IJMMS 25:5 (2001) 357–360
PII. S0161171201003027
http://ijmms.hindawi.com

REGULAR-UNIFORM CONVERGENCE AND THE OPEN-OPEN TOPOLOGY

KATHRYN F. PORTER

(Received 12 February 1999)

Abstract. In 1994, Bânzaru introduced the concept of regular-uniform, or r-uniform, convergence on a family of functions. We discuss the relationship between this topology and the open-open topology, which was described in 1993 by Porter, on various collections of functions.

2000 Mathematics Subject Classification. Primary 54C35, 54E15; Secondary 54A10, 54A20.

1. Introduction. In [1], Bânzaru introduced the concept of regular-uniform, or r-uniform, convergence on a family of functions $F \subset Y^X$ and proved a number of facts about the topological space (F, T_r) where T_r is the topology induced by this convergence. Porter introduced the open-open topology [5] in 1993 and proved that on families of self-homeomorphisms on X that the open-open topology is equivalent to the topology of Pervin quasi-uniform convergence [3]; this in fact is true on $C(X,Y)$, the collection of all continuous functions from X to Y. We shall show that the topology of r-uniform convergence on any subfamily F of the class of all continuous functions on X into Y is equivalent to the open-open topology [5], T_{oo}, on F and hence, equivalent to the topology of Pervin quasi-uniform convergence on F.

Throughout this paper let (X,T) and (Y,T') be topological spaces. We will use Y^X to mean the collection of all functions from X into Y while $C(X,Y)$ will represent the collection of all continuous functions from X into Y, and $H(X)$ is the collection of all self-homeomorphisms on X.

2. Preliminaries. A net of functions $\{f_\alpha:(X,T)\to(Y,T')\}_{\alpha\in I}$ *converges r-uniformly (or regular uniformly)* to $f \in Y^X$ [1] if and only if for any $O \in T'$ such that $f^{-1}(O) \neq \phi$, there exists $i_\theta \in I = [0,1]$ such that $f_i(x) \in O$ for all $i \in I$ with $i \geq i_\theta$ and for all $x \in f^{-1}(O)$. This convergence defines a topology on F called the *topology of r-uniform* or *regular uniform convergence.*

In the same paper, Bânzaru also defined a topology, T_r, on $F \subset Y^X$ as follows: let $f \in F$ and $O \in T'$. Set

$$S(f;O) = \{g \in F : g(f^{-1}(O)) \subset O\}, \tag{2.1}$$

then $S = \{S(f;O) : f \in F \text{ and } O \in T'\}$ is a subbasis for a topology T_r on F. Bânzaru then proved that this topology T_r on F is actually equivalent to the topology of r-uniform convergence on F.

Now let $O \in T$ and $U \in T'$ and define

$$(U,V) = \{h \in F : h(O) \subset U\}. \tag{2.2}$$

Then $S_{\text{oo}} = \{(O,U) : O \in T \text{ and } U \in T'\}$ is a subbasis for the *open-open topology*, T_{oo}, [5] on F.

In addition, the set $S_{\text{co}} = \{(C,U) \subset F : C \text{ is compact in } X \text{ and } U \text{ is open in } Y\}$ is a subbasis for the well-known *compact-open topology*, T_{co}, on F.

Let X be a nonempty set and let Q be a collection of subsets of $X \times X$ such that

(1) for all $U \in Q$, $\triangle = \{(x,x) \in X \times X : x \in X\} \subset U$,

(2) for all $U \in Q$, if $U \subset V$ then $V \in Q$,

(3) for all $U, V \in Q$, $U \cap V \in Q$, and

(4) for all $U \in Q$, there exists some $W \in Q$ such that $W \circ W \subset U$ where $W \circ W = \{(p,q) \in X \times X : \text{there exists some } r \in X \text{ with } (p,r), (r,q) \in W\}$ then Q is a *quasi-uniformity on* X.

A quasi-uniformity, Q, on X induces a topology, T_Q, on X, where for each $x \in X$, the set $\{U[x] : U \in Q\}$ is a neighborhood system at x where $U[x]$ is defined by $U[x] = \{y \in X : (x,y) \in U\}$.

A family, S of subsets of $X \times X$ which satisfies

(i) for all $R \in S$, $\triangle \subset R$, and

(ii) for all $R \in S$, there exists $T \in S$ such that $T \circ T \subset R$, is a *subbasis* for a quasi-uniformity, Q, on X. This subbasis S generates a *basis*, B, for the quasi-uniformity, Q, where B is the collection of all finite intersections of elements of S. The basis, B, generates the quasi-uniformity $Q = \{U \subset X \times X : \hat{B} \subset U \text{ for some } \hat{B} \in B\}$.

For a more thorough background on quasi-uniform spaces, see [2].

In 1962, Pervin [4] constructed a specific quasi-uniformity which induces a compatible topology for a given topological space. His construction is as follows: Let (X,T) be a topological space. For $O \in T$ define

$$S_O = (O \times O) \cup \big((X \setminus O) \times X\big). \tag{2.3}$$

One can show that for $O \in T$, $S_O \circ S_O = S_O$ and $\triangle \subset S_O$, hence, the collection $\{S_O : O \in T\}$ is a subbasis for a quasi-uniformity, P, on X, called the *Pervin quasi-uniformity.*

Let Q be a compatible quasi-uniformity for (X,T) and let $F \subset C(X,Y)$. For $U \in Q$, define the set

$$W(U) = \{(f,g) \in F \times F : \big(f(x), g(x)\big) \in U \text{ for all } x \in X\}. \tag{2.4}$$

Then the collection $B = \{W(U) : U \in Q\}$ is a basis for a quasi-uniformity, Q^*, on F, called *the quasi-uniformity of quasi-uniform convergence with respect to* Q [3]. The topology, T_{Q^*}, induced by Q^* on F, is called *the topology of quasi-uniform convergence with respect to* Q. If Q is the Pervin quasi-uniformity, P, then T_{P^*} is called the *topology of Pervin quasi-uniform convergence.*

3. The topologies. We first extend, to subsets of $C(X,Y)$, the result from [5] that the open-open topology is equivalent to the topology of Pervin quasi-uniform convergence on a subgroup G of $H(X)$.

Theorem 3.1. *Let $F \subset C(X,Y)$. The open-open topology, T_{oo}, is equivalent to the topology of Pervin quasi-uniform convergence, T_{P*}, on F.*

Proof. Assume $F \subset C(X,Y)$. Let (O,U) be a subbasic open set in T_{oo} and let $f \in F$. Then $f(O) \subset U$. So $f \in W(S_U)[f]$ where

$$W(S_U)[f] = \{g \in F : (f(x),g(x)) \in S_U = U \times U \cup (X \setminus U) \times X, \forall x \in X\}. \tag{3.1}$$

Hence, if $g \in W(S_U)[f]$ and $x \in O$, then $f(x) \in U$ so $g(x) \in U$. Thus, $g \in (O,U)$ and $W(S_U)[f] \subset (O,U)$. Therefore, $T_{oo} \subset T_{P*}$.

Now let $V \in T_{P*}$ and $f \in V$. Then there exists $U \in P$ such that $f \in W(U)[f] \subset V$. Since $U \in P$, there exists some finite collection, $\{U_i : i = 1,2,\dots,n\} \subset T$ such that $\cap_{i=1}^n S_{U_i} \subset U$. Define $A = \cap_{i=1}^n (f^{-1}(U_i),U_i)$. Then A is an open set in T_{oo} and $f \in A$. Assume $g \in A$ and let $x \in X$. If $f(x) \in U_j$ for some $j \in \{1,2,\dots,n\}$, then $x \in f^{-1}(U_j)$. Then, since $g \in A$, $g(x) \in U_j$, hence, $(f(x),g(x)) \in U_j \times U_j \subset S_{U_j}$. If $f(x) \notin U_j$ for some $j \in \{1,2,\dots,n\}$, then $(f(x),g(x)) \in (X - U_j) \times X \subset S_{U_j}$. Thus, $g \in W(\cap_{i=1}^n S_{U_i})[f] \subset W(U)[f] \subset V$ so that $A \subset V$. Therefore, $T_{oo} = T_{P*}$ on F. □

Next we show that the regular-uniform topology is equivalent to the open-open topology on any subset, F, of $C(X,Y)$, and hence, also to the topology of Pervin quasi-uniform convergence on F.

Theorem 3.2. *For $F \subset C(X,Y)$, $T_{oo} = T_r$ on F.*

Proof. Note that a subbasic open set in T_r, $S(f;O) = \{g \in F : g(f^{-1}(O)) \subset O\}$ is equal to $(f^{-1}(O),O)$. Hence, if $f^{-1}(O)$ is open in X, which is the case when f is continuous, $S(f;O)$ is a subbasic open set in T_{oo}. Therefore, $T_r \subset T_{oo}$.

Now let (O,U) be a subbasic open set in T_{oo} and let $f \in (O,U)$. Then $f(O) \subset U$ which implies that $O \subset f^{-1} \circ f(O) \subset f^{-1}(U)$. Since $f \circ f^{-1}(U) = U$, $f \in (f^{-1}(U),U) = S(f;U) \in T_r$. If $g \in (f^{-1}(U),U)$, then $g(f^{-1}(U)) \subset U$. If $x \in O$, then $x \in f^{-1}(U)$ so that $g(x) \in U$ giving us that $g \in (O,U)$, whence $T_{oo} \subset T_r$ and we are done. □

While it is always true that $T_{oo} \subset T_r$ on $F \subset Y^X$, it is not necessarily true that $T_r = T_{oo}$ for $F \subset Y^X$ as the following example shows.

Example 3.3. Define the sets $X = \{1,2,3\}$, $T = \{\{1\},\phi,X\}$, $Y = \{1,2,3,4\}$, $T' = \{\{1,2\},\{3,4\},\phi,Y\}$ and $F = \{f_1,f_2,f_3,f_4\}$ which are given in Table 3.1. Then $T_{oo} = \{\phi,F,\{f_1,f_2,f_3\},\{f_4\}\}$. But $S(f_3;\{3,4\}) = \{f_3\} \notin T_{oo}$. In fact, T_r is the discrete topology on F.

Bânsaru proved that for any $F \subset Y^X$, the compact-open topology, T_{co}, is coarser than T_r. However, although $T_{co} \subset T_{oo}$ on F when $F \subset C(X,Y)$, it is not necessarily true that $T_{co} \subset T_{oo}$ for $F \subset Y^X$. Consider Example 3.3 again. We have that $(\{2\},\{3,4\})$ is in T_{co} and equals $\{f_3\}$, but $\{f_3\} \notin T_{oo}$. In this example, the compact-open topology on F is also the discrete topology and thus equals the regular-uniform topology on F.

TABLE 3.1.

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$
1	1	1	1	3
2	2	1	4	1
3	3	1	1	4

TABLE 3.2.

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$	$f_9(x)$
1	1	1	1	2	2	2	3	3	3
2	2	1	3	2	1	3	1	3	2

Another fact that has been proved in [1] about the regular-uniform topology is that if the topology for Y is regular, then $(C(X,Y),T_r)$ is closed in (Y^X,T_r). However, this is not true when Y^X is given the open-open topology; that is, let (X,T) and (Y,T') be topological spaces such that (Y,T') is regular. Then $(C(X,Y),T_r)$, which is the same as $(C(X,Y),T_{oo})$ is not necessarily closed in (Y^X,T_{oo}). The following example illustrates this.

EXAMPLE 3.4. Let $X = \{1,2\}$, $T = \{\phi,X,\{1\}\}$, $Y = \{1,2,3\}$, and $T' = \{\phi,Y,\{1\},\{2,3\}\}$. The collection Y^X is given in Table 3.2. Note that T' is a partition topology and is thus regular. Also note that $f_1^{-1}(\{2,3\}) = \{2\}$ and so f_1 is not continuous. The only open sets in (Y^X,T_{oo}) that contain f_1 are $(\phi,Y) = Y^X$ and $(\{1\},\{1\}) = \{f_1,f_2,f_3\}$. Both of these sets contain the function f_2 which is continuous. Thus, $C(X,Y)$ is not closed in (Y^X,T_{oo}), even though (Y,T') is regular.

ACKNOWLEDGEMENT. Professor Porter would like to thank the Faculty Development Committee of Saint Mary's College of California for their support.

REFERENCES

[1] T. Bânzaru, *On topological structures of r-uniform convergence*, Proceedings of 23rd Conference on Geometry and Topology (Cluj-Napoca, 1993) (D. Andrica et al., ed.), "Babeş-Bolyai" Univ., Cluj, Napoca, 1994, pp. 27–29. MR 96k:54021. Zbl 846.54012.

[2] M. G. Murdeshwar and S. A. Naimpally, *Quasi-uniform Topological Spaces*, P. Noordhoff Ltd., Groningen, 1966. MR 35#2267. Zbl 139.40501.

[3] S. A. Naimpally, *Function spaces of quasi-uniform spaces*, Nederl. Akad. Wetensch. Proc. Ser. A 68 **27** (1966), 768–771. MR 32#4653. Zbl 134.41703.

[4] W. J. Pervin, *Quasi-uniformization of topological spaces*, Math. Ann. **147** (1962), 316–317. MR 25#3506b. Zbl 101.40501.

[5] K. F. Porter, *The open-open topology for function spaces*, Int. J. Math. Math. Sci. **16** (1993), no. 1, 111–116. MR 94b:54057. Zbl 807.54017.

KATHRYN F. PORTER: DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, SAINT MARY'S COLLEGE OF CALIFORNIA, MORAGA, CA 94575, USA

E-mail address: kporter@stmarys-ca.edu