$, or else $T_y = \infty$ and $\lim_{t\to\infty} p(t)y'(t)$ exists in $\mathbb{R} \cup \{\infty\}$. In the former case $y(t)$ is called a singular solution, and in the latter $y(t)$ is called a proper solution. The set of proper solutions of (1.1), (1.2), and (1.3) is further classified into the following four classes:

(i) the class of strongly increasing solutions consisting of all solutions $y(t)$ such that $\lim_{t\to\infty} p(t)y'(t) = \infty$;

(ii) the class of weakly increasing solutions consisting of all solutions $y(t)$ such that $\lim_{t\to\infty} p(t)y'(t) \in (0,\infty)$;

(iii) the class of weakly decreasing solutions consisting of all solutions $y(t)$ such that $\lim_{t\to\infty} p(t)y'(t) = 0$;

(iv) the class of strongly decreasing solutions consisting of all solutions $y(t)$ such that $\lim_{t\to\infty} p(t)y'(t) \in (-\infty, 0)$.

The main objective of this paper is to give explicit sufficient conditions for existence of some or all of these classes of proper solutions of (1.1), (1.2), and (1.3) defined on the given interval $[0,\infty)$.

2. Main results.

We begin by giving a condition under which (1.1), (1.2), and (1.3) have strongly decreasing solutions.

Theorem 2.1. *Suppose that there exist constants $c > 0$ and $I_0 > 0$ such that*

$$\int_0^\infty f\left(t, -cP(t), -\frac{c}{p(t)}\right) dt < \infty, \qquad \sum_{k=1}^\infty I_k(\cdot) < I_0, \qquad \sum_{k=1}^\infty p(t_k) J_k\left(-\frac{c}{p(t_k)}\right) < \infty. \tag{2.1}$$

Then, for any $b \in (c,\infty)$ and $\gamma \in \mathbb{R}$, (1.1), (1.2), and (1.3) have a strongly decreasing solution $y(t)$ satisfying

$$y(0) = \gamma, \qquad \lim_{t\to\infty} p(t)y'(t) = -b. \tag{2.2}$$

Proof. From (2.1), (A_2), and (A_3), we have

$$\int_0^\infty f\left(t, \gamma + I_0 - bP(t), -\frac{b}{p(t)}\right) dt < \infty, \qquad \sum_{k=1}^\infty p(t_k) J_k\left(-\frac{b}{p(t_k)}\right) < \infty. \tag{2.3}$$

Let Ω denote the Frechet space of all functions $y(t) : [0,\infty) \to \mathbb{R}$, such that $y(t)$ is twice continuously differentiable for $t \neq t_k$, $y(t^-)$, $y(t^+)$, $y'(t^-)$, $y'(t^+)$ exist and $y(t^-) = y(t)$, $y'(t^-) = y'(t)$ at $t = t_k$, with the usual metric topology, and M be the set of all $y(t) \in \Omega$ that satisfy the following inequalities:

$$\begin{aligned} \gamma - bP(t) &- \int_0^t \frac{1}{p(s)} \int_s^\infty f\left(\tau, \gamma + I_0 - bP(\tau), -\frac{b}{p(\tau)}\right) d\tau\, ds \\ &- P(t) \sum_{t \le t_k} p(t_k) J_k\left(-\frac{b}{p(t_k)}\right) - \sum_{t \le t_k} I_k(\gamma + I_0 - bP(t_k)) \le y(t) \le \gamma + I_0 - bP(t), \end{aligned}$$

$$-b - \int_t^\infty f\left(s, \gamma + I_0 - bP(s), -\frac{b}{p(s)}\right) ds - \sum_{t \le t_k} p(t_k) J_k\left(-\frac{b}{p(t_k)}\right) \le p(t)y'(t) \le -b, \quad t \ge 0. \tag{2.4}$$

Clearly, M is a nonempty closed convex subset of Ω. Define the operator $U : M \to \Omega$ by

$$\begin{aligned} Uy(t) = \gamma - bP(t) &- \int_0^t \frac{1}{p(s)} \int_s^\infty f(\tau, y(\tau), y'(\tau))\, d\tau\, ds \\ &- P(t) \sum_{t \le t_k} p(t_k) J_k(y'(t_k)) + \sum_{t_k < t} I_k(y(t_k)), \quad t \ge 0. \end{aligned} \tag{2.5}$$

It is easy to verify that $UM \subset M$, U is continuous and $\overline{UM}$ is compact. So, the Schauder-Tychonoff fixed point theorem implies that U has a fixed point y in M. This fixed point $y(t)$ is a strongly decreasing solution of (1.1), (1.2), and (1.3) satisfying (2.2). This completes the proof. □

Example 2.2. Consider the equation

$$\begin{aligned} &y'' = a(t)e^{y'}, \quad t \neq t_k,\ t \geq 0, \\ &y(t_k^+) - y(t_k) = m_k\left(\frac{\pi}{2} + \arctan y(t_k)\right), \\ &y'(t_k^+) - y'(t_k) = \ln\left(1 + M_k e^{y'(t_k)}\right), \quad k = 1, 2, \dots. \end{aligned} \tag{2.6}$$

If $\int_0^\infty a(t)\,dt < \infty$, and $\sum_{k=1}^\infty m_k < \infty$ and $\sum_{k=1}^\infty M_k < \infty$, then for any $b \in (c, \infty)$ and $\gamma \in \mathbb{R}$, (2.6) has a strongly decreasing solution

$$y(t) = \gamma - \int_0^t \ln\left[e^b + \int_s^\infty a(u)\,du + \sum_{t_k \geq s} \frac{M_k}{1 + M_k e^{y'(t_k)}}\right] ds + \sum_{t_k < t} m_k\left(\frac{\pi}{2} + \arctan y(t_k)\right) \tag{2.7}$$

satisfying $y(0) = \gamma$ and $\lim_{t\to\infty} y'(t) = -b$.

We now give a simple lemma which will be useful in the following discussions, and the proof of the lemma is straightforward by induction and will be omitted.

Lemma 2.3. *Together with (1.1), (1.2), and (1.3) we consider the equation*

$$\begin{aligned} &(p(t)z'(t))' = g(t, z(t), z'(t)), \quad t \neq t_k,\ t \geq 0, \\ &z(t_k^+) - z(t_k) = I_k^*(z(t_k)), \\ &z'(t_k^+) - z'(t_k) = J_k^*(z'(t_k)), \quad k = 1, 2, \dots, \end{aligned} \tag{2.8}$$

where $p(t)$ is as in (1.1), $g : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to (0,\infty)$ is continuous and nondecreasing in the last two variables, I_k^*, J_k^* are also continuous and nondecreasing from $\mathbb{R}$ to $(0,\infty)$, and

$$\begin{aligned} &f(t,u,v) \geq g(t,u,v), \quad (t,u,v) \in [0,\infty) \times \mathbb{R} \times \mathbb{R}, \\ &I_k(u(t_k)) \geq I_k^*(u(t_k)), \\ &J_k(u'(t_k)) \geq J_k^*(u'(t_k)), \quad k = 1, 2, \dots. \end{aligned} \tag{2.9}$$

Let $y(t)$ and $z(t)$ be solutions of (1.1), (1.2), (1.3), and (2.8), respectively, satisfying $z(a^+) \leq y(a^+)$ and $z'(a^+) < y'(a^+)$. If $y(t)$ is defined on $[a,b)$, then $z(t)$ exists on $[a,b)$ and satisfies $z(t) < y(t)$ and $z'(t) < y'(t)$ for $t \in (a,b)$.

Theorem 2.4. *Suppose that (2.1) hold for all $c > 0$. Then for any $\gamma \in \mathbb{R}$, (1.1), (1.2), and (1.3) have a unique weakly decreasing solution $y(t)$ satisfying $y(0) = \gamma$.*

Proof. We fix $\gamma \in \mathbb{R}$. Let $y_\alpha(t)$ denote the solution of (1.1), (1.2), and (1.3) satisfying $y(0) = \gamma$ and $p(0)y'(0) = \alpha$. We define the set $A \subset \mathbb{R}$ by

$$A = \{\alpha \in \mathbb{R} : y_\alpha(t) \text{ is a strongly decreasing solution}\}, \tag{2.10}$$

which is nonempty by Theorem 2.1. Now we show that A is an open set which is bounded above. Let $\alpha \in A$. If $\beta < \alpha$, then by Lemma 2.3 ($g \equiv f$, $I_k^* \equiv I_k$ and $J_k^* \equiv J_k$), $y_\beta(t)$ is a strongly decreasing solution, that is, $\beta \in A$. Suppose that $\beta > \alpha$. Since $\alpha \in A$, there exists an $l > 0$ such that $\lim_{n\to\infty} p(t)y'_\alpha = -l$. We choose $t_l > 0$ large enough so that

$$\int_{t_l}^{\infty} f\left(t, y + I_0 - \frac{l}{2}P(t), -\frac{l}{2p(t)}\right) dt < \frac{l}{4}, \qquad \sum_{t_k > t_l} p(t_k) J_k\left(-\frac{l}{2p(t_k)}\right) < \frac{l}{4}. \tag{2.11}$$

By the continuous dependence on initial conditions, for all $\beta > \alpha$ sufficiently close to α, $y_\beta(t)$ exist on $[0, t_l]$ and satisfy $p(t)y'_\beta(t) < -l$ for $t \in [0, t_l]$. It can be shown that for such a $\beta > \alpha$, $y_\beta(t)$ can be extended to $[0, \infty)$, and satisfies

$$p(t)y'_\beta(t) < -\frac{l}{2} \quad \text{for } t \geq 0. \tag{2.12}$$

In fact, if (2.12) fails, then there exists $t_m > t_1$ such that

$$p(t_m)y'_\beta(t_m) = -\frac{l}{2}, \qquad p(t)y'_\beta(t) < -\frac{l}{2} \tag{2.13}$$

for $t \in [0, t_m)$ and $t \neq t_k$, $t_k \in [0, t_m)$. Integrating (1.1) and using (2.11), (2.12), and (2.13), we have

$$\begin{aligned} -\frac{l}{2} &= p(t_m)y'_\beta(t_m) = p(t_l)y'_\beta(t_l) + \int_{t_l}^{t_m} f(t, y(t), y'(t))\, dt + \sum_{t_l \leq t_k < t_m} p(t_k) J_k(y(t_k)) \\ &\leq -l + \int_{t_l}^{t_m} f\left(t, y + I_0 - \frac{l}{2}P(t), -\frac{l}{2p(t)}\right) dt + \sum_{t_l \leq t_k < t_m} p(t_k) J_k\left(-\frac{l}{2p(t_k)}\right) \\ &\leq -l + \int_{t_l}^{\infty} f\left(t, y + I_0 - \frac{l}{2}P(t), -\frac{l}{2p(t)}\right) dt + \sum_{t_l \leq t_k} p(t_k) J_k\left(-\frac{l}{2p(t_k)}\right) \\ &< -l + \frac{l}{4} + \frac{l}{4} = -\frac{l}{2}. \end{aligned} \tag{2.14}$$

This contradiction proves that (2.12) holds, and this implies $\beta \in A$. Thus A is open. On the other hand, if $\alpha \geq 0$, then $\alpha \notin A$, so that A is bounded from above, we put $\alpha^* = \sup A$. It is obvious that $\alpha^* \notin A$ and $\alpha^* \leq 0$.

We consider the solution $y_{\alpha^*}(t)$. By the continuous dependence on the initial conditions, $y_{\alpha^*}(t)$ is not a singular solution, that is $y_{\alpha^*}(t)$ exists on $[0, \infty)$ and satisfies $\lim_{n\to\infty} p(t)y'_{\alpha^*}(t) = \eta^* \geq 0$ (η^* may be ∞). The continuous dependence on initial conditions precludes the possibility that η^* is positive, and so we must have $\eta^* = 0$. This means that y_{α^*} is a weakly decreasing solution passing through $(0, y)$.

To prove the uniqueness of the weakly decreasing solution passing through $(0, y)$, let $y_1(t)$ and $y_2(t)$ be two weakly decreasing solutions of (1.1), (1.2), and (1.3) such that $y_1(0) = y_2(0) = y$ but $y'_1(0) < y'_2(0)$. Lemma 2.3 ($g \equiv f$), $I_k^* \equiv I_k$ and $J_k^* \equiv J_k$ implies that $y_1(t) \leq y_2(t)$ and $y'_1(t) \leq y'_2(t)$ for $t \geq 0$. It follows from (1.1) that $[p(t)(y'_2(t) - y'_1(t))]' = f(t, y_2, y'_2) - f(t, y_1, y'_1) \geq 0$ for $t \geq 0$, $t \neq t_k$. So $p(t)y'_2(t) - p(t)y'_1(t) \geq p(0)[y'_2(0) - y'_1(0)] > 0$ for $t \geq 0$. Since the left-hand side of this inequality tends to 0 as $t \to \infty$, we have a contradiction. This completes the proof. □

The following theorem gives a useful information about the asymptotic behavior of weakly decreasing solutions of (1.1), (1.2), and (1.3).

Theorem 2.5. *All weakly decreasing solutions of (1.1), (1.2), and (1.3), if any, are either simultaneously bounded or simultaneously unbounded.*

Proof. Let $y_1(t)$ and $y_2(t)$ be the weakly decreasing solutions satisfying $y_1(0) = \gamma_1$ and $y_2(0) = \gamma_2$ with $\gamma_1 < \gamma_2$. It suffices to prove that the difference $y_2(t) - y_1(t)$ is a positive nonincreasing function on $[0,\infty)$. First we show that $y_2(t) > y_1(t)$ for $t \geq 0$. Otherwise, there exists $t^* > 0$ such that $y_1(t^*) = y_2(t^*)$ and $y_1'(t^*) > y_2'(t^*)$. We choose $t^{**} > t^*$ sufficiently close to t^*, such that $y_1(t^{**}) > y_2(t^{**})$ and fix it. By the continuous dependence on initial data, a solution $\tilde{y}(t)$ of (1.1), (1.2), and (1.3) with $\tilde{y}(0) = \gamma_2$ satisfies $y_2(t^{**}) < \tilde{y}(t^{**}) < y_1(t^{**})$ and $y_2'(t^{**}) < \tilde{y}'(t^{**}) < y_1'(t^{**})$, provided $\tilde{y}'(0) - y_2'(0) > 0$ is sufficiently small. By Lemma 2.3, $\tilde{y}(t)$ exists on $[0,\infty)$ and satisfies $y_2'(t) < \tilde{y}'(t) < y_1'(t)$, for $t \geq t^{**}$, that is, $p(t)y_2'(t) < p(t)\tilde{y}'(t) < p(t)y_1'(t)$ for $t \geq t^{**}$. This fact means that $\tilde{y}$ is a weakly decreasing solution passing through $(0,\gamma_2)$, which contradicts the uniqueness of the weakly decreasing solution passing through $(0,\gamma_2)$. Thus we obtain $y_2(t) > y_1(t)$ for $t \geq 0$. Next, if there exists $\tau \geq 0$ such that $y_1'(\tau) < y_2'(\tau)$, then the same argument as above leads us to the conclusion that there is a weakly decreasing solution different from $y_2(t)$ passing through $(0,\gamma_2)$. This again is a contradiction, and so we have $y_2'(t) \leq y_1'(t)$ for $t \geq 0$. It follows that $y_2(t) - y_1(t)$ is a positive nonincreasing function for $t \geq 0$, and the proof is complete. □

We now obtain conditions guaranteeing the existence of singular solutions of (1.1), (1.2), and (1.3).

Theorem 2.6. *Suppose $\sum_{k=1}^{\infty} I_k(\cdot) < I_0$ and that there exists a positive continuous function $f_*(t,u,v)$ on $[0,\infty)\times\mathbb{R}\times\mathbb{R}$ which is nonincreasing in t and nondecreasing in u and v, and satisfies $f(t,u,v) \geq f_*(t,u,v)$ on $[0,\infty)\times\mathbb{R}\times\mathbb{R}$. Moreover suppose that $p(t)P(t)$ is nondecreasing. We define*

$$F_\gamma(t,u) = \int_\gamma^u f_*\left(t,s,\frac{s-\gamma-I_0}{p(t)P(t)}\right)ds \quad \text{for } \gamma\in\mathbb{R},\ t>0,\ u>\gamma. \tag{2.15}$$

If

$$\int^\infty \left(F_\gamma(t,u)\right)^{-1/2} du < \infty \quad \text{for any } t > 0, \tag{2.16}$$

then for every $t_0 \geq 0$, there exists a singular solution $y(t)$ of (1.1), (1.2), and (1.3) satisfying $y(t_0) = \gamma$.

Proof. We fix $t^* > t_0 \geq 0$. Let m and M be positive constants such that $m \leq p(t) \leq M$ for $t \in [t_0,t^*]$. Choose $\delta = \delta(\gamma,t_0) > 0$ large enough so that

$$M\int_\gamma^{+\infty} \left(2mF_\gamma(t^*,u)+\delta^2\right)^{-1/2} du < t^* - t_0. \tag{2.17}$$

Now we show that the solution $y(t)$ of (1.1), (1.2), and (1.3) satisfying the initial conditions $y(t_0) = \gamma$ and $p(t_0)y'(t_0) \geq \delta$ cannot exist on $[t_0,t^*]$. Suppose the contrary, then from (1.1) and the monotonicity of $p(t)y'(t)$, we see that

$$\left((p(t)y'(t))^2\right)' \geq 2p(t)y'(t)f_*(t,y(t),y'(t)), \quad t \in [t_0,t^*],\ t \neq t_k \in [t_0,t^*]. \tag{2.18}$$

On the other hand, we have

$$\begin{aligned} y(t) &= \int_{t_0}^{t} y'(s)\,ds + \gamma + \sum_{t_k<t} I_k(y(t_k)) \\ &= \int_{t_0}^{t} \frac{p(s)y'(s)}{p(s)}\,ds + \gamma + \sum_{t_k<t} I_k(y(t_k)) \\ &\leq p(t)y'(t)\int_{t_0}^{t} \frac{1}{p(s)}\,ds + \gamma + \sum_{k=1}^{\infty} I_k(y(t_k)) \\ &\leq p(t)P(t)y'(t) + \gamma + I_0, \quad t \in [t_0,t^*], \end{aligned} \tag{2.19}$$

that is,

$$y'(t) \geq \frac{y(t)-\gamma-I_0}{p(t)P(t)}, \quad t \in [t_0,t^*]. \tag{2.20}$$

Integrating (2.18) from t_0 to $t \in [t_0,t^*]$ and using (2.20) and the monotonicity condition imposed on f_*, we obtain

$$\begin{aligned} (p(t)y'(t))^2 &\geq 2m\int_{t_0}^{t} y'(s)f_*\left(s,y(s),\frac{y(s)-\gamma-I_0}{p(s)P(s)}\right)ds + \delta^2 \\ &\quad + \sum_{t_k<t} p^2(t_k)\left[2y'(t_k)J_k(y'(t_k)) + J_k^2(y'(t_k))\right] \\ &\geq 2m\int_{t_0}^{t} y'(s)f_*\left(t,y(s),\frac{y(s)-\gamma-I_0}{p(t)P(t)}\right)ds + \delta^2 \end{aligned} \tag{2.21}$$

for $t \in [t_0,t^*]$, which is equivalent to

$$My'(t) \geq \left(2m\int_{\gamma}^{y(t)} f_*\left(t,s,\frac{s-\gamma-I_0}{p(t)P(t)}\right)ds + \delta^2\right)^{1/2}, \quad t \in [t_0,t^*]. \tag{2.22}$$

In view of the monotonicity of f_*, this implies

$$My'(t)\left(2mF_\gamma(t^*,y(t)) + \delta^2\right)^{-1/2} \geq 1, \quad t \in [t_0,t^*]. \tag{2.23}$$

Integrating from t_0 to t^*, and using (2.17) we obtain

$$\begin{aligned} t^*-t_0 &> M\int_{\gamma}^{\infty} \left(2mF_\gamma(t^*,u)+\delta^2\right)^{-1/2} du \\ &\geq M\int_{\gamma}^{y(t^*)} \left(2mF_\gamma(t^*,u)+\delta^2\right)^{-1/2} du \geq t^*-t_0, \end{aligned} \tag{2.24}$$

which is a contradiction. Thus this solution $y(t)$ must be singular. □

We now turn to the problem of finding conditions for (1.1), (1.2), and (1.3) to have weakly and strongly increasing solutions.

Theorem 2.7. *Suppose that there exist constants $c > 0$ and $I_0 > 0$ such that*

$$\int_0^\infty f\left(t, cP(t), \frac{c}{p(t)}\right) dt < \infty,$$
$$\sum_{k=1}^{\infty} I_k(\cdot) < I_0, \qquad \sum_{k=1}^{\infty} p(t_k) J_k\left(\frac{c}{p(t_k)}\right) < \infty. \tag{2.25}$$

Then for any $b \in (0, c)$ and any $\gamma \in \mathbb{R}$, equations (1.1), (1.2), and (1.3) have a weakly increasing solution $y(t)$ satisfying

$$y(0) = \gamma, \qquad \lim_{t\to\infty} p(t) y'(t) = b. \tag{2.26}$$

Proof. We omit the proof as it is virtually the same as that of Theorem 2.1. □

Theorem 2.8. *Suppose that the assumptions of Theorem 2.6 are satisfied for any $\gamma \in \mathbb{R}$. If (2.25) hold for all $c > 0$, then for any $\gamma \in \mathbb{R}$, (1.1), (1.2), and (1.3) have a strongly increasing solution $y(t)$ satisfying $y(0) = \gamma$.*

Proof. Let $\gamma \in \mathbb{R}$ be fixed and let $y_\alpha(t)$ be the solution of (1.1), (1.2), and (1.3) satisfying $y(0) = \gamma$ and $p(0)y'(0) = \alpha$. We define the sets A, $B \subset \mathbb{R}$ by

$$A = \{\alpha \in \mathbb{R} : y_\alpha(t) \text{ is a weakly increasing solution}\}$$
$$B = \{\alpha \in \mathbb{R} : y_\alpha(t) \text{ is a singular solution}\}. \tag{2.27}$$

By Theorems 2.6 and 2.7, we see that $B \neq \emptyset$ and $A \neq \emptyset$. Lemma 2.3 implies that $\alpha \le \beta$ for any $\alpha \in A$ and $\beta \in B$. Similar to the proof of Theorem 2.4 we can show that A and B are disjoint open subsets of $\mathbb{R}$. We put $\alpha^* = \sup A$ and $\beta_* = \inf B$. It is easily seen that $\alpha^* \notin A$, $\beta_* \notin B$, and $\alpha^* \le \beta_*$. Then, for any $\alpha \in [\alpha^*, \beta_*]$ (which may be reduced to one point), $y_\alpha(t)$ is a strongly increasing solution of (1.1), (1.2), and (1.3) satisfying $y(0) = \gamma$. This completes the proof. □

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