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EXPONENTIAL FORMS AND PATH INTEGRALS FOR COMPLEX NUMBERS IN n DIMENSIONS

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ABSTRACT. Two distinct systems of commutative complex numbers in n dimensions are described, of polar and planar types. Exponential forms of n-complex numbers are given in each case, which depend on geometric variables. Azimuthal angles, which are cyclic variables, appear in these forms at the exponent, and this leads to the concept of residue for path integrals of n-complex functions. The exponential function of an n-complex number is expanded in terms of functions called in this paper cosexponential functions, which are generalizations to n dimensions of the circular and hyperbolic sine and cosine functions. The factorization of n-complex polynomials is discussed.

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1. Introduction. Hypercomplex numbers are a generalization to several dimensions of the regular complex numbers in 2 dimensions. A well-known example of hypercomplex numbers are the quaternions of Hamilton, which are a system of hypercomplex numbers in 4 dimensions, the multiplication being a non-commutative operation [1]. Many other hypercomplex systems are possible [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13], but these interesting systems do not have all the required properties of regular, 2-dimensional complex numbers which rendered possible the development of the theory of functions of a 2-dimensional complex variable.

Two distinct systems of complex numbers in n dimensions are described in this paper, for which the multiplication is associative and commutative, and which are rich enough in properties such that exponential forms exist and the concepts of analytic n-complex function, contour integration and residue can be defined. The first type of n-complex numbers described in this article is characterized by the presence, in an odd number of dimensions, of one polar axis, and by the presence, in an even number of dimensions, of two polar axes. Therefore, these numbers will be called polar n-complex numbers. The other type of n-complex numbers described in this paper exists as a distinct entity only when the number of dimensions n of the space is even. These numbers will be called planar n-complex numbers. The planar hypercomplex numbers become for $n = 2$ the usual complex numbers $x + iy$.

The central result of this paper is the existence of an exponential form of n-complex numbers, which is expressed in terms of geometric variables. The exponential form provides the link between the algebraic side of the operations and the analytic properties of the functions of n-complex variables. The azimuthal angles ϕ_k, which are cyclic variables, appear in these forms at the exponent, and this leads to the concept of n-complex residue for path integrals of n-complex functions. Expressions are given for the elementary functions of n-complex variables. The exponential function of an n-complex number is expanded in terms of functions called in this paper n-dimensional cosexponential functions of the polar and planar type, respectively. The polar cosexponential functions are a generalization to n dimensions of the hyperbolic functions $\cosh y$, $\sinh y$, and the planar cosexponential functions are a generalization to n dimensions of the trigonometric functions $\cos y$, $\sin y$. Addition theorems and other relations are obtained for the n-dimensional cosexponential functions.

In the case of polar n-complex numbers, a polynomial can be written as a product of linear or quadratic factors, although several factorizations are in general possible. In the case of planar n-complex numbers, a polynomial can always be written as a product of linear factors, although, again, several factorizations are in general possible.

A study of commutative complex numbers in $2,3,4,5$, and 6 dimensions and further properties of polar and planar complex numbers in n dimensions can be found in [8].

2. Polar n-complex numbers

2.1. Operations with polar n-complex numbers. A hypercomplex number in n dimensions is determined by its n components $(x_0,x_1,\dots,x_{n-1})$. The polar n-complex numbers and their operations discussed in this paper can be represented by writing the n-complex number $(x_0,x_1,\dots,x_{n-1})$ as $u = x_0+h_1x_1+h_2x_2+\cdots+h_{n-1}x_{n-1}$, where $h_1,h_2,\dots,h_{n-1}$ are bases for which the multiplication rules are

$$h_jh_k = h_l, \quad l = j+k-n[(j+k)/n], \tag{2.1}$$

for $j,k,l = 0,1,\dots,n-1$, where $h_0 = 1$. In this relation, $[(j+k)/n]$ denotes the integer part of $(j+k)/n$, defined as $[a] \le a < [a]+1$, so that $0 \le j+k-n[(j+k)/n] \le n-1$. In this paper, brackets larger than the regular brackets, $[\,]$, do not have the meaning of integer part. The significance of the composition laws in (2.1) can be understood by representing the bases h_j, h_k by points on a circle at the angles $\alpha_j = 2\pi j/n$, $\alpha_k = 2\pi k/n$, as shown in Figure 2.1, and the product h_jh_k by the point of the circle at the angle $2\pi(j+k)/n$. If $2\pi \le 2\pi(j+k)/n < 4\pi$, the point represents the basis h_l of the angle $\alpha_l = 2\pi(j+k)/n-2\pi$.

Two polar n-complex numbers $u = x_0+h_1x_1+h_2x_2+\cdots+h_{n-1}x_{n-1}$, $u' = x_0'+h_1x_1'+h_2x_2'+\cdots+h_{n-1}x_{n-1}'$ are equal if and only if $x_j = x_j'$, $j = 0,1,\dots,n-1$. The sum of the polar n-complex numbers u and u' is

$$u+u' = x_0+x_0'+h_1(x_1+x_1')+\cdots+h_{n-1}(x_{n-1}+x_{n-1}'). \tag{2.2}$$

The product of the polar n-complex numbers u,u' is

$$\begin{aligned} uu' = {} & x_0x_0'+x_1x_{n-1}'+x_2x_{n-2}'+x_3x_{n-3}'+\cdots+x_{n-1}x_1' \\ & +h_1(x_0x_1'+x_1x_0'+x_2x_{n-1}'+x_3x_{n-2}'+\cdots+x_{n-1}x_2') \\ & +h_2(x_0x_2'+x_1x_1'+x_2x_0'+x_3x_{n-1}'+\cdots+x_{n-1}x_3') \\ & \vdots \\ & +h_{n-1}(x_0x_{n-1}'+x_1x_{n-2}'+x_2x_{n-3}'+x_3x_{n-4}'+\cdots+x_{n-1}x_0'). \end{aligned} \tag{2.3}$$

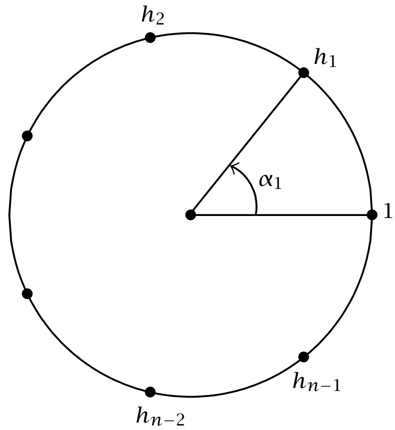

FIGURE 2.1. Representation of the polar n-complex bases $1, h_1, \dots, h_{n-1}$ by points on a circle at the angles $\alpha_k = 2\pi k/n$.

The product uu' can be written as

$$uu' = \sum_{k=0}^{n-1} h_k \sum_{l=0}^{n-1} x_l x'_{k-l+n[(n-k-1+l)/n]}. \tag{2.4}$$

If u, u', u'' are n-complex numbers, the multiplication is associative $(uu')u'' = u(u'u'')$ and commutative $uu' = u'u$ because the product of the bases, defined in (2.1), is associative and commutative.

The inverse of the polar n-complex number u is the n-complex number u' having the property that $uu' = 1$. This equation has a solution provided that the corresponding determinant ν is not equal to zero, $\nu \neq 0$. If n is an even number, it can be shown that

$$\nu = v_+ v_- \prod_{k=1}^{n/2-1} \rho_k^2, \tag{2.5}$$

and if n is an odd number,

$$\nu = v_+ \prod_{k=1}^{(n-1)/2} \rho_k^2, \tag{2.6}$$

where

$$\rho_k^2 = v_k^2 + \tilde{v}_k^2, \qquad v_k = \sum_{p=0}^{n-1} x_p \cos\left(\frac{2\pi kp}{n}\right), \qquad \tilde{v}_k = \sum_{p=0}^{n-1} x_p \sin\left(\frac{2\pi kp}{n}\right). \tag{2.7}$$

Thus, in an even number of dimensions n, an n-complex number has an inverse unless it lies on one of the nodal hypersurfaces $v_+ = 0$, or $v_- = 0$, or $\rho_1 = 0$, or ... or $\rho_{n/2-1} = 0$. In an odd number of dimensions n, an n-complex number has an inverse unless it lies on one of the nodal hypersurfaces $v+ = 0$, or $\rho_1 = 0$, or ... or $\rho_{(n-1)/2} = 0$.

For even n,

$$d^2 = \frac{1}{n} v_+^2 + \frac{1}{n} v_-^2 + \frac{2}{n} \sum_{k=1}^{n/2-1} \rho_k^2, \tag{2.8}$$

and for odd n,

$$d^2 = \frac{1}{n}v_+^2 + \frac{2}{n}\sum_{k=1}^{(n-1)/2}\rho_k^2. \tag{2.9}$$

From these relations it results that if the product of two n-complex numbers is zero, $uu' = 0$, then $\rho_+\rho'_+ = 0$, $\rho_-\rho'_- = 0$, $\rho_k\rho'_k = 0$, $k = 1,\dots,n/2$, which means that either $u = 0$, or $u' = 0$, or u, u' belong to orthogonal hypersurfaces in such a way that the afore-mentioned products of components should be equal to zero.

2.2. Geometric representation of polar n-complex numbers. The polar n-complex number $x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$ can be represented by the point A of coordinates $(x_0, x_1, \dots, x_{n-1})$. If O is the origin of the n-dimensional space, the distance from the origin O to the point A of coordinates $(x_0, x_1, \dots, x_{n-1})$ has the expression

$$d^2 = x_0^2 + x_1^2 + \cdots + x_{n-1}^2. \tag{2.10}$$

The quantity d will be called modulus of the polar n-complex number $u = x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$. The modulus of an n-complex number u will be designated by $d = |u|$. If $\nu > 0$, the quantity $\rho = \nu^{1/n}$ will be called amplitude of the polar n-complex number u.

The exponential and trigonometric forms of the polar n-complex number u can be obtained conveniently in a rotated system of axes defined by the transformation

$$v_+ = \sqrt{n}\xi_+, \qquad v_- = \sqrt{n}\xi_-, \qquad v_k = \sqrt{\frac{n}{2}}\xi_k, \qquad \tilde{v}_k = \sqrt{\frac{n}{2}}\eta_k, \tag{2.11}$$

for $k = 1,\dots,[(n-1)/2]$. This transformation from the coordinates $x_0, \dots, x_{n-1}$ to the variables $\xi_+, \xi_-, \xi_k, \eta_k$ is unitary.

The position of the point A of coordinates $(x_0, x_1, \dots, x_{n-1})$ can also be described with the aid of the distance d, equation (2.10), and of $n-1$ angles defined further. Thus, in the plane of the axes $v_k, \tilde{v}_k$, the azimuthal angles ϕ_k can be introduced by the relations

$$\cos\phi_k = \frac{v_k}{\rho_k}, \qquad \sin\phi_k = \frac{\tilde{v}_k}{\rho_k}, \tag{2.12}$$

where $0 \le \phi_k < 2\pi$, so that there are $[(n-1)/2]$ azimuthal angles. If the projection of the point A on the plane of the axes $v_k, \tilde{v}_k$ is A_k, and the projection of the point A on the 4-dimensional space defined by the axes $v_1, \tilde{v}_1, v_k, \tilde{v}_k$ is A_{1k}, the angle ψ_{k-1} between the line OA_{1k} and the 2-dimensional plane defined by the axes $v_k, \tilde{v}_k$ is

$$\tan\psi_{k-1} = \frac{\rho_1}{\rho_k}, \tag{2.13}$$

for $0 \le \psi_k \le \pi/2$, $k = 2,\dots,[(n-1)/2]$, so that there are $[(n-3)/2]$ planar angles. Moreover, there is a polar angle θ_+, which can be defined as the angle between the line OA_{1+} and the axis v_+, where A_{1+} is the projection of the point A on the 3-dimensional space generated by the axes $v_1, \tilde{v}_1, v_+$,

$$\tan\theta_+ = \frac{\sqrt{2}\rho_1}{v_+}, \tag{2.14}$$

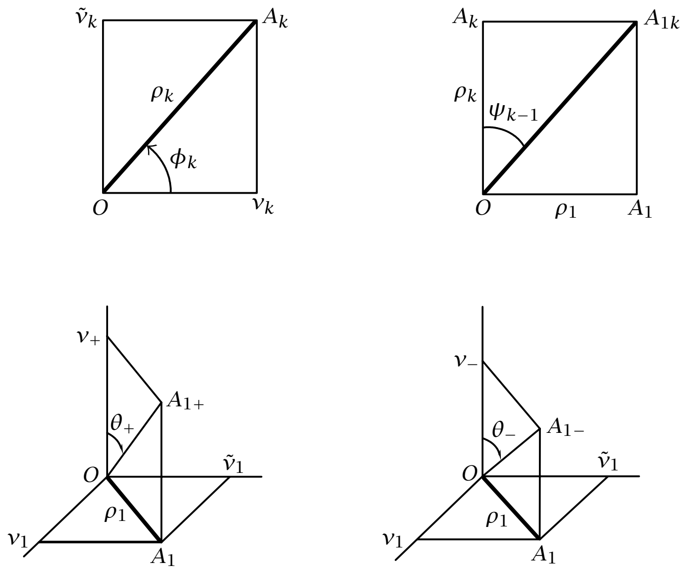

FIGURE 2.2. Angular variables for the description of n-complex numbers.

where $0 \le \theta_+ \le \pi$, and in an even number of dimensions n there is also a polar angle θ_-, which can be defined as the angle between the line OA_{1-} and the axis v_-, where A_{1-} is the projection of the point A on the 3-dimensional space generated by the axes $v_1, \tilde{v}_1, v_-$,

$$\tan\theta_- = \frac{\sqrt{2}\rho_1}{v_-}, \tag{2.15}$$

where $0 \le \theta_- \le \pi$. Thus, the position of the point A is described, in an even number of dimensions, by the distance d, by $n/2-1$ azimuthal angles, by $n/2-2$ planar angles, and by 2 polar angles. In an odd number of dimensions, the position of the point A is described by $(n-1)/2$ azimuthal angles, by $(n-3)/2$ planar angles, and by 1 polar angle. These angles are shown in Figure 2.2. The variables $v, \rho, \rho_k, \tan\theta_+/\sqrt{2}, \tan\theta_-/\sqrt{2}$, $\tan\psi_k$ are multiplicative and the azimuthal angles ϕ_k are additive upon the multiplication of polar n-complex numbers.

2.3. The n-dimensional polar cosexponential functions. The exponential function of the polar n-complex variable u can be defined by the series $\exp u = 1 + u + u^2/2! + u^3/3! + \cdots$. It can be checked by direct multiplication of the series that $\exp(u+u') = \exp u \cdot \exp u'$, so that $\exp u = \exp x_0 \cdot \exp(h_1 x_1) \cdots \exp(h_{n-1}x_{n-1})$.

It can be seen with the aid of the representation in Figure 2.1 that

$$h_k^{n+p} = h_k^p \tag{2.16}$$

for p integer, $k = 1, \ldots, n-1$. Then $e^{h_k y}$ can be written as

$$e^{h_k y} = \sum_{p=0}^{n-1} h_{kp-n[kp/n]} g_{nl}(y), \tag{2.17}$$

where the functions g_{nl}, which will be called polar cosexponential functions in n dimensions, are

$$g_{nl}(y) = \sum_{p=0}^{\infty} \frac{y^{l+pn}}{(l+pn)!} \tag{2.18}$$

for $l = 0, 1, \ldots, n-1$. If n is even, the polar cosexponential functions of even index k are even functions, $g_{n,2l}(-y) = g_{n,2l}(y)$, and the polar cosexponential functions of odd index are odd functions, $g_{n,2l+1}(-y) = -g_{n,2l+1}(y)$, $l = 0, 1, \ldots, n/2-1$. For odd values of n, the polar cosexponential functions do not have a definite parity. It can be checked that

$$\sum_{l=0}^{n-1} g_{nl}(y) = e^{y} \tag{2.19}$$

and, for even n,

$$\sum_{l=0}^{n-1} (-1)^k g_{nl}(y) = e^{-y}. \tag{2.20}$$

The expression of the polar n-dimensional cosexponential functions is

$$g_{nk}(y) = \frac{1}{n} \sum_{l=0}^{n-1} \exp\left[y \cos\left(\frac{2\pi l}{n}\right)\right] \cos\left[y \sin\left(\frac{2\pi l}{n}\right) - \frac{2\pi k l}{n}\right] \tag{2.21}$$

for $k = 0, 1, \ldots, n-1$. It can be shown from (2.21) that

$$\sum_{k=0}^{n-1} g_{nk}^2(y) = \frac{1}{n} \sum_{l=0}^{n-1} \exp\left[2y \cos\left(\frac{2\pi l}{n}\right)\right]. \tag{2.22}$$

It can be seen that the right-hand side of (2.22) does not contain oscillatory terms. If n is a multiple of 4, it can be shown by replacing y by iy in (2.22) that

$$\sum_{k=0}^{n-1} (-1)^k g_{nk}^2(y) = \frac{2}{n} \left\{1 + \cos 2y + \sum_{l=1}^{n/4-1} \cos\left[2y \cos\left(\frac{2\pi l}{n}\right)\right]\right\} \tag{2.23}$$

which does not contain exponential terms.

Addition theorems for the polar n-dimensional cosexponential functions can be obtained from the relation $\exp h_1(y+z) = \exp h_1 y \cdot \exp h_1 z$, by substituting the expression of the exponentials as given by $e^{h_1 y} = \sum_{p=0}^{n-1} h_p g_{np}(y)$,

$$\begin{aligned} g_{nk}(y+z) &= g_{n0}(y)g_{nk}(z) + g_{n1}(y)g_{n,k-1}(z) + \cdots + g_{nk}(y)g_{n0}(z) \\ &\quad + g_{n,k+1}(y)g_{n,n-1}(z) + g_{n,k+2}(y)g_{n,n-2}(z) + \cdots + g_{n,n-1}(y)g_{n,k+1}(z) \end{aligned} \tag{2.24}$$

for $k = 0, 1, \ldots, n-1$.

It can also be shown that

$$\begin{aligned} &\{g_{n0}(y) + h_1 g_{n1}(y) + \cdots + h_{n-1} g_{n,n-1}(y)\}^l \\ &\quad = g_{n0}(ly) + h_1 g_{n1}(ly) + \cdots + h_{n-1} g_{n,n-1}(ly). \end{aligned} \tag{2.25}$$

The polar n-dimensional cosexponential functions are solutions of the nth-order differential equation

$$\frac{d^n \zeta}{du^n} = \zeta \tag{2.26}$$

whose solutions are of the form $\zeta(u) = A_0 g_{n0}(u) + A_1 g_{n1}(u) + \cdots + A_{n-1} g_{n,n-1}(u)$. It can be checked that the derivatives of the polar cosexponential functions are related by

$$\frac{dg_{n0}}{du} = g_{n,n-1}, \quad \frac{dg_{n1}}{du} = g_{n0}, \ \ldots, \ \frac{dg_{n,n-2}}{du} = g_{n,n-3}, \quad \frac{dg_{n,n-1}}{du} = g_{n,n-2}. \tag{2.27}$$

For $n = 1$ and $n = 2$ the polar cosexponential functions are $g_{10}(y) = e^y$, $g_{20}(y) = \cosh y$, and $g_{21}(y) = \sinh y$. For $n = 3$ the cosexponential functions are $g_{30}(y) = 1 + y^3/3! + y^6/6! + \cdots$, $g_{31}(y) = y + y^4/4! + y^7/7! + \cdots$, $g_{32}(y) = y^2/2! + y^5/5! + y^8/8! + \cdots$, and they fulfill the identity $g_{30}^3 + g_{31}^3 + g_{30}^3 - 3g_{30}g_{31}g_{32} = 1$.

2.4. Exponential and trigonometric forms of polar n-complex numbers. In order to obtain the exponential and trigonometric forms of polar n-complex numbers, a new set of hypercomplex bases will be introduced for even n by the relations

$$e_+ = \frac{1}{n}\sum_{p=0}^{n-1} h_p, \qquad e_k = \frac{2}{n}\sum_{p=0}^{n-1} h_p \cos\left(\frac{2\pi kp}{n}\right), \qquad \tilde{e}_k = \frac{2}{n}\sum_{p=0}^{n-1} h_p \sin\left(\frac{2\pi kp}{n}\right), \tag{2.28}$$

where $k = 1, \ldots, [(n-1)/2]$ and, if n is even,

$$e_- = \frac{1}{n}\sum_{p=0}^{n-1} (-1)^p h_p. \tag{2.29}$$

The multiplication relations for the new hypercomplex bases are

$$\begin{gathered} e_+^2 = e_+, \quad e_-^2 = e_-, \quad e_+e_- = 0, \quad e_+e_k = 0, \quad e_+\tilde{e}_k = 0, \quad e_-e_k = 0, \quad e_-\tilde{e}_k = 0, \\ e_k^2 = e_k, \quad \tilde{e}_k^2 = -e_k, \quad e_k\tilde{e}_k = \tilde{e}_k, \quad e_ke_l = 0, \quad e_k\tilde{e}_l = 0, \quad \tilde{e}_k\tilde{e}_l = 0, \quad k \neq l, \end{gathered} \tag{2.30}$$

where $k, l = 1, \ldots, [(n-1)/2]$. It can be shown that, for even n,

$$u = e_+v_+ + e_-v_- + \sum_{k=1}^{n/2-1} (e_kv_k + \tilde{e}_k\tilde{v}_k), \tag{2.31}$$

and for odd n,

$$u = e_+v_+ + \sum_{k=1}^{(n-1)/2} (e_kv_k + \tilde{e}_k\tilde{v}_k). \tag{2.32}$$

The exponential form of the polar n-complex number u is

$$\begin{aligned} u = \rho \exp\Bigg\{ \sum_{p=1}^{n-1} h_p \Bigg[&\frac{1}{n}\ln\frac{\sqrt{2}}{\tan\theta_+} + F(n)\frac{(-1)^p}{n}\ln\frac{\sqrt{2}}{\tan\theta_-} \\ &- \frac{2}{n}\sum_{k=2}^{[(n-1)/2]} \cos\left(\frac{2\pi kp}{n}\right)\ln\tan\psi_{k-1}\Bigg] + \sum_{k=1}^{[(n-1)/2]} \tilde{e}_k\phi_k \Bigg\}, \end{aligned} \tag{2.33}$$

where $F(n) = 1$ for even n and $F(n) = 0$ for odd n, and

$$\rho = \left(v_+v_-\rho_1^2 \cdots \rho_{n/2-1}^2\right)^{1/n} \tag{2.34}$$

for even n, and

$$\rho=\left(v_+\rho_1^2\cdots\rho_{(n-1)/2}^2\right)^{1/n} \tag{2.35}$$

for odd n.

The trigonometric form of the polar n-complex number u is

$$\begin{aligned}u={}&d\left(\frac{n}{2}\right)^{1/2}\left(\frac{1}{\tan^2\psi_+}+\frac{F(n)}{\tan^2\psi_-}+1+\frac{1}{\tan^2\psi_1}+\frac{1}{\tan^2\psi_2}+\cdots+\frac{1}{\tan^2\psi_{[(n-3)/2]}}\right)^{-1/2}\\&\times\left(\frac{e_+\sqrt{2}}{\tan\theta_+}+F(n)\frac{e_-\sqrt{2}}{\tan\theta_-}+e_1+\sum_{k=2}^{[(n-1)/2]}\frac{e_k}{\tan\psi_{k-1}}\right)\exp\left(\sum_{k=1}^{[(n-1)/2]}\tilde{e}_k\phi_k\right).\end{aligned} \tag{2.36}$$

2.5. Elementary functions of a polar n-complex variable. The logarithm u_1 of the polar n-complex number u, $u_1=\ln u$, can be defined as the solution of the equation $u=e^{u_1}$. For even n, $\ln u$ exists as an n-complex function with real components if $v_+>0$ and $v_->0$. For odd n $\ln u$ exists as an n-complex function with real components if $v_+>0$. The expression of the logarithm is

$$\begin{aligned}\ln u={}&\ln\rho+\sum_{p=1}^{n-1}h_p\left[\frac{1}{n}\ln\frac{\sqrt{2}}{\tan\theta_+}+F(n)\frac{(-1)^p}{n}\ln\frac{\sqrt{2}}{\tan\theta_-}\right.\\&\left.-\frac{2}{n}\sum_{k=2}^{[(n-1)/2]}\cos\left(\frac{2\pi kp}{n}\right)\ln\tan\psi_{k-1}\right]+\sum_{k=1}^{[(n-1)/2]}\tilde{e}_k\phi_k.\end{aligned} \tag{2.37}$$

The function $\ln u$ is multivalued because of the presence of the terms $\tilde{e}_k\phi_k$.

The power function u^m of the polar n-complex variable u can be defined for real values of m as $u^m=e^{m\ln u}$. It can be shown that

$$u^m=e_+v_+^m+F(n)e_-v_-^m+\sum_{k=1}^{[(n-1)/2]}\rho_k^m\left(e_k\cos m\phi_k+\tilde{e}_k\sin m\phi_k\right). \tag{2.38}$$

For integer values of m, this expression is valid for any $x_0,\dots,x_{n-1}$. The power function is multivalued unless m is an integer.

2.6. Power series of polar n-complex numbers. A power series of the polar n-complex variable u is a series of the form

$$a_0+a_1u+a_2u^2+\cdots+a_lu^l+\cdots. \tag{2.39}$$

Using the inequality

$$|u'u''|\le\sqrt{n}|u'||u''| \tag{2.40}$$

which replaces the relation of equality extant for 2-dimensional complex numbers, it can be shown that the series (2.39) is absolutely convergent for $|u|<c$, where $c=\lim_{l\to\infty}|a_l|/\sqrt{n}|a_{l+1}|$.

The convergence of the series (2.39) can also be studied with the aid of the formulas (2.38), which for integer values of m are valid for any values of $x_0,\dots,x_{n-1}$. If $a_l=\sum_{p=0}^{n-1}h_pa_{lp}$, and

$$A_{l+}=\sum_{p=0}^{n-1}a_{lp},\qquad A_{lk}=\sum_{p=0}^{n-1}a_{lp}\cos\left(\frac{2\pi kp}{n}\right),\qquad \tilde{A}_{lk}=\sum_{p=0}^{n-1}a_{lp}\sin\left(\frac{2\pi kp}{n}\right), \tag{2.41}$$

for $k = 1, \ldots, [(n-1)/2]$, and for even n

$$A_{l-} = \sum_{p=0}^{n-1} (-1)^p a_{lp}, \tag{2.42}$$

the series (2.39) can be written as

$$\sum_{l=0}^{\infty} \left[e_+ A_{l+} v_+^l + F(n) e_- A_{l-} v_-^l + \sum_{k=1}^{[(n-1)/2]} (e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk})(e_k v_k + \tilde{e}_k \tilde{v}_k)^l \right]. \tag{2.43}$$

The series in (2.39) is absolutely convergent for

$$|v_+| < c_+, \qquad |v_-| < c_-, \qquad \rho_k < c_k, \tag{2.44}$$

for $k = 1, \ldots, [(n-1)/2]$, where

$$c_+ = \lim_{l \to \infty} \frac{|A_{l+}|}{|A_{l+1,+}|}, \qquad c_- = \lim_{l \to \infty} \frac{|A_{l-}|}{|A_{l+1,-}|}, \qquad c_k = \lim_{l \to \infty} \frac{(A_{lk}^2 + \tilde{A}_{lk}^2)^{1/2}}{(A_{l+1,k}^2 + \tilde{A}_{l+1,k}^2)^{1/2}}. \tag{2.45}$$

These relations show that the region of convergence of the series (2.39) is an n-dimensional cylinder.

2.7. Analytic functions of polar n-complex variables. The derivative of a function $f(u)$ of the n-complex variable u is defined as a function $f'(u)$ having the property that

$$|f(u) - f(u_0) - f'(u_0)(u - u_0)| \longrightarrow 0 \quad \text{as } |u - u_0| \longrightarrow 0. \tag{2.46}$$

If the difference $u - u_0$ is not parallel to one of the nodal hypersurfaces, the definition in (2.46) can also be written as

$$f'(u_0) = \lim_{u \to u_0} \frac{f(u) - f(u_0)}{u - u_0}. \tag{2.47}$$

The derivative of the function $f(u) = u^m$, with m an integer, is $f'(u) = m u^{m-1}$, as can be seen by developing $u^m = [u_0 + (u - u_0)]^m$ as

$$u^m = \sum_{p=0}^{m} \frac{m!}{p!(m-p)!} u_0^{m-p} (u - u_0)^p, \tag{2.48}$$

and using the definition (2.46).

If the function $f'(u)$ defined in (2.46) is independent of the direction in space along which u is approaching u_0, the function $f(u)$ is said to be analytic, analogously to the case of functions of regular complex variables [14]. The function u^m, with m an integer, of the n-complex variable u is analytic, because the difference $u^m - u_0^m$ is always proportional to $u - u_0$, as can be seen from (2.48). Then series of integer powers of u will also be analytic functions of the n-complex variable u, and this result holds in fact for any commutative algebra.

If the n-complex function $f(u)$ of the polar n-complex variable u is written in terms of the real functions $P_k(x_0, \ldots, x_{n-1})$, $k = 0, 1, \ldots, n-1$ of the real variables $x_0, x_1, \ldots, x_{n-1}$ as

$$f(u) = \sum_{k=0}^{n-1} h_k P_k(x_0, \ldots, x_{n-1}), \tag{2.49}$$

then relations of equality exist between the partial derivatives of the functions P_k. The derivative of the function f can be written as

$$\lim_{\Delta u\to 0}\frac{1}{\Delta u}\sum_{k=0}^{n-1}\left(h_k\sum_{l=0}^{n-1}\frac{\partial P_k}{\partial x_l}\Delta x_l\right), \tag{2.50}$$

where $\Delta u=\sum_{k=0}^{n-1}h_l\Delta x_l$.

The relations between the partial derivatives of the functions P_k are obtained by setting successively in (2.50) $\Delta u=h_l\Delta x_l$, for $l=0,1,\dots,n-1$, and equating the resulting expressions. The relations are

$$\frac{\partial P_k}{\partial x_0}=\frac{\partial P_{k+1}}{\partial x_1}=\cdots=\frac{\partial P_{n-1}}{\partial x_{n-k-1}}=\frac{\partial P_0}{\partial x_{n-k}}=\cdots=\frac{\partial P_{k-1}}{\partial x_{n-1}} \tag{2.51}$$

for $k=0,1,\dots,n-1$. The relations (2.51) are analogous to the Riemann relations for the real and imaginary components of a complex function. It can be shown from (2.51) that the components P_k fulfill the second-order equations

$$\begin{aligned}\frac{\partial^2 P_k}{\partial x_0\partial x_l}&=\frac{\partial^2 P_k}{\partial x_1\partial x_{l-1}}=\cdots=\frac{\partial^2 P_k}{\partial x_{[l/2]}\partial x_{l-[l/2]}}=\frac{\partial^2 P_k}{\partial x_{l+1}\partial x_{n-1}}\\&=\frac{\partial^2 P_k}{\partial x_{l+2}\partial x_{n-2}}=\cdots=\frac{\partial^2 P_k}{\partial x_{l+1+[(n-l-2)/2]}\partial x_{n-1-[(n-l-2)/2]}}\end{aligned} \tag{2.52}$$

for $k,l=0,1,\dots,n-1$.

2.8. Integrals of polar n-complex functions. The singularities of polar n-complex functions arise from terms of the form $1/(u-u_0)^m$, with $m>0$. Functions containing such terms are singular not only at $u=u_0$, but also at all points of the hypersurfaces passing through u_0 and which are parallel to the nodal hypersurfaces.

The integral of a polar n-complex function between two points A,B along a path situated in a region free of singularities is independent of path, which means that the integral of an analytic function along a loop situated in a region free of singularities is zero,

$$\oint_\Gamma f(u)\,du=0, \tag{2.53}$$

where it is supposed that a surface $\sum$ spanning the closed loop Γ is not intersected by any of the hypersurfaces associated with the singularities of the function $f(u)$. Using the expression, equation (2.49), for $f(u)$ and the fact that $du=\sum_{k=0}^{n-1}h_k\,dx_k$, the explicit form of the integral in (2.53) is

$$\oint_\Gamma f(u)\,du=\oint_\Gamma\sum_{k=0}^{n-1}h_k\sum_{l=0}^{n-1}P_l\,dx_{k-l+n[(n-k-1+l)/n]}. \tag{2.54}$$

If the functions P_k are regular on a surface $\sum$ spanning the loop Γ, the integral along the loop Γ can be transformed in an integral over the surface $\sum$ of terms of the form $\partial P_l/\partial x_{k-m+n[(n-k+m-1)/n]}-\partial P_m/\partial x_{k-l+n[(n-k+l-1)/n]}$. The integrals of these terms are equal to zero by (2.51), and this proves (2.53).

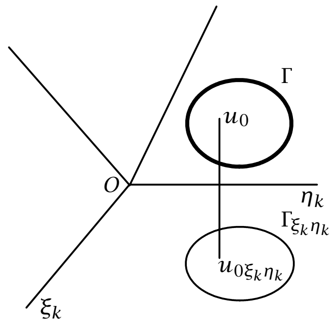

FIGURE 2.3. Integration path Γ and pole u_0, and their projections $\Gamma_{\xi_k\eta_k}$ and $u_{0\xi_k\eta_k}$ on the plane $\xi_k\eta_k$.

The quantity $du/(u-u_0)$ is

$$\frac{du}{u-u_0}=\frac{d\rho}{\rho}+\sum_{p=1}^{n-1}h_p\left[\frac{1}{n}d\ln\frac{\sqrt{2}}{\tan\theta_+}+F(n)\frac{(-1)^p}{n}d\ln\frac{\sqrt{2}}{\tan\theta_-}\right.$$
$$\left.-\frac{2}{n}\sum_{k=2}^{[(n-1)/2]}\cos\left(\frac{2\pi kp}{n}\right)d\ln\tan\psi_{k-1}\right]+\sum_{k=1}^{[(n-1)/2]}\tilde{e}_k\,d\phi_k. \tag{2.55}$$

Since $\rho,\ln(\sqrt{2}/\tan\theta_+),\ln(\sqrt{2}/\tan\theta_-)$, and $\ln(\tan\psi_{k-1})$ are single-valued variables, it follows that $\oint_\Gamma d\rho/\rho=0$, $\oint_\Gamma d(\ln\sqrt{2}/\tan\theta_+)=0$, $\oint_\Gamma d(\ln\sqrt{2}/\tan\theta_-)=0$, and $\oint_\Gamma d(\ln\tan\psi_{k-1})=0$. On the other hand, since ϕ_k are cyclic variables, they may give contributions to the integral around the closed loop Γ.

The expression of $\oint_\Gamma du/(u-u_0)$ can be written with the aid of a functional which will be called $\operatorname{int}(M,C)$, defined for a point M and a closed curve C in a 2-dimensional plane, such that

$$\operatorname{int}(M,C)=\begin{cases}1 & \text{if } M \text{ is an interior point of } C,\\ 0 & \text{if } M \text{ is exterior to } C.\end{cases} \tag{2.56}$$

If $f(u)$ is an analytic function of a polar n-complex variable which can be expanded in a series in the region of the curve Γ and on a surface spanning Γ, then

$$\oint_\Gamma\frac{f(u)\,du}{u-u_0}=2\pi f(u_0)\sum_{k=1}^{[(n-1)/2]}\tilde{e}_k\operatorname{int}\left(u_{0\xi_k\eta_k},\Gamma_{\xi_k\eta_k}\right), \tag{2.57}$$

where $u_{0\xi_k\eta_k}$ and $\Gamma_{\xi_k\eta_k}$ are respectively the projections of the point u_0 and of the loop Γ on the plane defined by the axes ξ_k and η_k, as shown in Figure 2.3.

2.9. Factorization of polar n-complex polynomials. A polynomial of degree m of the polar n-complex variable u has the form

$$P_m(u)=u^m+a_1u^{m-1}+\cdots+a_{m-1}u+a_m, \tag{2.58}$$

where a_l, $l = 1,\dots,m$, are in general polar n-complex constants. If $a_l = \sum_{p=0}^{n-1} h_p a_{lp}$, and with the notations of (2.41) and (2.42) applied for $l = 1,\dots,m$, the polynomial $P_m(u)$ can be written as

$$\begin{aligned} P_m = {} & e_+\left(v_+^m + \sum_{l=1}^{m} A_{l+}v_+^{m-l}\right) + F(n)e_-\left(v_-^m + \sum_{l=1}^{m} A_{l-}v_-^{m-l}\right) \\ & + \sum_{k=1}^{[(n-1)/2]}\left[(e_k v_k + \tilde{e}_k \tilde{v}_k)^m + \sum_{l=1}^{m}(e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk})(e_k v_k + \tilde{e}_k \tilde{v}_k)^{m-l}\right], \end{aligned} \tag{2.59}$$

where the constants $A_{l+}, A_{l-}, A_{lk}, \tilde{A}_{lk}$ are real numbers.

These relations can be written with the aid of (2.28) and (2.29) as

$$P_m(u) = \prod_{p=1}^{m}(u - u_p), \tag{2.60}$$

where

$$u_p = e_+ v_{p+} + F(n)e_- v_{p-} + \sum_{k=1}^{[(n-1)/2]}(e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}), \tag{2.61}$$

for $p = 1,\dots,m$. The roots v_{p+}, the roots v_{p-} and, for a given k, the roots $e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}$ defined in (2.59) may be ordered arbitrarily. This means that (2.61) gives sets of m roots $u_1,\dots,u_m$ of the polynomial $P_m(u)$, corresponding to the various ways in which the roots $v_{p+}, v_{p-}, e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}$ are ordered according to p in each group. Thus, while the polar hypercomplex components in (2.59) taken separately have unique factorizations, the polynomial $P_m(u)$ can be written in many different ways as a product of linear factors.

For example, $u^2 - 1 = (u - u_1)(u - u_2)$, where for even n, $u_1 = \pm e_+ \pm e_- \pm e_1 \pm e_2 \pm \cdots \pm e_{n/2-1}$, $u_2 = -u_1$, so that there are $2^{n/2}$ independent sets of roots u_1, u_2 of $u^2 - 1$. It can be checked that $(\pm e_+ \pm e_- \pm e_1 \pm e_2 \pm \cdots \pm e_{n/2-1})^2 = e_+ + e_- + e_1 + e_2 + \cdots + e_{n/2-1} = 1$. For odd n, $u_1 = \pm e_+ \pm e_1 \pm e_2 \pm \cdots \pm e_{(n-1)/2}$, $u_2 = -u_1$, so that there are $2^{(n-1)/2}$ independent sets of roots u_1, u_2 of $u^2 - 1$. It can be checked that $(\pm e_+ \pm e_1 \pm e_2 \pm \cdots \pm e_{(n-1)/2})^2 = e_+ + e_1 + e_2 + \cdots + e_{(n-1)/2} = 1$.

2.10. Representation of polar n-complex numbers by irreducible matrices. The polar n-complex number u can be represented by the matrix

$$U = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \cdots & x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 & \cdots & x_{n-3} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_0 \end{pmatrix}. \tag{2.62}$$

The product $u = u'u''$ is represented by the matrix multiplication $U = U'U''$. It can be shown that the irreducible form [15] of the matrix U in terms of matrices with real coefficients is, for even n,

$$\begin{pmatrix} v_+ & 0 & 0 & \cdots & 0 \\ 0 & v_- & 0 & \cdots & 0 \\ 0 & 0 & V_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & V_{n/2-1} \end{pmatrix} \tag{2.63}$$

and, for odd n,

$$\begin{pmatrix} v_+ & 0 & 0 & \cdots & 0 \\ 0 & V_1 & 0 & \cdots & 0 \\ 0 & 0 & V_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & V_{(n-1)/2} \end{pmatrix}, \tag{2.64}$$

where

$$V_k = \begin{pmatrix} v_k & \tilde{v}_k \\ -\tilde{v}_k & v_k \end{pmatrix}, \tag{2.65}$$

$k = 1,\dots,[(n-1)/2]$. The relations between the variables $v_+, v_-, v_k, \tilde{v}_k$ for the multiplication of polar n-complex numbers are $v_+ = v'_+ v''_+$, $v_- = v'_- v''_-$, $v_k = v'_k v''_k - \tilde{v}'_k \tilde{v}''_k$, $\tilde{v}_k = v'_k \tilde{v}''_k + \tilde{v}'_k v''_k$.

3. Planar hypercomplex numbers in even n dimensions

3.1. Operations with planar n-complex numbers. A planar hypercomplex number in n dimensions is determined by its n components $(x_0, x_1, \dots, x_{n-1})$. The planar n-complex numbers and their operations discussed in this paper can be represented by writing the n-complex number $(x_0, x_1, \dots, x_{n-1})$ as $u = x_0 + h_1 x_1 + h_2 x_2 + \cdots + h_{n-1} x_{n-1}$, where $h_1, h_2, \dots, h_{n-1}$ are bases for which the multiplication rules are

$$h_j h_k = (-1)^{[(j+k)/n]} h_l, \quad l = j+k-n[(j+k)/n], \tag{3.1}$$

for $j,k,l = 0,1,\dots,n-1$, where $h_0 = 1$. The rules for the planar bases differ from the rules for the polar bases by the minus sign which appears when $n \le j+k \le 2n-2$. The significance of the composition laws in (3.1) can be understood by representing the bases h_j, h_k by points on a circle at the angles $\alpha_j = \pi j/n$, $\alpha_k = \pi k/n$, as shown in Figure 3.1, and the product $h_j h_k$ by the point of the circle at the angle $\pi(j+k)/n$. If $\pi \le \pi(j+k)/n < 2\pi$, the point is opposite to the basis h_l of angle $\alpha_l = \pi(j+k)/n - \pi$.

In an odd number of dimensions n, a transformation of coordinates according to $x_{2l} = x'_l$, $x_{2m-1} = -x'_{(n-1)/2+m}$, and of the bases according to $2l = h'_l$, $h_{2m-1} = -h'_{(n-1)/2+m}$, $l = 0,\dots,(n-1)/2$, $m = 1,\dots,(n-1)/2$, leaves the expression of an n-complex number unchanged, $\sum_{k=0}^{n-1} h_k x_k = \sum_{k=0}^{n-1} h'_k x'_k$, and the products of the bases h'_k are $h'_j h'_k = h'_l$, $l = j+k-n[(j+k)/n]$, $j,k,l = 0,1,\dots,n-1$. Thus, the planar n-complex numbers with the rules are equivalent in an odd number of dimensions to the polar n-complex numbers. Therefore, in this section it will be supposed that n is an even number, unless otherwise stated.

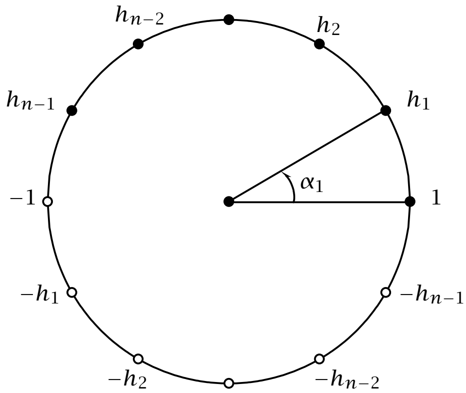

FIGURE 3.1. Representation of the planar n-complex bases $1, h_1, \ldots, h_{n-1}$ by points on a circle at the angles $\alpha_k = \pi k/n$.

Two planar n-complex numbers $u = x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$, $u' = x'_0 + h_1x'_1 + h_2x'_2 + \cdots + h_{n-1}x'_{n-1}$ are equal if and only if $x_j = x'_j$, $j = 0, 1, \ldots, n-1$. The sum of the planar n-complex numbers u and u' is

$$u + u' = x_0 + x'_0 + h_1(x_1 + x'_1) + \cdots + h_{n-1}(x_{n-1} + x'_{n-1}). \tag{3.2}$$

The product of the planar numbers u, u' is

$$\begin{aligned} uu' = {} & x_0x'_0 - x_1x'_{n-1} - x_2x'_{n-2} - x_3x'_{n-3} - \cdots - x_{n-1}x'_1 \\ & + h_1(x_0x'_1 + x_1x'_0 - x_2x'_{n-1} - x_3x'_{n-2} - \cdots - x_{n-1}x'_2) \\ & + h_2(x_0x'_2 + x_1x'_1 + x_2x'_0 - x_3x'_{n-1} - \cdots - x_{n-1}x'_3) \\ & \vdots \\ & + h_{n-1}(x_0x'_{n-1} + x_1x'_{n-2} + x_2x'_{n-3} + x_3x'_{n-4} + \cdots + x_{n-1}x'_0). \end{aligned} \tag{3.3}$$

The product uu' can be written as

$$uu' = \sum_{k=0}^{n-1} h_k \sum_{l=0}^{n-1} (-1)^{[(n-k-1+l)/n]} x_l x'_{k-l+n[(n-k-1+l)/n]}. \tag{3.4}$$

If u, u', u'' are n-complex numbers, the multiplication is associative, $(uu')u'' = u(u'u'')$, and is commutative, $uu' = u'u$, because the product of the bases, defined in (3.1), is both associative and commutative. For $n = 2$ the product is $uu' = x_0x'_0 - x_1x'_1 + h_1(x_0x'_1 + x_1x'_0)$. In 2 dimensions, the notation for h_1 is $h_1 = i$, i being the conventional imaginary unit.

The inverse of the planar n-complex number u is the n-complex number u' having the property that $uu' = 1$. This equation has a solution provided that the corresponding determinant ν is not equal to zero, $\nu \neq 0$. For planar n-complex numbers $\nu \geq 0$, and the quantity $\rho = \nu^{1/n}$ will be called amplitude of the planar n-complex number u.

It can be shown that

$$\nu = \prod_{k=1}^{n/2} \rho_k^2, \tag{3.5}$$

where

$$\rho_k^2 = v_k^2 + \tilde{v}_k^2, \quad v_k = \sum_{p=0}^{n-1} x_p \cos\left(\frac{\pi(2k-1)p}{n}\right), \quad \tilde{v}_k = \sum_{p=0}^{n-1} x_p \sin\left(\frac{\pi(2k-1)p}{n}\right). \tag{3.6}$$

Thus, a planar n-complex number has an inverse unless it lies on one of the nodal hypersurfaces $\rho_1 = 0$, or $\rho_2 = 0$, or ... or $\rho_{n/2} = 0$. It can also be shown that

$$d^2 = \frac{2}{n} \sum_{k=1}^{n/2} \rho_k^2. \tag{3.7}$$

From this relation it results that if the product of two n-complex numbers is zero, $uu' = 0$, then $\rho_k \rho_k' = 0$, $k = 1, \ldots, n/2$, which means that either $u = 0$, or $u' = 0$, or u, u' belong to orthogonal hypersurfaces in such a way that the afore-mentioned products of components should be equal to zero. For $n = 2$, $v_1 = x_0$, $\tilde{v}_1 = x_1$, that is, v_1 and $\tilde{v}_1$ are the real and imaginary parts of a 2-dimensional complex number.

3.2. Geometric representation of planar n-complex numbers. The planar n-complex number $x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$ can be represented by the point A of coordinates $(x_0, x_1, \ldots, x_{n-1})$. If O is the origin of the n-dimensional space, the distance from the origin O to the point A of coordinates $(x_0, x_1, \ldots, x_{n-1})$ has the expression written in (2.10). The quantity d will be now called modulus of the planar n-complex number $u = x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$. The modulus of an n-complex number u will be designated by $d = |u|$. The quantity $\rho = \nu^{1/n}$ will be called amplitude of the n-complex number u.

The exponential and trigonometric forms of the planar n-complex number u can be obtained conveniently in a rotated system of axes defined by the transformation

$$v_k = \sqrt{\frac{n}{2}}\,\xi_k, \qquad \tilde{v}_k = \sqrt{\frac{n}{2}}\,\eta_k, \tag{3.8}$$

for $k = 1, \ldots, n/2$. This transformation from the coordinates $x_0, \ldots, x_{n-1}$ to the variables ξ_k, η_k is unitary.

The position of the point A of coordinates $(x_0, x_1, \ldots, x_{n-1})$ can also be described with the aid of the distance d, equation (2.10), and of $n-1$ angles defined further. Thus, in the plane of the axes $v_k, \tilde{v}_k$, the radius ρ_k and the azimuthal angle ϕ_k can be introduced by the relations

$$\cos\phi_k = \frac{v_k}{\rho_k}, \qquad \sin\phi_k = \frac{\tilde{v}_k}{\rho_k}, \tag{3.9}$$

for $0 \le \phi_k < 2\pi$, $k = 1, \ldots, n/2$, so that there are $n/2$ azimuthal angles. If the projection of the point A on the plane of the axes $v_k, \tilde{v}_k$ is A_k, and the projection of the point A on the 4-dimensional space defined by the axes $v_1, \tilde{v}_1, v_k, \tilde{v}_k$ is A_{1k}, the angle

ψ_{k-1} between the line OA_{1k} and the 2-dimensional plane defined by the axes $v_k, \tilde{v}_k$ is

$$\tan\psi_{k-1} = \frac{\rho_1}{\rho_k}, \tag{3.10}$$

where $0 \le \psi_k \le \pi/2$, $k = 2,\dots,n/2$, so that there are $n/2-1$ planar angles. Thus, the position of the point A is described by the distance d, by $n/2$ azimuthal angles and by $n/2-1$ planar angles. The variables $v, \rho, \rho_k, \tan\psi_k$ are multiplicative and the azimuthal angles ϕ_k are additive upon the multiplication of polar n-complex numbers.

3.3. The planar n-dimensional cosexponential functions. The exponential function of the planar n-complex variable u can be defined by the series $\exp u = 1 + u + u^2/2! + u^3/3! + \cdots$. It can be checked by direct multiplication of the series that $\exp(u+u') = \exp u \cdot \exp u'$, so that $\exp u = \exp x_0 \cdot \exp(h_1x_1)\cdots\exp(h_{n-1}x_{n-1})$.

It can be seen with the aid of the representation in Figure 3.1 that

$$h_k^{n+p} = -h_k^p, \tag{3.11}$$

for integer p, $k = 1,\dots,n-1$. For k even, $e^{h_k y}$ can be written as

$$e^{h_k y} = \sum_{p=0}^{n-1}(-1)^{[kp/n]}h_{kp-n[kp/n]}g_{np}(y), \tag{3.12}$$

where g_{np} are the polar n-dimensional cosexponential functions. For odd k, $e^{h_k y}$ is

$$e^{h_k y} = \sum_{p=0}^{n-1}(-1)^{[kp/n]}h_{kp-n[kp/n]}f_{np}(y), \tag{3.13}$$

where the functions f_{nk}, which will be called planar cosexponential functions in n dimensions, are

$$f_{nk}(y) = \sum_{p=0}^{\infty}(-1)^p\frac{y^{k+pn}}{(k+pn)!}, \tag{3.14}$$

for $k = 0,1,\dots,n-1$.

The planar cosexponential functions of even index k are even functions, $f_{n,2l}(-y) = f_{n,2l}(y)$, and the planar cosexponential functions of odd index are odd functions, $f_{n,2l+1}(-y) = -f_{n,2l+1}(y)$, $l = 0,\dots,n/2-1$.

The planar n-dimensional cosexponential function $f_{nk}(y)$ is related to the polar n-dimensional cosexponential function $g_{nk}(y)$ by the relation

$$f_{nk}(y) = e^{-i\pi k/n}g_{nk}(e^{i\pi/n}y), \tag{3.15}$$

for $k = 0,\dots,n-1$. The expression of the planar n-dimensional cosexponential functions is then

$$f_{nk}(y) = \frac{1}{n}\sum_{l=1}^{n}\exp\left[y\cos\left(\frac{\pi(2l-1)}{n}\right)\right]\cos\left[y\sin\left(\frac{\pi(2l-1)}{n}\right) - \frac{\pi(2l-1)k}{n}\right], \tag{3.16}$$

$k = 0,1,\dots,n-1$. The planar cosexponential function defined in (3.14) has the expression given in (3.16) for any natural value of n, this result is not restricted to even values of n.

It can be shown from (3.16) that

$$\sum_{k=0}^{n-1} f_{nk}^2(y) = \frac{1}{n}\sum_{l=1}^{n} \exp\left[2y\cos\left(\frac{\pi(2l-1)}{n}\right)\right]. \tag{3.17}$$

It can be seen that the right-hand side of (3.17) does not contain oscillatory terms. If n is a multiple of 4, it can be shown by replacing y by iy in (3.17) that

$$\sum_{k=0}^{n-1} (-1)^k f_{nk}^2(y) = \frac{4}{n}\sum_{l=1}^{n/4} \cos\left[2y\cos\left(\frac{\pi(2l-1)}{n}\right)\right], \tag{3.18}$$

which does not contain exponential terms.

For odd n, the planar n-dimensional cosexponential function $f_{nk}(y)$ is related to the n-dimensional cosexponential function $g_{nk}(y)$ also by the relation

$$f_{nk}(y) = (-1)^k g_{nk}(-y), \tag{3.19}$$

as can be seen by comparing the series for the two classes of functions.

Addition theorems for the planar n-dimensional cosexponential functions can be obtained from the relation $\exp h_1(y+z) = \exp h_1 y \cdot \exp h_1 z$, by substituting the expression of the exponentials as given by $e^{h_1 y} = \sum_{p=0}^{n-1} h_p f_{np}(y)$,

$$\begin{aligned} f_{nk}(y+z) &= f_{n0}(y)f_{nk}(z) + f_{n1}(y)f_{n,k-1}(z) + \cdots + f_{nk}(y)f_{n0}(z) \\ &\quad - f_{n,k+1}(y)f_{n,n-1}(z) - f_{n,k+2}(y)f_{n,n-2}(z) - \cdots - f_{n,n-1}(y)f_{n,k+1}(z), \end{aligned} \tag{3.20}$$

for $k = 0, 1, \ldots, n-1$. It can also be shown that

$$\{f_{n0}(y) + h_1 f_{n1}(y) + \cdots + h_{n-1} f_{n,n-1}(y)\}^l = f_{n0}(ly) + h_1 f_{n1}(ly) + \cdots + h_{n-1} f_{n,n-1}(ly). \tag{3.21}$$

For $n = 2$, equation (3.13) has the form $e^{h_1 y} = f_{20} + h_1 f_{21}$, and the planar cosexponential functions are $f_{20}(y) = \cos y$ and $f_{21}(y) = \sin y$.

The planar n-dimensional cosexponential functions are solutions of the nth-order differential equation

$$\frac{d^n \zeta}{du^n} = -\zeta \tag{3.22}$$

whose solutions are of the form $\zeta(u) = A_0 f_{n0}(u) + A_1 f_{n1}(u) + \cdots + A_{n-1} f_{n,n-1}(u)$. It can be checked that the derivatives of the planar cosexponential functions are related by

$$\frac{df_{n0}}{du} = -f_{n,n-1}, \quad \frac{df_{n1}}{du} = f_{n0}, \quad \ldots, \quad \frac{df_{n,n-2}}{du} = f_{n,n-3}, \quad \frac{df_{n,n-1}}{du} = f_{n,n-2}. \tag{3.23}$$

3.4. Exponential and trigonometric forms of planar n-complex numbers. In order to obtain the exponential and trigonometric forms of planar n-complex numbers, a new set of hypercomplex bases will be introduced by the relations

$$e_k = \frac{2}{n}\sum_{p=0}^{n-1} h_p \cos\left(\frac{\pi(2k-1)p}{n}\right), \qquad \tilde{e}_k = \frac{2}{n}\sum_{p=0}^{n-1} h_p \sin\left(\frac{\pi(2k-1)p}{n}\right), \tag{3.24}$$

for $k=1,\ldots,n/2$. The multiplication relations for the new hypercomplex bases are

$$e_k^2=e_k,\quad \tilde e_k^2=-e_k,\quad e_k\tilde e_k=\tilde e_k,\quad e_ke_l=0,\quad e_k\tilde e_l=0,\quad \tilde e_k\tilde e_l=0,\quad k\neq l, \tag{3.25}$$

for $k,l=1,\ldots,n/2$. It can be shown that

$$x_0+h_1x_1+\cdots+h_{n-1}x_{n-1}=\sum_{k=1}^{n/2}\left(e_kv_k+\tilde e_k\tilde v_k\right). \tag{3.26}$$

The exponential form of the planar n-complex number u is

$$u=\rho\exp\left\{\sum_{p=1}^{n-1}h_p\left[-\frac{2}{n}\sum_{k=2}^{n/2}\cos\left(\frac{\pi(2k-1)p}{n}\right)\ln\tan\psi_{k-1}\right]+\sum_{k=1}^{n/2}\tilde e_k\phi_k\right\}, \tag{3.27}$$

where the amplitude is

$$\rho=\left(\rho_1^2\cdots\rho_{n/2}^2\right)^{1/n}. \tag{3.28}$$

The trigonometric form of the planar n-complex number u is

$$\begin{aligned}u=d\left(\frac{n}{2}\right)^{1/2}&\left(1+\frac{1}{\tan^2\psi_1}+\frac{1}{\tan^2\psi_2}+\cdots+\frac{1}{\tan^2\psi_{n/2-1}}\right)^{-1/2}\\&\times\left(e_1+\sum_{k=2}^{n/2}\frac{e_k}{\tan\psi_{k-1}}\right)\exp\left(\sum_{k=1}^{n/2}\tilde e_k\phi_k\right).\end{aligned} \tag{3.29}$$

For $n=2$, $e_1=1$, $\tilde e_1=h_1$, $\rho=d$, there is no planar angle, and (3.27) and (3.29) have both the form $u=\rho\exp(h_1\phi_1)$.

3.5. Elementary functions of a planar n-complex variable. The logarithm u_1 of the planar n-complex number u, $u_1=\ln u$, can be defined as the solution of the equation $u=e^{u_1}$. The logarithm exists as a planar n-complex function with real components for all values of $x_0,\ldots,x_{n-1}$ for which $\rho\neq 0$. The expression of the logarithm is

$$\ln u=\ln\rho+\sum_{p=1}^{n-1}h_p\left[-\frac{2}{n}\sum_{k=2}^{n/2}\cos\left(\frac{\pi(2k-1)p}{n}\right)\ln\tan\psi_{k-1}\right]+\sum_{k=1}^{n/2}\tilde e_k\phi_k. \tag{3.30}$$

The function $\ln u$ is multivalued because of the presence of the terms $\tilde e_k\phi_k$.

The power function u^m of the planar n-complex variable u can be defined for real values of m as $u^m=e^{m\ln u}$. It can be shown that

$$u^m=\sum_{k=1}^{n/2}\rho_k^m\left(e_k\cos m\phi_k+\tilde e_k\sin m\phi_k\right). \tag{3.31}$$

The power function is multivalued unless m is an integer.

3.6. Power series of planar n-complex numbers. A power series of the planar n-complex variable u is a series of the form

$$a_0+a_1u+a_2u^2+\cdots+a_lu^l+\cdots. \tag{3.32}$$

Using the inequality

$$|u'u''| \le \sqrt{\frac{n}{2}}|u'||u''|, \tag{3.33}$$

which replaces the relation of equality extant for 2-dimensional complex numbers, it can be shown that the series (3.32) is absolutely convergent for $|u| < c$, where $c = \lim_{l\to\infty}|a_l|/\sqrt{n/2}|a_{l+1}|$.

The convergence of the series (3.32) can be also studied with the aid of the formulas (3.31), which for integer values of m are valid for any values of $x_0,\dots,x_{n-1}$. If $a_l = \sum_{p=0}^{n-1} h_p a_{lp}$, and

$$A_{lk} = \sum_{p=0}^{n-1} a_{lp}\cos\frac{\pi(2k-1)p}{n}, \qquad \tilde{A}_{lk} = \sum_{p=0}^{n-1} a_{lp}\sin\frac{\pi(2k-1)p}{n}, \tag{3.34}$$

for $k = 1,\dots,n/2$, the series (3.32) can be written as

$$\sum_{l=0}^{\infty}\sum_{k=1}^{n/2}\left(e_k A_{lk} + \tilde{e}_k\tilde{A}_{lk}\right)\left(e_k v_k + \tilde{e}_k\tilde{v}_k\right)^l. \tag{3.35}$$

The series in (3.32) is absolutely convergent for

$$\rho_k < c_k, \tag{3.36}$$

for $k = 1,\dots,n/2$, where

$$c_k = \lim_{l\to\infty}\frac{\left[A_{lk}^2 + \tilde{A}_{lk}^2\right]^{1/2}}{\left[A_{l+1,k}^2 + \tilde{A}_{l+1,k}^2\right]^{1/2}}. \tag{3.37}$$

These relations show that the region of convergence of the series (3.32) is an n-dimensional cylinder.

3.7. Analytic functions of planar n-complex variables. If the n-complex function $f(u)$ of the planar n-complex variable u is written in terms of the real functions $P_k(x_0,\dots,x_{n-1})$, $k = 0,1,\dots,n-1$ of the real variables $x_0,x_1,\dots,x_{n-1}$ as

$$f(u) = \sum_{k=0}^{n-1} h_k P_k(x_0,\dots,x_{n-1}), \tag{3.38}$$

then relations of equality exist between the partial derivatives of the functions P_k,

$$\frac{\partial P_k}{\partial x_0} = \frac{\partial P_{k+1}}{\partial x_1} = \cdots = \frac{\partial P_{n-1}}{\partial x_{n-k-1}} = -\frac{\partial P_0}{\partial x_{n-k}} = \cdots = -\frac{\partial P_{k-1}}{\partial x_{n-1}}, \tag{3.39}$$

for $k = 0,1,\dots,n-1$. The relations (3.39) are analogous to the Riemann relations for the real and imaginary components of a complex function. It can be shown from (3.39) that the components P_k fulfill the second-order equations

$$\begin{aligned}\frac{\partial^2 P_k}{\partial x_0\partial x_l} &= \frac{\partial^2 P_k}{\partial x_1\partial x_{l-1}} = \cdots = \frac{\partial^2 P_k}{\partial x_{[l/2]}\partial x_{l-[l/2]}} = -\frac{\partial^2 P_k}{\partial x_{l+1}\partial x_{n-1}}\\ &= -\frac{\partial^2 P_k}{\partial x_{l+2}\partial x_{n-2}} = \cdots = -\frac{\partial^2 P_k}{\partial x_{l+1+[(n-l-2)/2]}\partial x_{n-1-[(n-l-2)/2]}},\end{aligned} \tag{3.40}$$

for $k,l = 0,1,\dots,n-1$.

3.8. Integrals of planar n-complex functions. The singularities of planar n-complex functions arise from terms of the form $1/(u-u_0)^m$, with $m > 0$. Functions containing such terms are singular not only at $u = u_0$, but also at all points of the hypersurfaces passing through u_0 and which are parallel to the nodal hypersurfaces.

The integral of a planar n-complex function between two points A, B along a path situated in a region free of singularities is independent of path, which means that the integral of an analytic function along a loop situated in a region free of singularities is zero,

$$\oint_\Gamma f(u)\,du = 0, \tag{3.41}$$

where it is supposed that a surface $\sum$ spanning the closed loop Γ is not intersected by any of the hypersurfaces associated with the singularities of the function $f(u)$. Using the expression, equation (3.38), for $f(u)$ and the fact that $du = \sum_{k=0}^{n-1} h_k\,dx_k$, the explicit form of the integral in (3.41) is

$$\oint_\Gamma f(u)\,du = \oint_\Gamma \sum_{k=0}^{n-1} h_k \sum_{l=0}^{n-1} (-1)^{[(n-k-1+l)/n]} P_l\,dx_{k-l+n[(n-k-1+l)/n]}. \tag{3.42}$$

If the functions P_k are regular on a surface $\sum$ spanning the loop Γ, the integral along the loop Γ can be transformed in an integral over the surface $\sum$ of terms of the form $\partial P_l/\partial x_{k-m+n[(n-k+m-1)/n]} - (-1)^s \partial P_m/\partial x_{k-l+n[(n-k+l-1)/n]}$, where $s = [(n-k+m-1)/n] - [(n-k+l-1)/n]$. The integrals of these terms are equal to zero by (3.39), and this proves (3.41).

The quantity $du/(u-u_0)$ is

$$\frac{du}{u-u_0} = \frac{d\rho}{\rho} + \sum_{p=1}^{n-1} h_p \left[-\frac{2}{n} \sum_{k=2}^{n/2} \cos\left(\frac{2\pi kp}{n}\right) d\ln\tan\psi_{k-1} \right] + \sum_{k=1}^{n/2} \tilde{e}_k\,d\phi_k. \tag{3.43}$$

Since ρ and $\ln(\tan\psi_{k-1})$ are single-valued variables, it follows that $\oint_\Gamma d\rho/\rho = 0$, and $\oint_\Gamma d(\ln\tan\psi_{k-1}) = 0$. On the other hand, since ϕ_k are cyclic variables, they may give contributions to the integral around the closed loop Γ.

If $f(u)$ is an analytic function of a polar n-complex variable which can be expanded in a series which holds on the curve Γ and on a surface spanning Γ, then

$$\oint_\Gamma \frac{f(u)\,du}{u-u_0} = 2\pi f(u_0) \sum_{k=1}^{n/2} \tilde{e}_k \operatorname{int}(u_{0\xi_k\eta_k}, \Gamma_{\xi_k\eta_k}). \tag{3.44}$$

For $n = 2$, equation (3.44) becomes $\oint_\Gamma f(u)\,du/(u-u_0) = 2\pi h_1 f(u_0)\operatorname{int}(u_0, \Gamma)$, which is the theorem of Cauchy for 2-complex numbers.

3.9. Factorization of planar n-complex polynomials. A polynomial of degree m of the planar n-complex variable u has the form

$$P_m(u) = u^m + a_1 u^{m-1} + \cdots + a_{m-1}u + a_m, \tag{3.45}$$

where a_l, $l = 1, \ldots, m$, are in general planar n-complex constants. If $a_l = \sum_{p=0}^{n-1} h_p a_{lp}$, and with the notations of (3.34) applied for $l = 1, \ldots, m$, the polynomial $P_m(u)$ can be written as

$$P_m = \sum_{k=1}^{n/2} \left\{ (e_k v_k + \tilde{e}_k \tilde{v}_k)^m + \sum_{l=1}^{m} (e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk})(e_k v_k + \tilde{e}_k \tilde{v}_k)^{m-l} \right\}, \tag{3.46}$$

where the constants $A_{lk}, \tilde{A}_{lk}$ are real numbers.

These relations can be written with the aid of (3.24) as

$$P_m(u) = \prod_{p=1}^{m} (u - u_p), \tag{3.47}$$

where

$$u_p = \sum_{k=1}^{n/2} (e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}), \tag{3.48}$$

for $p = 1, \ldots, m$. For a given k, the roots $e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}$ defined in (3.46) may be ordered arbitrarily. This means that (3.48) gives sets of m roots $u_1, \ldots, u_m$ of the polynomial $P_m(u)$, corresponding to the various ways in which the roots $e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}$ are ordered according to p for each value of k. Thus, while the planar hypercomplex components in (3.46) taken separately have unique factorizations, the polynomial $P_n(u)$ can be written in many different ways as a product of linear factors.

For example, $u^2 + 1 = (u - u_1)(u - u_2)$, where $u_1 = \pm\tilde{e}_1 \pm \tilde{e}_2 \pm \cdots \pm \tilde{e}_{n/2}, u_2 = -u_1$, so that there are $2^{n/2-1}$ independent sets of roots u_1, u_2 of $u^2 + 1$. It can be checked that $(\pm\tilde{e}_1 \pm \tilde{e}_2 \pm \cdots \pm \tilde{e}_{n/2})^2 = -e_1 - e_2 - \cdots - e_{n/2} = -1$.

3.10. Representation of planar n-complex numbers by irreducible matrices. The planar n-complex number u can be represented by the matrix

$$U = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ -x_{n-1} & x_0 & x_1 & \cdots & x_{n-2} \\ -x_{n-2} & -x_{n-1} & x_0 & \cdots & x_{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -x_1 & -x_2 & -x_3 & \cdots & x_0 \end{pmatrix}. \tag{3.49}$$

The product $u = u'u''$ is, for even n, represented by the matrix multiplication $U = U'U''$. It can be shown that the irreducible form [15] of the matrix U, in terms of matrices with real coefficients, is

$$\begin{pmatrix} v_+ & 0 & \cdots & 0 \\ 0 & V_1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & V_{n/2} \end{pmatrix}, \tag{3.50}$$

where

$$V_k = \begin{pmatrix} v_k & \tilde{v}_k \\ -\tilde{v}_k & v_k \end{pmatrix}, \tag{3.51}$$

for $k = 1, \ldots, n/2$. The relations between the variables $v_k, \tilde{v}_k$ for the multiplication of planar n-complex numbers are $v_k = v'_k v''_k - \tilde{v}'_k \tilde{v}''_k$, $\tilde{v}_k = v'_k \tilde{v}''_k + \tilde{v}'_k v''_k$.

4. Conclusions. The polar and planar n-complex numbers described in this paper have a geometric representation based on modulus, amplitude, and angular variables. The n-complex numbers have exponential and trigonometric forms, which can be expressed with the aid of geometric variables. The exponential function of an n-complex variable can be developed in terms of the cosexponential functions. The n-complex functions defined by series of powers are analytic, and the partial derivatives of the real components of n-complex functions are closely related. The integrals of n-complex functions are independent of path in regions where the functions are regular. The fact that the exponential form of the n-complex numbers depends on the cyclic azimuthal variables leads to the concept of pole and residue for n-complex integrals on closed paths. The polynomials of polar n-complex variables can be written as products of linear or quadratic factors, and the polynomials of planar n-complex variables can be written as products of linear factors.

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