IJMMS 25:6 (2001) 417–420
PII. S0161171201001181
http://ijmms.hindawi.com

ON PERIODIC RINGS

XIANKUN DU and QI YI

(Received 23 April 1997 and in revised form 31 October 1997)

ABSTRACT. It is proved that a ring is periodic if and only if, for any elements x and y, there exist positive integers k, l, m, and n with either $k \neq m$ or $l \neq n$, depending on x and y, for which $x^k y^l = x^m y^n$. Necessary and sufficient conditions are established for a ring to be a direct sum of a nil ring and a J-ring.

2000 Mathematics Subject Classification. Primary 16U99, 16N20, 16D70.

1. Introduction. Throughout, R is a ring. This paper is concerned with the rings satisfying the following condition.

$$\text{For any } x, y \in R, \text{ there exist distinct 2-tuples } (k,l) \text{ and } (m,n) \text{ of positive integers depending on } x \text{ and } y \text{ such that } x^k y^l = x^m y^n. \tag{1.1}$$

Such rings with $l = m = 1$ and $k = n > 1$ were handled in Yaqub [8] and Luh [7]. Recently, with the goal of generalizing the work of Yaqub [8] and Luh [7], Guo [5] studied such rings for $l = m = 1$, $k > 1$, and $n > 1$, under an additional assumption on periodicity. Our aim is to prove that a ring with (1.1) is periodic. This enables us to remove the hypothesis on periodicity from the results of Guo [5]. We also establish necessary and sufficient conditions for a ring to be a direct sum of a nil ring and a J-ring, which improves the results of Guo [5]. Of course, they also extend the results of Yaqub [8] and Luh [7].

Recall that a semigroup S is periodic if, for any $x \in S$, there exist distinct positive integers $m = m(x)$ and $n = n(x)$ such that $x^m = x^n$. A ring R is called periodic if its multiplicative semigroup is periodic. An element x of R is said to be potent if $x^n = x$ for some integer $n = n(x) > 1$. Let P denote the set of potent elements of R, and let N be the set of nilpotent elements of R. If $R = N + P$, R is called weakly periodic, according to Grosen et al. [4]. By Bell [1], a periodic ring is weakly periodic, but the converse is open. If $R = P$, R is called a J-ring. It is well known that J-rings are commutative.

For a ring R, $(R, \circ)$ is a semigroup with identity 0, under the operation $a \circ b = a + b - ab$ for $a, b \in R$. This semigroup is called the adjoint semigroup of R. Also, $(R, \circ)$ is a group if and only if R is a (Jacobson) radical ring. For $a \in R$ and a positive integer k, $a^{[k]}$ denotes the kth power of a in $(R, \circ)$. Making use of formal identity 1, we get $a^{[k]} = 1 - (1-a)^k = af(a)$ for some $f(t) \in \mathbb{Z}[t]$, where $\mathbb{Z}[t]$ is the set of polynomials with integer coefficients.

2. Main results. The following Theorem 2.1 is needed in the proof of our main theorem, and it is of independent interest as well.

Theorem 2.1. *$(R, \circ)$ is a periodic semigroup if and only if R is a periodic torsion ring.*

Proof. If R is periodic and torsion, then for any $a \in R$, the subring $[a]$ generated by a is finite, and so $\{a, a^{[2]}, a^{[3]}, \ldots\}$ is finite. It follows that $(R, \circ)$ is periodic.

Conversely, suppose that $(R, \circ)$ is periodic. We first assume that R is a torsion ring. For each $a \in R$, we have positive integers m and n with $m < n$ such that $a^{[m]} = a^{[n]}$, whence a meets a monic polynomial of degree n of $\mathbb{Z}[t]$ and $[a]$ is a finite subring. It follows that R is periodic. Next, we have to prove that R is a torsion ring. To do this, it suffices to assume that R is torsion-free and then, show that $R = 0$ because R/T is a torsion-free ring and $(R/T, \circ)$ is a periodic semigroup, where T is the torsion ideal of R. For any $a \in R$, it is easy to see that $a^{[k]}$ is an idempotent of R for some positive integer k. Let $e = a^{[k]}$and let m and n be distinct positive integers such that $(3e)^{[m]} = (3e)^{[n]}$. Then we have

$$((-2)^m - (-2)^n)e = (e - 3e)^m - (e - 3e)^n = (3e)^{[n]} - (3e)^{[m]} = 0, \tag{2.1}$$

forcing $e = 0$. Consequently, $(R, \circ)$ is actually a periodic group. Clearly, $([a], \circ)$ is a subgroup of $(R, \circ)$. This shows that $[a]$ is a finitely generated commutative radical ring, which implies that $[a]$ is nilpotent. By Eldridge [3], a is contained in the torsion ideal of R and, hence, $a = 0$ as required. □

Lemma 2.2. *If R satisfies (1.1), then for each $a \in R$, either*
(1) *$a^{[r]} - a$ is nilpotent for some integer $r = r(a) > 1$ or*
(2) *$a^r = a^{r+1} f(a)$ for some positive integer $r = r(a)$ and some $f(t) \in \mathbb{Z}[t]$.*

Proof. Let (k, l) and (m, n) be distinct 2-tuples of positive integers such that

$$a^k (a - a^2)^l = a^m (a - a^2)^n. \tag{2.2}$$

Hence, we have

$$a^{k+l} (1 - a)^l = a^{m+n} (1 - a)^n. \tag{2.3}$$

Without loss of generality, we suppose that $k + l \le m + n$. If $k + l < m + n$, then by (2.3), a simple calculation yields $a^{k+l} = a^{k+l+1} f(a)$ for some $f(t) \in \mathbb{Z}[t]$, which proves (2). If $k + l = m + n$, then $l \neq n$ and $l < n$, say. From (2.3), we have

$$a^{k+l} (1 - a)^l (1 - (1 - a)^{n-l}) = 0. \tag{2.4}$$

Since

$$a^{[n-l]} = 1 - (1 - a)^{n-l} = a f(a), \quad \text{for some } f(t) \in \mathbb{Z}[t], \tag{2.5}$$

we have

$$a^{k+l+1} (1 - a)^l f(a) = 0, \text{ and so } (a(1 - a) f(a))^{k+l+1} = 0. \tag{2.6}$$

Noting that

$$a^{[n-l+1]} - a = (1 - a) a^{[n-l]} = a(1 - a) f(a), \tag{2.7}$$

one sees that $a^{[n-l+1]} - a$ is nilpotent, which proves (1). □

Lemma 2.3. *If R satisfies (1.1) and R contains no nonzero idempotents, then R is nil.*

Proof. Since condition (1.1) is inherited by subrings, it suffices to prove Lemma 2.3 for the subring $[a]$ for any $a \in R$. Thus, for notational convenience, we assume that R is commutative. In this case, N is an ideal of R. Since condition (1.1) is also inherited by R/N, and idempotents lift modulo N, we may suppose that R does not contain any nonzero nilpotent elements and have to prove that $R = 0$. For $a \in R$, if $a^r = a^{r+1}f(a)$ is as in Lemma 2.2(2), then $(af(a))^r$ is an idempotent. Thus, $(af(a))^r = 0$, from which $a^r = a^r(af(a))^r = 0$. This implies that $a = 0$. By Lemma 2.2, it follows that, for any $a \in R$, we have $a^{[r]} = a$ for some integer $r = r(a) > 1$. By Theorem 2.1, R is periodic. Since R contains nonzero idempotents, R is nil, and so $R = 0$. □

We are now in a position to prove our main theorem.

Theorem 2.4. *A ring is periodic if and only if it satisfies (1.1).*

Proof. It is clear that a periodic ring satisfies (1.1). Conversely, let R satisfy (1.1). If R is nil, we are done. Otherwise, Lemma 2.3 implies that R contains a nonzero idempotent e. By (1.1), we have $(2e)^i(3e)^j = (2e)^p(3e)^q$ for distinct 2-tuples (i,j) and (p,q) of positive integers, whence $(2^i3^j - 2^p3^q)e = 0$. It follows that e is contained in the torsion ideal T of R. Since R/T is torsion-free and satisfies (1.1), R/T contains no nonzero idempotents. By Lemma 2.3, R/T is nil, which implies that, for any $a \in R$, there exist positive integers m and n such that $ma^n = 0$. Set $b = a^n$. If $b^{[r]} - b$ is nilpotent as in Lemma 2.2, then b fulfils a monic polynomial with integer coefficients. Hence, the subring $[b]$ generated by b is finite, which yields $b^k = b^l$ for some positive integers k and l with $k < l$. Thus, $a^{nk} = a^{nl} = a^{nk+1}f(a)$, where $f(t) = t^{nl-nk-1}$. It follows, from Lemma 2.2 and Chacron [2], that R is periodic. □

In what follows, we consider conditions for a ring to be a direct sum of a J-ring and a nil ring.

Lemma 2.5. *For a ring R, the following statements are equivalent.*

(1) *For any $x,y \in R$, there exist positive integers $m > 1$, p, and q depending on x and y such that $x^my^p = xy^q$.*

(2) *$R = N + P$ and $NP = 0$.*

Proof. (1)⇒(2). By Theorem 2.1, R is periodic and so $R = N + P$. Suppose that $NP \neq 0$. Then $au \neq 0$ for some $a \in N$ and $u \in P$. Let $u^k = u$, $a^{n+1}u = 0$, and $a^nu \neq 0$ for some integers $k > 1$ and $n \geq 1$. Since u^{k-1} is an idempotent, by (1), we have $a^nu^{k-1} = (a^n)^mu^{k-1} = 0$ for some integer $m > 1$, whence $a^nu = a^nu^k = a^nu^{k-1}u = 0$, a contradiction.

(2)⇒(1). For $a,b \in N$ and $u,v \in P$, there exist an integer $m > 1$ such that $u^m = u$ and $a^m = b^m = 0$. From this and $NP = 0$, we get

$$(a+u)^m(b+v)^m = (a+u)(b+v)^m. \tag{2.8}$$

Hence, R satisfies (1). □

Remark 2.6. We cannot expect $PN = 0$ in Lemma 2.5(2). A simple counterexample due to Bell [1] is the ring on the Klein 4-group $\{0,a,b,c\}$ with multiplication such that $0x = cx = 0$ and $ax = bx = x$ for all $x \in R$. The ring satisfies the identity $xy = x^2y$ and $PN \neq 0$.

Theorem 2.7. *For a ring R, the following statements are equivalent.*

(1) *R is a direct sum of a nil ring and a J-ring.*

(2) *For any $x, y \in R$, there exists an integer $n = n(x,y) > 1$ such that $x^n y = xy^n$.*

(3) *For any $x, y \in R$, there exist integers $m = m(x,y) > 1$ and $n = n(x,y) > 1$ such that $x^m y = xy^n$.*

(4) *For any $x, y \in R$, there exist positive integers k, l, m, n, p, and q with $k > 1$ and $p > 1$ depending on x and y such that $x^k y^l = xy^m$ and $x^n y^p = x^q y$.*

Proof. (4)⇒(1) follows from Lemma 2.5 and Hirano et al. [6, Theorem 1]. The rest is trivial. □

Remark 2.8. The equivalence between (1) and (2) was pointed out in Bell [1]. The equivalence between (1) and (3) was also presented in Guo [5] under an additional hypothesis. Theorem 2.7 extends and sharpens the results of Guo [5].

Acknowledgement. The first author would like to thank Symbol Computation and Knowledge Engineering Open Laboratory of Education Ministry of State for its hospitality and to Visiting Scholar Foundation of Key Laboratory at Jilin University for its support.

References

[1] H. E. Bell, *A commutativity study for periodic rings*, Pacific J. Math. **70** (1977), no. 1, 29–36. MR 58#793. Zbl 336.16034.

[2] M. Chacron, *On a theorem of Herstein*, Canad. J. Math. **21** (1969), 1348–1353. MR 41#6905. Zbl 213.04302.

[3] K. E. Eldridge, *Artinian q-torsion rings with p-groups*, Math. Ann. **185** (1970), 13–16. MR 41#5418. Zbl 185.09201.

[4] J. Grosen, H. Tominaga, and A. Yaqub, *On weakly periodic rings, periodic rings and commutativity theorems*, Math. J. Okayama Univ. **32** (1990), 77–81. MR 92c:16035. Zbl 741.16014.

[5] X. Z. Guo, *Direct sums of J-rings and radical rings*, Int. J. Math. Math. Sci. **18** (1995), no. 3, 531–534. CMP 1 331 953. Zbl 832.16030.

[6] Y. Hirano, H. Tominaga, and A. Yaqub, *On rings in which every element is uniquely expressible as a sum of a nilpotent element and a certain potent element*, Math. J. Okayama Univ. **30** (1988), 33–40. MR 90d:16028. Zbl 665.16016.

[7] J. Luh, *On the structure of pre-J-rings*, Hung-Ching Chow Sixty-fifth Anniversary Volume, Math. Res. Center Nat. Taiwan Univ., Taipei, 1967, pp. 47–52. MR 37#250. Zbl 206.04601.

[8] A. Yaqub, *The structures of pre-p^k-rings and generalized pre-p-rings*, Amer. Math. Monthly **71** (1964), 1010–1014. MR 30#2048.

Xiankun Du: Department of Mathematics, Jilin University, Changchun 130012, China

E-mail address: duxk@public.cc.jl.cn

Qi Yi: Jilin Commercial College, Changchun 130062, China