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ON WEAK CENTER GALOIS EXTENSIONS OF RINGS

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ABSTRACT. Let B be a ring with 1, C the center of B, G a finite automorphism group of B, and B^G the set of elements in B fixed under each element in G. Then, the notion of a center Galois extension of B^G with Galois group G (i.e., C is a Galois algebra over C^G with Galois group $G|_C \cong G$) is generalized to a weak center Galois extension with group G, where B is called a weak center Galois extension with group G if $BI_i = Be_i$ for some idempotent in C and $I_i = \{c - g_i(c) \mid c \in C\}$ for each $g_i \neq 1$ in G. It is shown that B is a weak center Galois extension with group G if and only if for each $g_i \neq 1$ in G there exists an idempotent e_i in C and $\{b_k e_i \in Be_i; c_k e_i \in Ce_i, k = 1,2,\dots,m\}$ such that $\sum_{k=1}^m b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i$ and g_i restricted to $C(1-e_i)$ is an identity, and a structure of a weak center Galois extension with group G is also given.

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1. Introduction. Galois theory for fields was generalized for rings in the sixties and seventies [3, 4, 7, 8]. Let B be a ring with 1, $G = \{g_1 = 1, g_2, \dots, g_n\}$ an automorphism group of B of order n for some integer n, C the center of B, and B^G the set of elements in B fixed under each element in G. There are several well-known classes of noncommutative Galois extensions: (1) the DeMeyer-Kanzaki Galois extension B (i.e., B is an Azumaya C-algebra which is a Galois algebra with Galois group $G|_C \cong G$) [3, 7], (2) the H-separable Galois extension B (i.e., B is a Galois and a H-separable extension of B^G) [8], (3) the Azumaya Galois extension B (i.e., B is a Galois extension of B^G which is an Azumaya C^G-algebra) [1], (4) the central Galois algebra [3, 4, 7], and (5) the center Galois extension B (i.e., C is a Galois algebra over C^G with Galois group $G|_C \cong G$) [11]. We note that a commutative Galois extension is a DeMeyer-Kanzaki Galois extension which is a center Galois extension. It is well know that C is a Galois extension of C^G if and only if the ideals generated by $\{c - g(c) \mid c \in C\}$ is C for each $g \neq 1$ in G [2, Proposition 1.2, page 80]. This fact was generalized in [11] to a center Galois extension; that is, B is a center Galois extension of B^G if and only if the ideals of B generated by $\{c - g(c) \mid c \in C\}$ is B, that is, $BI_i = B$, where $I_i = \{c - g_i(c) \mid c \in C\}$ for each $g_i \neq 1$ in G (for more about center Galois extensions, see [5, 6, 9, 10, 11]). Generalizing the condition that $BI_i = B = B1$ to that $BI_i = Be_i$ for some idempotent e_i in C for each $g_i \neq 1$ in G, we obtain a broader class of rings B than the class of center Galois extensions. This class of rings is called weak center Galois extensions. The purpose of the present paper is to give a characterization and a structure of a weak center Galois extension B with group G. We shall show that B is a weak center Galois extension with group G if and only if for each $g_i \neq 1$ in G there exists an idempotent e_i in C and $\{b_k e_i \in Be_i; c_k e_i \in Ce_i, k = 1,2,\dots,m\}$ such that $\sum_{k=1}^m b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i$

and g_i restricted to $C(1-e_i)$ is an identity. Next, we call B a T-Galois extension of B^T if there exist elements $\{a_i, b_i \text{ in } B,\ i=1,2,\dots,m\}$ for some integer m such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ for $g \in T \cup \{1\}$. We note that T is not necessarily a subgroup of G. Let B be a weak center Galois extension with group G. Then, we show that there exists a partition of $G-\{1\}$, $\{T_j \subset G,\ j=1,2,\dots,h$ for some integer $h\}$ and some idempotents $e_j \in C$, $j=1,2,\dots,h$ such that Be_j is a T_j-Galois extension of $(Be_j)^{T_j}$. So $B = \sum_{j=1}^h Be_j \oplus B(1-\vee_{j=1}^h e_j)$ such that Be_j is a T_j-Galois extension of $(Be_j)^{T_j}$ for $j=1,2,\dots,h$, where $\vee$ is the sum of the Boolean algebra of the idempotents in C. Moreover, when G is abelian, e_j can be taken as orthogonal idempotents in C so that $\sum_{j=1}^h Be_j$ is a direct sum. Furthermore, a sufficient condition is given for the existence of a subgroup $H_j \subset T_j \cup \{1\}$ for $j=1,2,\dots,h$. In this case, Be_j is a H_j-Galois extension of $(Be_j)^{H_j}$ with Galois group H_j.

2. Definitions and notation. Throughout this paper, B represents a ring with 1, $G=\{g_1=1,g_2,\dots,g_n\}$ an automorphism group of B of order n for some integer n, C the center of B, and B^G the set of elements in B fixed under each element in G. We denote $I_i=\{c-g_i(c)\mid c\in C\}$ and BI_i the ideal of B generated by I_i for $g_i \in G$.

B is called a G-Galois extension of B^G if there exist elements $\{a_i,b_i \text{ in } B,\ i=1,2,\dots,m\}$ for some integer m such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$. Such a set $\{a_i,b_i\}$ is called a G-Galois system for B. B is called a weak center Galois extension of B^G with group G if $BI_i = Be_i$ for some idempotent in C for each $g_i \neq 1$ in G. For a subset T (not necessary a subgroup) of G, B is called a T-Galois extension of B^T if there exist elements $\{a_i,b_i \text{ in } B,\ i=1,2,\dots,m\}$ for some integer m such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ for $g \in T\cup\{1\}$. Such a set $\{a_i,b_i\}$ is called a T-Galois system for B. For a B-module M, we denote $\mathrm{Ann}_B(M)=\{b\in B\mid bm=0 \text{ for all } m\in M\}$.

3. Weak center Galois extensions. In [11], the present authors showed that a center Galois extension B is equivalent to each of the following statements: (i) $BI_i = B$ for each $g_i \neq 1$ in G and (ii) B is a Galois extension of B^G with a Galois system $\{b_i \in B,\ c_i \in C,\ i=1,2,\dots,m\}$ for some integer m. In this section, we generalize this characterization to a weak center Galois extension B with group G. We begin with the following lemma.

Lemma 3.1. *If B is a weak center Galois extension with group G, then*
(1) *g_i restricted to Be_i is an automorphism of Be_i.*
(2) *Be_i is a $\{g_i\}$-Galois extension of $(Be_i)^{\{g_i\}}$.*

Proof. (1) For any $b=\sum_{k=1}^m b_k(c_k-g_i(c_k)) \in BI_i = Be_i$, where $b_k\in B$ and $c_k \in C$, $k=1,2,\dots,m$ for some integer m, we have $g_i(b)=g_i(\sum_{k=1}^m b_k(c_k-g_i(c_k))) = \sum_{k=1}^m g_i(b_k)(g_i(c_k)-g_i(g_i(c_k))) \in BI_i = Be_i$. Hence, $g_i(Be_i)\subset Be_i$. Thus, g_i restricted to Be_i is an automorphism of Be_i since g_i is an automorphism of B.

(2) Since $BI_i = Be_i$, there exist $\{b_k\in B,\ c_k\in C,\ k=1,2,\dots,m\}$ for some integer m such that $\sum_{k=1}^m b_k(c_k-g_i(c_k)) = e_i$. Therefore, $\sum_{k=1}^m b_kc_k = e_i + \sum_{k=1}^m b_k g_i(c_k)$. Let $b_{m+1} = -\sum_{k=1}^m b_k g_i(c_k)$ and $c_{m+1}=1$. Then $\sum_{k=1}^{m+1} b_kc_k = e_i$ and $\sum_{k=1}^{m+1} b_k g_i(c_k) = 0$. Noting that e_i is the identity of Be_i and g_i restricted to Be_i is an automorphism

of Be_i, we have $g_i(e_i) = e_i$. Hence, $\sum_{k=1}^{m+1} b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i$, that is, $\{b_k e_i; c_k e_i, k = 1,2,\ldots,m+1\}$ is a $\{g_i\}$-Galois system for Be_i. □

The following is an equivalent condition for a weak center Galois extension with group G.

Theorem 3.2. *B is a weak center Galois extension with group G (i.e., $BI_i = Be_i$ for some idempotent e_i in C for each $g_i \neq 1$ in G) if and only if for each $g_i \neq 1$ in G there exists an idempotent e_i in C and $\{b_k e_i \in Be_i; c_k e_i \in Ce_i, k = 1,2,\ldots,m\}$ such that $\sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i$ and g_i restricted to $C(1-e_i)$ is an identity.*

Proof. ($\Rightarrow$) By Lemma 3.1(2), BI_i $(= Be_i)$ contains a $\{g_i\}$-Galois system $\{b_k e_i \in Be_i; c_k e_i \in Ce_i, k = 1,2,\ldots,m\}$ such that $\sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i$. Next, we show that g_i restricted to $C(1-e_i)$ is an identity. In fact, by Lemma 3.1(1), $g_i(e_i) = e_i$. Hence, for any $c \in C$, $c(1-e_i) - g_i(c(1-e_i)) = (c - g_i(c))(1-e_i) \in Ce_i \cap C(1-e_i) = \{0\}$. Thus, $g_i(c(1-e_i)) = c(1-e_i)$ for all $c \in C$. This proves that g_i restricted to $C(1-e_i)$ is an identity.

($\Leftarrow$) By hypothesis, for each $g_i \neq 1$ in G there exists an idempotent e_i in C and $\{b_k e_i \in Be_i; c_k e_i \in Ce_i, k = 1,2,\ldots,m\}$ such that $\sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i$. Hence, $e_i = \sum_{k=1}^{m} b_k e_i (c_k e_i - g_i(c_k e_i)) \in BI_i$. Hence, $Be_i \subset BI_i$. But e_i is an idempotent, so $Be_i = Be_i e_i \subset BI_i e_i \subset Be_i$. Thus, $Be_i = BI_i e_i$. Since g_i restricted to $C(1-e_i)$ is an identity, $g_i(c(1-e_i)) = c(1-e_i)$ for all $c \in C$ (in particular, $g_i(e_i) = e_i$). Hence, $c - g_i(c) = ce_i - g_i(ce_i) = (c - g_i(c))e_i$ for all $c \in C$. This implies that $Be_i = BI_i e_i = BI_i$. □

Recall that B is called a T-Galois extension of B^T for a subset T (not necessary a subgroup) of G if B contains a T-Galois system. Next, we give a structure of a weak center Galois extension with group G.

Lemma 3.3. *Assume B is a weak center Galois extension with group G. Let $T_j = \{g_i \in G \mid BI_i = Be_j,$ i.e., $e_i = e_j\}$. Then Be_j is a T_j-Galois extension of $(Be_j)^{T_j}$ for each $j \neq 1$.*

Proof. By the proof of Lemma 3.1(2), for each $g_i \in T_j$, there is a $\{g_i\}$-Galois system $\{b_k^{(i)} e_j; c_k^{(i)} e_j, k = 1,2,\ldots,m_i\}$ for Be_j, where $b_k^{(i)} \in B$ and $c_k^{(i)} \in C$, $k = 1,2,\ldots,m_i$ for some integer m_i. Denote the elements in T_j by $\{g_{i_1}, g_{i_2}, \ldots, g_{i_t}\}$ for some integer t. Let $b_{k_1,k_2,\ldots,k_t} = b_{k_1}^{(i_1)} b_{k_2}^{(i_2)} \cdots b_{k_t}^{(i_t)} e_j$ and $c_{k_1,k_2,\ldots,k_t} = c_{k_1}^{(i_1)} c_{k_2}^{(i_2)} \cdots c_{k_t}^{(i_t)} e_j$ for $k_l = 1,2,\ldots,m_{i_l}$ and $l = 1,2,\ldots,t$. Noting that $c_{k_l}^{(i_l)} \in C$, $l = 1,2,\ldots,t$, we have

$$\begin{aligned}
&\sum_{k_1=1}^{m_{i_1}} \sum_{k_2=1}^{m_{i_2}} \cdots \sum_{k_t=1}^{m_{i_t}} b_{k_1,k_2,\ldots,k_t} c_{k_1,k_2,\ldots,k_t} \\
&\quad = \sum_{k_1=1}^{m_{i_1}} \sum_{k_2=1}^{m_{i_2}} \cdots \sum_{k_t=1}^{m_{i_t}} \left(b_{k_1}^{(i_1)} b_{k_2}^{(i_2)} \cdots b_{k_t}^{(i_t)} e_j\right)\left(c_{k_1}^{(i_1)} c_{k_2}^{(i_2)} \cdots c_{k_t}^{(i_t)} e_j\right) \\
&\quad = \sum_{k_1=1}^{m_{i_1}} \left(b_{k_1}^{(i_1)} e_j\right)\left(c_{k_1}^{(i_1)} e_j\right) \sum_{k_2=1}^{m_{i_2}} \left(b_{k_2}^{(i_2)} e_j\right)\left(c_{k_2}^{(i_2)} e_j\right) \cdots \sum_{k_t=1}^{m_{i_t}} \left(b_{k_t}^{(i_t)} e_j\right)\left(c_{k_t}^{(i_t)} e_j\right) \\
&\quad = e_j,
\end{aligned} \tag{3.1}$$

and, for each $g_i \in T_j$,

$$\begin{aligned}
&\sum_{k_1=1}^{m_{i_1}}\sum_{k_2=1}^{m_{i_2}}\cdots\sum_{k_t=1}^{m_{i_t}} b_{k_1,k_2,\dots,k_t} g_i\left(c_{k_1,k_2,\dots,k_t}\right) \\
&\quad = \sum_{k_1=1}^{m_{i_1}}\sum_{k_2=1}^{m_{i_2}}\cdots\sum_{k_t=1}^{m_{i_t}} \left(b_{k_1}^{(i_1)} b_{k_2}^{(i_2)}\cdots b_{k_t}^{(i_t)} e_j\right) g_i\left(c_{k_1}^{(i_1)} c_{k_2}^{(i_2)}\cdots c_{k_t}^{(i_t)} e_j\right) \\
&\quad = \sum_{k_1=1}^{m_{i_1}} \left(b_{k_1}^{(i_1)} e_j\right) g_i\left(c_{k_1}^{(i_1)} e_j\right) \sum_{k_2=1}^{m_{i_2}} \left(b_{k_2}^{(i_2)} e_j\right) g_i\left(c_{k_2}^{(i_2)} e_j\right) \cdots \sum_{k_t=1}^{m_{i_t}} \left(b_{k_t}^{(i_t)} e_j\right) g_i\left(c_{k_t}^{(i_t)} e_j\right) \\
&\quad = 0.
\end{aligned} \tag{3.2}$$

Thus, $\{b_{k_1,k_2,\dots,k_t};\ c_{k_1,k_2,\dots,k_t},\ k_l = 1,2,\dots,m_{i_l}$ and $l = 1,2,\dots,t\}$ is a T_j-Galois system for Be_j. This completes the proof. □

Theorem 3.4. *If B is a weak center Galois extension with group G, then there exists a partition $\{T_j \subset G,\ j = 1,2,\dots,m\}$ of $G-\{1\}$ and a finite set of central idempotents $\{e'_i \mid i = 1,2,\dots,m$ for some integer $m\}$ such that* (1) *Be'_j is a T_j-Galois extension of $(Be'_j)^{T_j}$,* (2) *$B = \sum_{j=1}^{m} Be'_j \oplus B(1-\vee_{j=1}^{m} e'_j)$, where $\vee_{j=1}^{m} e'_j$ is the sum of $e'_1, e'_2, \dots, e'_m$ in the Boolean algebra of all idempotents in C, and* (3) *$G|_{C(1-\vee_{j=1}^{m} e'_j)} = \{1\}$.*

Proof. (1) Since $BI_i = Be_i$ for some idempotent e_i in C for each $g_i \neq 1$ in G, we have a set of central idempotents $\{e_i \mid g_i \neq 1$ in $G\}$. Let $E = \{e'_j \mid j = 1,2,\dots,m\}$ be the set of all distinct idempotents in $\{e_i \mid g_i \neq 1$ in $G\}$ and let $T_j = \{g_i \in G \mid BI_i = Be'_j$, i.e., $e_i = e'_j\}$. Then Be'_j is a T_j-Galois extension of $(Be'_j)^{T_j}$ for each $j = 1,2,\dots,m$ by Lemma 3.3. Moreover, since $E = \{e'_j \mid j = 1,2,\dots,m\}$ is the set of all distinct idempotents in $\{e_i \mid BI_i = Be_i$ for $g_i \neq 1$ in $G\}$, it is easy to see that $T_i \cap T_j = \emptyset$, the empty set for $i \neq j$ and that $\cup_{j=1}^{m} T_j = G-\{1\}$, that is, $\{T_j \subset G,\ j = 1,2,\dots,m\}$ is a partition of $G-\{1\}$.

Part (2) is an immediate consequence of part (1), and Theorem 3.2 implies part (3).

We remark that the partition of $G-\{1\}$, $\{T_j \subset G,\ j = 1,2,\dots,m\}$ is determined by the set of all distinct idempotents in $\{e_i \mid BI_i = Be_i$ for $g_i \neq 1$ in $G\}$. □

When G is abelian, we obtain a stronger structure of a weak center Galois extension with group G.

Lemma 3.5. *Assume that B is a weak center Galois extension with group G. If G is abelian, then $g_j(e_i) = e_i$ for all $i, j = 2,3,\dots,n$.*

Proof. For any $c - g_i(c) \in I_i$, $g_j(c - g_i(c)) = g_j(c) - g_i(g_j(c)) \in I_i$. Hence, $g_j(BI_i) \subset BI_i$. Thus, g_j restricted to BI_i $(= Be_i)$ is an automorphism of Be_i since g_j is an automorphism of B. Therefore, $g_j(e_i) = e_i$. □

Theorem 3.6. *Assume that B is a weak center Galois extension with group G. If G is abelian, then there exist orthogonal idempotents $\{f_i \mid i = 1,2,\dots,p$ for some integer $p\}$ and some subset $T^{(i)}$ of G, $i = 1,2,\dots,p$ such that $B = \oplus\sum_{i=1}^{p} Bf_i \oplus B(1-\vee_{i=1}^{p} f_i)$, where $\vee_{i=1}^{p} f_i$ is the sum of $f_1, f_2, \dots, f_p$ in the Boolean algebra of all idempotents in C and Bf_i is a $T^{(i)}$-Galois extension of $(Bf_i)^{T^{(i)}}$ for $i = 1,2,\dots,p$.*

Proof. By Theorem 3.4, there exists a set of distinct idempotents $E = \{e'_j \mid j = 1,2,\dots,m\}$ in C and a partition $\{T_j \mid j = 1,2,\dots,m\}$ of $G-\{1\}$ such that Be'_j is a T_j-Galois extension of $(Be'_j)^{T_j}$ for $j = 1,2,\dots,m$. Now, let S be the Boolean subalgebra generated by E with all nonzero minimal elements $f_1,f_2,\dots,f_p$ in S. Then, it is easy to see that $f_if_j = 0$ for $i \neq j$, and so $f_1,f_2,\dots,f_p$ are orthogonal idempotents in C. For each f_i, $i = 1,2,\dots,p$, $f_i = e'_{j_1}e'_{j_2}\cdots e'_{j_{p_i}}$. By Theorem 3.4, Be'_{j_l} is a T_{j_l}-Galois extension of $(Be'_{j_l})^{T_{j_l}}$ for each $l = 1,2,\dots,p_i$ with a T_{j_l}-Galois system $\{b^{(l)}_{t_l}e'_{j_l}; c^{(l)}_{t_l}e'_{j_l} \mid b^{(l)}_{t_l} \in B, c^{(l)}_{t_l} \in C$, and $t_l = 1,2,\dots,m_l\}$. Hence, by using the same patching method as given in Lemma 3.3, $\{b_{t_1,t_2,\dots,t_{p_i}} = b^{(1)}_{t_1}b^{(2)}_{t_2}\cdots b^{(p_i)}_{t_{p_i}}f_i;\ c_{t_1,t_2,\dots,t_{p_i}} = c^{(1)}_{t_1}c^{(2)}_{t_2}\cdots c^{(p_i)}_{t_{p_i}}f_i \mid t_l = 1,2,\dots,m_l$ and $l = 1,2,\dots,p_i\}$ is a $T^{(i)}$-Galois system for Bf_i, where $T^{(i)} = \cup_{l=1}^{k_i}T_{j_l}$. Thus, $B = \oplus\sum_{i=1}^{p}Bf_i \oplus B(1-\vee_{i=1}^{p}f_i)$ such that Bf_i is a $T^{(i)}$-Galois extension of $(Bf_i)^{T^{(i)}}$ for $i = 1,2,\dots,p$ and $\{f_1,f_2,\dots,f_p\}$ is a set of orthogonal idempotents in C. □

4. Special cases. We note that the T_i's in Theorem 3.4 and $T^{(i)}$'s in Theorem 3.6 may not be subgroups of G. Next, we give a sufficient condition for each $T_i \cup \{1\}$ $(\subset G)$ containing a subgroup H_i so that Be_i is a H_i-Galois extension of $(Be_i)^{H_i}$ with Galois group H_i. Consequently, Be_i becomes a center Galois extension of $(Be_i)^{H_i}$ with Galois group H_i, and B is a center Galois extension of G with Galois group G if $e_i = 1$ for all $g_i \neq 1$. We first show a relation between $B(1-e_p)$, $B(1-e_q)$, and $B(1-e_t)$, where $g_pg_q = g_t \in G$.

Lemma 4.1. *Let $J_i = \{b \in B \mid bc = g_i(c)b$ for all $c \in C\}$ for each $g_i \in G$. Then, $J_pJ_q \subset J_t$ if $g_pg_q = g_t \in G$.*

Proof. Let $a \in J_p$ and $b \in J_q$. Then, for any $c \in C$, $(ab)c = ag_q(c)b = g_p(g_q(c))ab = g_t(c)(ab)$, where $g_pg_q = g_t$. Hence, $ab \in J_t$. Thus, $J_pJ_q \subset J_t$. □

Corollary 4.2. *If B is a weak center Galois extension with group G, then $B(1-e_p)B(1-e_q) \subset B(1-e_t)$, where $g_pg_q = g_t \in G$.*

Proof. Since B is a weak center Galois extension with group G, $BI_i = Be_i$ for some idempotent e_i in C for each $g_i \neq 1$ in G. But $I_i = \{c-g_i(c) \mid c \in C\}$, so $J_i = \{b \in B \mid bc = g_i(c)b$ for all $c \in C\} = \{b \in B \mid b(c-g_i(c)) = 0$ for all $c \in C\}$. Hence, $J_i = \mathrm{Ann}_B(I_i) = \mathrm{Ann}_B(BI_i) = \mathrm{Ann}_B(Be_i) = B(1-e_i)$. Thus, by Lemma 4.1, we have $B(1-e_p)B(1-e_q) \subset B(1-e_t)$, where $g_pg_q = g_t \in G$. □

Theorem 4.3. *Assume that B is a weak center Galois extension with group G. Let T_i, for each $i = 2,3,\dots,n$, be the subset of G as given in Theorem 3.4 such that Be_i is a T_i-Galois extension of $(Be_i)^{T_i}$, S the Boolean subalgebra generated by $\{e_i \mid g_i \neq 1$ in $G\}$ with all nonzero minimal elements $\{f_1,f_2,\dots,f_k\}$ in S, and $H_j = \{1\} \cup \{g_i \in G \mid e_if_j = f_j$ and $e_if_l = 0$ for all $l \neq j\}$. Then, H_j is a subgroup of G for each $j = 1,2,\dots,k$ such that $g_i(f_j) = f_j$ for each $g_i \in H_j$.*

Proof. (1) For any g_p and g_q in H_j, let $g_pg_q = g_t$ for some $g_t \in G$. We claim that $g_t \in H_j$ if $g_t \neq 1$. Since $g_t \neq 1$, $BI_t = Be_t$ for some idempotent $e_t \neq 0$ in C. By Corollary 4.2, $B(1-e_p)B(1-e_q) \subset B(1-e_t)$. Therefore, in the Boolean algebra of all idempotents in C with operations $\wedge$, $\vee$, complement, and the relation $<$, $(1-e_p)(1-e_q) < (1-e_t)$. So $e_t < e_p \vee e_q = e_p + e_q - e_p e_q$. Thus, $e_t = e_t(e_p + e_q - e_p e_q)$. Since $g_p, g_q \in H_j$, $e_p f_l = 0$ and $e_q f_l = 0$ for all $l \neq j$. Hence, $e_t f_l = e_t(e_p + e_q - e_p e_q) f_l = 0$ for all $l \neq j$. Moreover, since S is the Boolean subalgebra generated by $\{e_i \mid g_i \neq 1 \text{ in } G\}$, there is at least one nonzero minimal element in S less than e_t. But $e_t f_l = 0$ for all $l \neq j$, so f_j must be less than e_t. Hence, $e_t f_j = f_j$. Thus, $g_t (= g_p g_q) \in H_j$, and so H_j is a subgroup of G. Moreover, suppose $g_i \in H_j$. Then $e_i f_j = f_j$ and $e_i f_l = 0$ for all $l \neq j$. Hence, e_i is greater than f_j, but not greater than f_l for all $l \neq j$. Since $g_i(e_i) = e_i$, $g_i(f_j)$ is a nonzero minimal element in S less than e_i. Thus, $g_i(f_j) = f_j$. □

Corollary 4.4. *Keeping the notation in Theorem 4.3, if $H_j \neq \{1\}$ for $j = 1,2,\dots,p$, then $B = \oplus\sum_{j=1}^{p} Bf_j \oplus B(1 - \vee_{j=1}^{p} f_j)$, where $\vee_{j=1}^{p} f_j$ is the sum of $f_1, f_2, \dots, f_p$ in the Boolean algebra of all idempotents in C, such that Bf_j is a H_j-Galois extension of $(Bf_j)^{H_j}$ with Galois group H_j for $j = 1,2,\dots,p$.*

Corollary 4.5. *If $BI_j = B$ for each $g_j \neq 1$ in G, then B is a center Galois extension of B^G with Galois group G.*

Proof. Since $e_2 = e_3 = \cdots = e_n$, $T_2 = T_3 = \cdots = T_n = G - \{1\}$, so $T_j \cup \{1\} = G$. Thus, B is a Galois extension of B^G with a Galois system $\{b_i \in B; c_i \in C,\ i = 1,2,\dots,m\}$ for some integer m, that is, B is a center Galois extension of B^G with Galois group G. □

If the order of each nonidentity element in G has order 2 (hence, G is abelian), the following theorem shows that $T_i \cup \{1\}$ contains a subgroup of G for each $g_j \neq 1$ in T_i.

Theorem 4.6. *Assume that B is a weak center Galois extension with group G. If each nonidentity element g_i in G has order 2, then T_i contains a subgroup of H_i of order 2 for each $g_j \neq 1$ in G such that Be_i is a H_i-Galois extension of $(Be_i)^{H_i}$ with Galois group H_i.*

Proof. Let $BI_i = Be_i$ for $g_i \neq 1$ in G. Then $H_i = \{1, g_i\}$ is a subgroup contained in $T_i \cup \{1\}$, where $T_i = \{g_k \in G \mid BI_k = Be_i\}$ as defined in Theorem 3.4. Since Be_i is a T_i-Galois extension of $(Be_i)^{T_i}$, Be_i is a H_i-Galois extension of $(Be_i)^{H_i}$ with Galois group H_i. □

Theorem 3.4 shows that a weak center Galois extension is a sum of T_i-Galois extensions for some $T_i \subset G$ and Theorem 4.6 states a weak center Galois extension as a direct sum of center Galois extensions. The following is an example of a weak center Galois extension with group G as given in Theorem 4.6, but not a Galois extension.

Example 4.7. *Let $\mathbb{Q}$ be the rational field, $B = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$, and $G = \{g_1 = 1, g_2, g_3, g_4 = g_2 g_3\}$ such that $g_2(a_1,a_2,a_3,a_4,a_5) = (a_2,a_1,a_3,a_4,a_5)$ and $g_3(a_1,a_2,a_3,a_4,a_5) = (a_1,a_2,a_4,a_3,a_5)$ for all $(a_1,a_2,a_3,a_4,a_5) \in B$. Then,*

(1) $BI_i = Be_i$ for each $g_i \neq 1$ in G, where $e_2 = (1,1,0,0,0)$, $e_3 = (0,0,1,1,0)$, and $e_4 = (1,1,1,1,0)$. Hence, B is a weak center Galois extension with group G.

(2) B is not a Galois extension since G restricted to $\{(0,0,0,0,a) \mid a \in \mathbb{Q}\}$ is identity.

(3) Let $H_i = \{1, g_i\}$, $i = 2,3,4$. Then H_i is a subgroup of G of order 2. Moreover, $BI_2 = Be_2$ is a center H_2-Galois extension of $(Be_2)^{H_2}$ with Galois system $\{b_1 = (1,0,0,0,0)$, $b_2 = (0,1,0,0,0)$; $c_1 = (1,0,0,0,0)$, $c_2 = (0,1,0,0,0)\}$, $BI_3 = Be_3$ is a center H_3-Galois extension of $(Be_3)^{H_3}$ with Galois system $\{b_1 = (0,0,1,0,0)$, $b_2 = (0,0,0,1,0)$; $c_1 = (0,0,$

$1,0,0)$, $c_2 = (0,0,0,1,0)\}$, *and* $BI_4 = Be_4$ *is a center* H_4*-Galois extension of* $(Be_4)^{H_4}$ *with Galois system* $\{b_1 = (1,0,0,0,0)$, $b_2 = (0,1,0,0,0)$, $b_3 = (0,0,1,0,0)$, $b_4 = (0,0,0,1,0)$; $c_1 = (1,0,0,0,0)$, $c_2 = (0,1,0,0,0)$, $c_3 = (0,0,1,0,0)$, $c_4 = (0,0,0,1,0)\}$.

(4) $S = \{0 = (0,0,0,0,0), e_2, e_3, e_4, 1 = (1,1,1,1,1)\}$ *is the Boolean subalgebra generated by* $E = \{e_2, e_3, e_4\}$ *in the Boolean algebra of all idempotents in the center of* B. *The minimal elements in* S *are* $f_1 = e_2$ *and* $f_2 = e_3$, *and* $f_1 \vee f_2 = e_4$. *We have that* $Bf_1 = \{(a_1, a_2, 0, 0, 0) \mid a_1, a_2 \in \mathbb{Q}\}$, $Bf_2 = \{(0,0,a_3,a_4,0) \mid a_3, a_4 \in \mathbb{Q}\}$, *and* $B(1 - f_1 \vee f_2) = \{(0,0,0,0,a_5) \mid a_5 \in \mathbb{Q}\}$. *So* $B = Bf_1 \oplus Bf_2 \oplus B(1 - f_1 \vee f_2)$ *and* Bf_j *is a* H_j*-Galois extension of* $(Bf_j)^{H_j}$ *for* $j = 1, 2$.

(5) *Since* $e_2 = (1,1,0,0,0)$, $e_3 = (0,0,1,1,0)$, *and* $e_4 = (1,1,1,1,0)$, *we have* $C(1-e_2) = \{(0,0,a_3,a_4,a_5) \mid a_3,a_4,a_5 \in \mathbb{Q}\}$, $C(1-e_3) = \{(a_1,a_2,0,0,a_5) \mid a_1,a_2,a_5 \in \mathbb{Q}\}$, *and* $C(1-e_4) = \{(0,0,0,0,a_5) \mid a_5 \in \mathbb{Q}\}$. *So* g_i *restricted to* $C(1-e_i)$ *is an identity for each* $g_i \neq 1$ *in* G.

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