PERISTALTIC VISCOELASTIC FLUID MOTION IN A TUBE

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ABSTRACT. Peristaltic motion of viscoelastic incompressible fluid in an axisymmetric tube with a sinusoidal wave is studied theoretically in the case that the radius of the tube is small relative to the wavelength. Oldroyd flow has been considered in this study and the problem is formulated and analyzed using a perturbation expansion in terms of the variation of the wave number. This analysis can model the chyme movement in the small intestine by considering the chyme as an Oldroyd fluid. We found out that the pumping rate of Oldroyd fluid is less than that for a Newtonian fluid. Further, the effects of Reynolds number, Weissenberg number, amplitude ratio and wave number on the pressure rise and friction force have been discussed. It is found that the pressure rise does not depend on Weissenberg number at a certain value of flow rate. The results are studied for various values of the physical parameters of interest.

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1. Introduction. A peristaltic pump is a device for pumping fluids, generally from a region of lower to higher pressure, by means of a contraction wave traveling along a tube-like structure. This traveling-wave phenomenon is referred to as (peristaltic). This phenomenon is now well known to physiologists to be one of the major mechanisms for fluid transport in many biological systems. The study of the mechanism of peristalsis, in both mechanical and physiological situations, has recently become the object of scientific research. Since the first investigation of Latham [6], several theoretical and experimental attempts have been made to understand peristaltic action in different situations. A review of much of the early literature is presented in an article by Jaffrin and Shapiro [5]. A summary of most of the experimental and theoretical investigations reported, with details of the geometry, fluid, Reynolds number, wavelength parameter, wave amplitude parameter, and wave shape has been given by Srivastava and Srivastava [11].

Most theoretical investigations have been carried out for Newtonian fluids, although it is known that most physiological fluids behave as non-Newtonian fluids. However, limited studies for non-Newtonian fluids have been made by Srivastava and Srivastava [12, 13], Srivastava [10], and Elshehawey and Mekheimer [4]. Bohme and Friedrich [1] have investigated peristaltic flow of second-order viscoelastic liquid assuming that the relevant Reynolds number is small enough to neglect inertial forces, and that the ratio of the wavelength and the channel height is large so that the pressure is constant over the cross-section. Peristaltic motion of a third-order fluid in a planar channel has been studied by Siddiqui and Schwarz [9], under the long-wavelength approximation assumption. El Misery et al. [2] studied the peristaltic motion of Carreau fluid in a channel. They developed the solution in a perturbation series in powers of Weissenberg

number using long-wavelength approximation and also the same problem was studied
by Elshehawey et al. [3] in the case of nonuniform channel.

The purpose of this paper is to study the peristaltic motion of Oldroyd fluid in a
tube. This problem can model the movement of the chyme, which may be considered
as Oldroyd fluid, through small intestine. In our analysis, we assumed that the velocity
components, the pressure, the shearing stress, and the flow rate may be expanded in a
regular perturbation series in the wave number. Expressions for pressure rise, velocity
components and friction force were obtained in terms of the flow rate, the occlusion,
the Reynolds number, the Weissenberg number, and the wave number.

2. Formulation and analysis. Consider the flow of an incompressible Oldroyd fluid
in a circular tube of radius a. We assume an infinite wave train traveling with velocity
c along the wall. Taking $\bar{R}$ and $\bar{Z}$ as cylindrical coordinates, the geometry of the wall
surface is

$$\bar{h}(\bar{Z},\bar{t}) = a + b \sin \frac{2\pi}{\lambda}(\bar{Z} - c\bar{t}), \tag{2.1}$$

where b is the wave amplitude, λ is the wave length, and $\bar{Z}$ is the same direction of
the wave propagation.

Choosing moving coordinates $(\bar{r},\bar{z})$ (wave frame) which travel in the $\bar{Z}$-direction
with the same speed as the wave, the unsteady flow in the laboratory frame $(\bar{R},\bar{Z})$ can
be treated as steady [8]. The coordinates frame are related through

$$\bar{z} = \bar{Z} - c\bar{t}, \qquad \bar{r} = \bar{R}, \qquad \bar{w} = \bar{W} - c, \qquad \bar{u} = \bar{U}, \tag{2.2}$$

where $\bar{U},\bar{W}$ and $\bar{u},\bar{w}$ are, respectively, the radial and the axial velocity components in
the corresponding coordinate systems.

Equations of motion in the moving coordinates are

$$\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}}(\bar{r}\bar{u}) + \frac{\partial \bar{w}}{\partial \bar{z}} = 0,$$

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}}\right) = -\frac{\partial \bar{p}}{\partial \bar{r}} - \left(\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}\bar{\tau}_{11}) - \frac{\bar{\tau}_{22}}{\bar{r}} + \frac{\partial \bar{\tau}_{13}}{\partial \bar{z}}\right), \tag{2.3}$$

$$\rho\left(\bar{u}\frac{\partial \bar{w}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{w}}{\partial \bar{z}}\right) = -\frac{\partial \bar{p}}{\partial \bar{z}} - \left(\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}\bar{\tau}_{13}) + \frac{\partial \bar{\tau}_{33}}{\partial \bar{z}}\right).$$

The constitutive equations of Oldroyd fluid are, [7],

$$\bar{\tau}_{11} + \Gamma\left[\bar{u}\frac{\partial \bar{\tau}_{11}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{\tau}_{11}}{\partial \bar{z}} - 2\bar{\tau}_{11}\frac{\partial \bar{u}}{\partial \bar{r}} - 2\bar{\tau}_{13}\frac{\partial \bar{u}}{\partial \bar{z}}\right] = -\mu\bar{\dot{\gamma}}_{11},$$

$$\bar{\tau}_{13} + \Gamma\left[\bar{u}\frac{\partial \bar{\tau}_{13}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{\tau}_{13}}{\partial \bar{z}} - \bar{\tau}_{33}\frac{\partial \bar{u}}{\partial \bar{z}} - \bar{\tau}_{11}\frac{\partial \bar{w}}{\partial \bar{r}} + \frac{\bar{u}}{\bar{r}}\bar{\tau}_{13}\right] = -\mu\bar{\dot{\gamma}}_{13}, \tag{2.4}$$

$$\bar{\tau}_{22} + \Gamma\left[\bar{u}\frac{\partial \bar{\tau}_{22}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{\tau}_{22}}{\partial \bar{z}} - \frac{2\bar{u}}{\bar{r}}\bar{\tau}_{22}\right] = -\mu\bar{\dot{\gamma}}_{22},$$

$$\bar{\tau}_{33} + \Gamma\left[\bar{u}\frac{\partial \bar{\tau}_{33}}{\partial \bar{r}} + \bar{w}\frac{\partial \bar{\tau}_{33}}{\partial \bar{z}} - 2\bar{\tau}_{33}\frac{\partial \bar{w}}{\partial \bar{z}} - 2\bar{\tau}_{13}\frac{\partial \bar{w}}{\partial \bar{r}}\right] = -\mu\bar{\dot{\gamma}}_{33},$$

where $\bar{p}$ is the pressure, $\bar{\tau}_{ij}$, $i, j = 1, 2, 3$, are the components of the extra stress tensor, Γ is the relaxation time, ρ is the viscosity, and $\bar{\dot{y}}_{ij}$, $i, j = 1, 2, 3$, are the components of strain-rate tensor and given by

$$\bar{\dot{y}}_{11} = 2\frac{\partial \bar{u}}{\partial \bar{r}}, \qquad \bar{\dot{y}}_{22} = 2\frac{\bar{u}}{\bar{r}}, \qquad \bar{\dot{y}}_{33} = 2\frac{\partial \bar{w}}{\partial \bar{z}}, \qquad \bar{\dot{y}}_{13} = \left(\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{r}}\right). \tag{2.5}$$

The boundary conditions are

$$\frac{\partial \bar{w}}{\partial \bar{r}} = 0, \quad \bar{u} = 0 \quad \text{for } \bar{r} = 0, \qquad \bar{w} = -c, \quad \bar{u} = -c\frac{d\bar{h}}{d\bar{z}} \quad \text{for } \bar{r} = \bar{h}. \tag{2.6}$$

Introducing the nondimensional variables and parameters

$$z = \frac{\bar{z}}{\lambda}, \qquad Z = \frac{\bar{Z}}{\lambda}, \qquad r = \frac{\bar{r}}{a}, \qquad R = \frac{\bar{R}}{a}, \qquad t = \frac{c\bar{t}}{\lambda}, \qquad p = \frac{a^2\bar{p}}{c\lambda\mu},$$

$$u = \frac{\lambda\bar{u}}{ac}, \qquad w = \frac{\bar{w}}{c}, \qquad W = \frac{\bar{W}}{c}, \qquad \delta = \frac{a}{\lambda}, \qquad \tau_{ij} = \frac{a\bar{\tau}_{ij}}{c\mu}, \tag{2.7}$$

$$\dot{y}_{ij} = \frac{a}{c}\bar{\dot{y}}_{ij}, \qquad \text{Wi} = \frac{c\Gamma}{a}, \qquad \text{Re} = \frac{\rho c a}{\mu},$$

where Wi is the Weissenberg number, δ is the wave number, Re is the Reynolds number, and $h = \bar{h}/a = 1 + (b/a)\sin 2\pi z = 1 + \varphi \sin 2\pi z$, $\varphi = b/a < 1$, is the amplitude ratio, equations (2.3), (2.4), after using (2.5), become

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{2.8}$$

$$\text{Re}\,\delta^3\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} - \delta\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{11}) - \frac{\tau_{22}}{r} + \delta\frac{\partial \tau_{13}}{\partial z}\right), \tag{2.9}$$

$$\text{Re}\,\delta\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{13}) + \delta\frac{\partial \tau_{33}}{\partial z}\right), \tag{2.10}$$

$$\tau_{11} + \text{Wi}\,\delta\left[u\frac{\partial \tau_{11}}{\partial r} + w\frac{\partial \tau_{11}}{\partial z} - 2\tau_{11}\frac{\partial u}{\partial r} - 2\delta\tau_{13}\frac{\partial u}{\partial z}\right] = -2\delta\left(\frac{\partial u}{\partial r}\right), \tag{2.11}$$

$$\tau_{13} + \text{Wi}\left[\delta\left(u\frac{\partial \tau_{13}}{\partial r} + w\frac{\partial \tau_{13}}{\partial z} - \delta\tau_{33}\frac{\partial u}{\partial z} + \frac{u}{r}\tau_{13}\right) - \tau_{11}\frac{\partial w}{\partial r}\right] = -\left(\delta^2\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right), \tag{2.12}$$

$$\tau_{22} + \text{Wi}\,\delta\left[u\frac{\partial \tau_{22}}{\partial r} + w\frac{\partial \tau_{22}}{\partial z} - \frac{2u}{r}\tau_{22}\right] = -2\delta\frac{u}{r}, \tag{2.13}$$

$$\tau_{33} + \text{Wi}\left[\delta\left(u\frac{\partial \tau_{33}}{\partial r} + w\frac{\partial \tau_{33}}{\partial z} - 2\tau_{33}\frac{\partial w}{\partial z}\right) - 2\tau_{13}\frac{\partial w}{\partial r}\right] = -2\delta\left(\frac{\partial w}{\partial z}\right), \tag{2.14}$$

The nondimensional boundary conditions are

$$\frac{\partial w}{\partial r} = 0, \quad u = 0 \quad \text{for } r = 0, \qquad w = -1, \quad u = -\frac{dh}{dz} \quad \text{for } r = h. \tag{2.15}$$

Eliminating the pressure from (2.9) and (2.10), we obtain

$$
\operatorname{Re}\delta\left[\delta^2\frac{\partial}{\partial z}\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)-\frac{\partial}{\partial r}\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)\right]
$$
$$
=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{13})+\delta\frac{\partial\tau_{33}}{\partial z}\right)-\delta\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{11})-\frac{\tau_{22}}{r}+\delta\frac{\partial\tau_{13}}{\partial z}\right).
$$
(2.16)

3. Rate of volume flow. The instantaneous volume flow rate in the fixed frame is given by

$$
Q=2\pi\int_0^{\bar{h}}\bar{W}\bar{R}\,d\bar{R},
$$
(3.1)

where $\bar{h}$ is a function of $\bar{Z}$ and $\bar{t}$.

The rate of volume flow in the moving frame (wave frame) is given by

$$
\bar{q}=2\pi\int_0^{\bar{h}}\bar{w}\bar{r}\,d\bar{r},
$$
(3.2)

where $\bar{h}$ is a function of $\bar{z}$.

Using (3.2), one finds that the two rates of volume flow are related by

$$
Q=\bar{q}+\pi c\bar{h}^2.
$$
(3.3)

The time-mean flow over a period $T=\lambda/c$ at a fixed position $\bar{Z}$ is defined as

$$
\bar{Q}=\frac{1}{T}\int_0^T Q\,d\bar{t},
$$
(3.4)

which can be written, using (2.1) and (3.3), as

$$
\bar{Q}=\bar{q}+\pi ca^2\left(1+\frac{\varphi^2}{2}\right).
$$
(3.5)

Defining the dimensionless time-mean flows θ and F in the fixed and wave frame, respectively, as

$$
\theta=\frac{\bar{Q}}{\pi ca^2},\qquad F=\frac{\bar{q}}{\pi ca^2},
$$
(3.6)

then making use of (3.6), equation (3.5) can be rewritten as

$$
\theta=F+1+\frac{\varphi^2}{2},
$$
(3.7)

where

$$
F=2\int_0^{h(z)}rw\,dr.
$$
(3.8)

4. Perturbation solution. Beginning by expanding the following quantities in a power series of the small parameter δ as follows:

$$u = u_0 + \delta u_1 + \delta^2 u_2 + O(\delta^3), \qquad w = w_0 + \delta w_1 + \delta^2 w_2 + O(\delta^3),$$

$$\frac{\partial p}{\partial z} = \frac{\partial p_0}{\partial z} + \delta \frac{\partial p_1}{\partial z} + \delta^2 \frac{\partial p_2}{\partial z} + O(\delta^3), \qquad \tau_{11} = \tau_{11}^{(0)} + \delta \tau_{11}^{(1)} + \delta^2 \tau_{11}^{(2)} + O(\delta^3),$$

$$\tau_{13} = \tau_{13}^{(0)} + \delta \tau_{13}^{(1)} + \delta^2 \tau_{13}^{(2)} + O(\delta^3), \qquad \tau_{22} = \tau_{11}^{(0)} + \delta \tau_{22}^{(1)} + \delta^2 \tau_{22}^{(2)} + O(\delta^3), \tag{4.1}$$

$$\tau_{33} = \tau_{33}^{(0)} + \delta \tau_{33}^{(1)} + \delta^2 \tau_{33}^{(2)} + O(\delta^3), \qquad F = F_0 + \delta F_1 + \delta^2 F_2 + O(\delta^3),$$

then using the perturbation expansions (4.1) in equations (2.8), (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), (2.16), and (3.8) and collecting terms of like powers of δ, we obtain three sets of coupled linear differential equations with their corresponding boundary conditions in u_0, w_0, u_1, w_1, and u_2, w_2 for the first three powers of δ. The first set of differential equations in u_0, w_0, subject to the corresponding boundary conditions, yields the following classical Poiseuille flow:

$$w_0 = c_1 + c_2 r^2, \tag{4.2a}$$

$$u_0 = -\frac{c_1'}{2} r - \frac{c_2'}{4} r^3, \tag{4.2b}$$

where

$$c_1 = 1 + \frac{2F_0}{h^2}, \qquad c_2 = -2\left(\frac{F_0 + h^2}{h^4}\right). \tag{4.3}$$

On substituting the zeroth-order solution (4.2) in the second set of differential equations and using its corresponding boundary conditions, the first-order solution can be obtained in the form

$$w_1 = c_3 + c_4 r^2 + c_5 r^4 + c_6 r^6, \tag{4.4a}$$

$$u_1 = -\frac{c_3'}{2} r - \frac{c_4'}{4} r^3 - \frac{c_5'}{6} r^5 - \frac{c_6'}{8} r^7, \tag{4.4b}$$

where

$$c_3 = \frac{2F_1}{h^2} + \mathrm{Re}\left(\frac{c_1 c_2' h^4}{48} + \frac{c_2 c_2' h^6}{144}\right), \qquad c_4 = -\frac{2F_1}{h^4} - \mathrm{Re}\left(\frac{c_1 c_2' h^2}{12} + \frac{c_2 c_2' h^4}{48}\right),$$

$$c_5 = \mathrm{Re}\left(\frac{c_1 c_2'}{16}\right), \qquad c_6 = \mathrm{Re}\left(\frac{c_2 c_2'}{72}\right). \tag{4.5}$$

We now solve the second-order system. Using the zeroth-order and the first-order solutions in the third set of differential equations and using the boundary conditions, we obtain

$$w_2 = b_1 + b_2 r^2 + b_3 r^4 + b_4 r^6 + b_5 r^8 + b_6 r^{10}, \tag{4.6}$$

where

$$b_1 = \frac{2F_2}{h^2} - \frac{c_2'' h^4}{24} + \mathrm{Re}\left(\frac{h^4 a_1}{48} + \frac{h^6 a_2}{72} + \frac{3 h^8 a_3}{320} + \frac{h^{10} a_4}{150}\right)$$

$$+ \mathrm{Wi}\left(\frac{h^4 a_5}{12} + \frac{h^6 a_6}{12} + \frac{3 h^8 a_7}{40}\right) - \mathrm{Wi}^2\left(\frac{h^4 a_8}{12} + \frac{h^6 a_9}{12}\right),$$

$$b_2 = -\frac{2F_2}{h^4} + \frac{c_2''h^2}{6} - \mathrm{Re}\left(\frac{h^2 a_1}{12} + \frac{h^4 a_2}{24} + \frac{h^6 a_3}{40} + \frac{h^8 a_4}{60}\right)$$

$$-W\left(\frac{h^2 a_5}{3} + \frac{h^4 a_6}{4} + \frac{h^6 a_7}{5}\right) + \mathrm{Wi}^2\left(\frac{h^2 a_8}{3} + \frac{h^4 a_9}{4}\right),$$

$$b_3 = -\frac{c_2''}{8} + \frac{1}{16}\left(\mathrm{Re}\,a_1 + 4\,\mathrm{Wi}\,a_5 - 4\,\mathrm{Wi}^2\,a_8\right),$$

$$b_4 = \frac{1}{36}\left(\mathrm{Re}\,a_2 + 6\,\mathrm{Wi}\,a_6 - 6\,\mathrm{Wi}^2\,a_9\right),$$

$$b_5 = \frac{1}{64}\left(\mathrm{Re}\,a_3 + 8\,\mathrm{Wi}\,a_7\right), \qquad b_6 = \frac{\mathrm{Re}\,a_4}{100},$$

$$a_1 = (c_3 c_2' + c_1 c_4'), \qquad a_2 = \left(c_1 c_5' - c_5 c_1' + \frac{c_4 c_2'}{2} + \frac{c_2 c_4'}{4}\right),$$

$$a_3 = \left(c_1 c_6' - 2 c_6 c_1' + \frac{2 c_2 c_5'}{3}\right), \qquad a_4 = \left(\frac{3 c_2 c_6'}{4} - \frac{c_6 c_2'}{2}\right), \qquad a_5 = 4(c_1 c_5' - c_5 c_1'),$$

$$a_6 = \left(6 c_1 c_6' - 12 c_6 c_1' + \frac{4 c_2 c_5'}{3} - \frac{4 c_5 c_2'}{3}\right), \qquad a_7 = (3 c_2 c_6' - 4 c_6 c_2'),$$

$$a_8 = (3 c_2 c_1' c_2' - 3 c_1 c_2'^2 + 3 c_1 c_2 c_2''), \qquad a_9 = (c_2^2 c_2'' - c_2 c_2'^2).$$

$$(4.7)$$

At this order, the perturbation solution for the axial velocity can be, using (4.2a), (4.4a), and (4.6), written as

$$w = c_1 + c_2 r^2 + \delta(c_3 + c_4 r^2 + c_5 r^4 + c_6 r^6)$$
$$+ \delta^2(b_1 + b_2 r^2 + b_3 r^4 + b_4 r^6 + b_5 r^8 + b_6 r^{10}).$$

$$(4.8)$$

A close look at (4.8) reveals that the axial velocity is affected by the wave number, the Reynolds number and the viscoelastic parameter (Weissenberg number).

5. Pressure gradient. An expression for the pressure gradient, $\partial p/\partial z$, can be obtained by substituting (4.1) into the dimensionless equation of motion (2.10) and equating the coefficients of like powers of δ, we obtain three sets of partial differential equations for $\partial p_0/\partial z, \partial p_1/\partial z$, and $\partial p_2/\partial z$. Using this form of $\partial p/\partial z$, the pressure rise and the friction force per wavelength can be obtained.

The nondimensional pressure rise and the nondimensional friction force per wavelength are defined, respectively, as

$$\Delta p_\lambda = \int_0^1 \frac{dp}{dz}\,dz, \qquad F_\lambda = \int_0^1 h^2\left(-\frac{dp}{dz}\right)dz.$$

$$(5.1)$$

Since $\partial p/\partial z$ is periodic in z, the pressure rise and the friction force per wavelength in the longitudinal direction are independent of r, [5]. Accordingly, the integrals in (5.1) can be evaluated on the axis at $r = 0$. Further, the pressure rise and friction force can be expanded as a power series in δ as

$$\Delta p_\lambda = \Delta p_{\lambda 0} + \delta \Delta p_{\lambda 1} + \delta^2 \Delta p_{\lambda 2} + \cdots,$$

$$F_\lambda = F_{\lambda 0} + \delta F_{\lambda 1} + \delta^2 F_{\lambda 2} + \cdots,$$

$$(5.2)$$

where

$$\Delta p_{\lambda 0} = \int_0^1 \frac{dp_0}{dz}\,dz, \qquad \Delta p_{\lambda 1} = \int_0^1 \frac{dp_1}{dz}\,dz, \qquad \Delta p_{\lambda 2} = \int_0^1 \frac{dp_2}{dz}\,dz,$$

$$F_{\lambda 0} = \int_0^1 h^2 \left(-\frac{dp_0}{dz}\right)dz, \qquad F_{\lambda 1} = \int_0^1 h^2 \left(-\frac{dp_1}{dz}\right)dz, \qquad F_{\lambda 2} = \int_0^1 h^2 \left(-\frac{dp_2}{dz}\right)dz.$$

(5.3)

We now use the zeroth-, first-, and second-order terms for the pressure gradient in (5.3), integrating from 0 to 1, then substituting in (5.2) we obtain

$$\Delta p_{\lambda}^{(2)} = -8F^{(2)}I_4 - 8I_2$$

$$+\delta^2\Bigg[-\frac{32\pi^2}{3}(3(\varphi^2-1)I_2 + 5I_1 - 2)$$

$$+F^{(2)}\left\{-\frac{16\pi^2}{3}(11(\varphi^2-1)I_4 + 21I_3 - 10I_2)\right\}$$

$$+\mathrm{Re}^2\Bigg\{(F^{(2)})^3\left[\frac{22\pi^2}{135}(11(\varphi^2-1)I_6 + 19I_5 - 8I_4)\right]$$

$$+(F^{(2)})^2\left[\frac{\pi^2}{135}(115(\varphi^2-1)I_4 + 137I_3 - 22I_2)\right]$$

$$-\frac{\pi^2 F^{(2)}}{135}(41(\varphi^2-1)I_2 + 121I_1 - 80) - \frac{\pi^2\varphi^2}{27}\Bigg\}$$

$$-\mathrm{Re}\,\mathrm{Wi}\Bigg\{(F^{(2)})^3\left[\frac{32\pi^2}{5}(7(\varphi^2-1)I_8 + 13I_7 - 6I_6)\right]$$

$$+(F^{(2)})^2\left[\frac{32\pi^2}{15}(19(\varphi^2-1)I_6 + 33I_5 - 14I_4)\right]$$

$$+F^{(2)}\left[\frac{8\pi^2}{15}(11(\varphi^2-1)I_4 + 13I_3 - 2I_2)\right]$$

$$+\frac{8\pi^2}{15}((\varphi^2-1)I_2 + 3I_1 - 2)\Bigg\}$$

$$-\mathrm{Wi}^2\Big\{(F^{(2)})^3[128\pi^2(19(\varphi^2-1)I_{10} + 35I_9 - 16I_8)]$$

$$+(F^{(2)})^2[64\pi^2(37(\varphi^2-1)I_8 + 67I_7 - 30I_6)]$$

$$+F^{(2)}[128\pi^2(6(\varphi^2-1)I_6 + 11I_5 - 5I_4)]$$

$$+128\pi^2((\varphi^2-1)I_4 + 2I_3 - I_2)\Big\}\Bigg],$$

(5.4)

$$F_\Lambda^{(2)} = 8F^{(2)}I_2$$

$$+8+\delta^2\left[\frac{64\pi^2\varphi^2}{3}+F^{(2)}\left\{\frac{16\pi^2}{3}\left(11(\varphi^2-1)I_2+21I_1-10\right)\right\}\right.$$

$$-\mathrm{Re}^2\left\{(F^{(2)})^3\left[\frac{22\pi^2}{135}\left(11(\varphi^2-1)I_4+19I_3-8I_2\right)\right]\right.$$

$$+\frac{\pi^2(F^{(2)})^2}{135}\left(115(\varphi^2-1)I_2+137I_1-22\right)$$

$$\left.-\frac{\pi^2\varphi^2F^{(2)}}{135}-\frac{\pi^2\varphi^2(\varphi^2+4)}{1080}\right\}$$

$$+\mathrm{Re\,Wi}\left\{(F^{(2)})^3\left[\frac{32\pi^2}{5}\left(7(\varphi^2-1)I_6+13I_5-6I_4\right)\right]\right.$$

$$+(F^{(2)})^2\left[\frac{32\pi^2}{15}\left(19(\varphi^2-1)I_4+33I_3-14I_2\right)\right]$$

$$\left.+(F^{(2)})^2\left[\frac{8\pi^2}{15}\left(11(\varphi^2-1)I_2+13I_1-2\right)\right]\right\}$$

$$+\mathrm{Wi}^2\left\{(F^{(2)})^3\left[128\pi^2\left(19(\varphi^2-1)I_8+35I_7-16I_6\right)\right]\right.$$

$$+(F^{(2)})^2\left[64\pi^2\left(37(\varphi^2-1)I_6+67I_5-30I_4\right)\right]$$

$$+F^{(2)}\left[128\pi^2\left(6(\varphi^2-1)I_4+11I_3-5I_2\right)\right]$$

$$\left.\left.+128\pi^2\left((\varphi^2-1)I_2+2I_1-1\right)\right\}\right],$$

(5.5)

where

$$I_n=\int_0^1\frac{1}{h^n}\,dz,\quad n=1,2,\ldots,10.$$

(5.6)

Here, we used the relation

$$F_0=F^{(2)}-\delta F_1-\delta^2 F_2,$$

(5.7)

and $\Delta p_\lambda^{(2)}, F_\lambda^{(2)}$, and $F^{(2)}$ are, respectively, the pressure rise, the friction force and the flow rate in the wave frame to the second order in δ, and where

$$I_1 = \frac{1}{(1-\varphi^2)^{1/2}}, \qquad I_2 = \frac{1}{(1-\varphi^2)^{3/2}}, \qquad I_3 = \frac{(1+\varphi^2/2)}{(1-\varphi^2)^{5/2}}, \qquad I_4 = \frac{(1+3\varphi^2/2)}{(1-\varphi^2)^{7/2}},$$

$$I_n = \frac{1}{(1-\varphi^2)}\left[\left(\frac{2n-3}{n-1}\right)I_{n-1} - \left(\frac{n-2}{n-1}\right)I_{n-2}\right], \quad n > 4.$$

(5.8)

The substitution of (5.8) into (5.4) and (5.5) yields

$$\Delta p_\lambda^{(2)} = -\frac{8(1+3\varphi^2/2)}{(1-\varphi^2)^{7/2}}\left(\theta^{(2)} - 1 - \frac{\varphi^2}{2}\right) - \frac{8}{(1-\varphi^2)^{3/2}}$$

$$+ \delta^2\left[-\frac{64\pi^2\varphi^2}{3(1-\varphi^2)^{5/2}}\left(\theta^{(2)} - 1 - \frac{\varphi^2}{2}\right) - \frac{64\pi^2}{3}\left\{\frac{1}{(1-\varphi^2)^{1/2}} - 1\right\}\right.$$

$$+ \mathrm{ReWi}\left\{\frac{8\pi^2\varphi^2(4-\varphi^2-3\varphi^4)(\theta^{(2)}-1-\varphi^2/2)^2}{5(1-\varphi^2)^{11/2}}\right.$$

$$\left. + \frac{64\pi^2\varphi^2(\theta^{(2)}-1-\varphi^2/2)}{15(1-\varphi^2)^{5/2}} - \frac{16\pi^2}{15}\left(1 - \frac{1}{(1-\varphi^2)^{1/2}}\right)\right\}$$

$$+ \mathrm{Re}^2\left\{\frac{11\pi^2\varphi^2(3\varphi^4+\varphi^2-4)(\theta^{(2)}-1-\varphi^2/2)^3}{135(1-\varphi^2)^{11/2}}\right.$$

(5.9)

$$- \frac{82\pi^2\varphi^2(\theta^{(2)}-1-\varphi^2/2)^2}{135(1-\varphi^2)^{5/2}}$$

$$\left. + \frac{16\pi^2}{27}\left(1 - \frac{1}{(1-\varphi^2)^{1/2}}\right)\left(\theta^{(2)} - 1 - \frac{\varphi^2}{2}\right) - \frac{\pi^2\varphi^2}{27}\right\}$$

$$+ \mathrm{Wi}^2\left\{\frac{8\pi^2\varphi^2(35\varphi^6+280\varphi^4+336\varphi^2+64)}{(1-\varphi^2)^{17/2}}\left(\theta^{(2)} - 1 - \frac{\varphi^2}{2}\right)\right.$$

$$+ \frac{48\pi^2\varphi^2(5\varphi^4+20\varphi^2+8)}{(1-\varphi^2)^{13/2}}\left(\theta^{(2)} - 1 - \frac{\varphi^2}{2}\right)^2$$

$$\left.\left. + \frac{16\pi^2\varphi^2(3\varphi^4+\varphi^2-4)}{(1-\varphi^2)^{11/2}}\left(\theta^{(2)} - 1 - \frac{\varphi^2}{2}\right) - \frac{64\pi^2\varphi^2}{(1-\varphi^2)^{5/2}}\right\}\right],$$

and

$$F_\lambda^{(2)} = \frac{8(\theta^{(2)} - 1 - \varphi^2/2)}{(1-\varphi^2)^{3/2}}$$

$$+ 8 + \delta^2 \Bigg[\frac{160\pi^2}{3} \left(\frac{1}{(1-\varphi^2)^{1/2}} - 1 \right) \left(\theta^{(2)} - 1 - \frac{\varphi^2}{2} \right) + \frac{64\pi^2\varphi^2}{3}$$

$$+ \operatorname{Re}\operatorname{Wi} \Bigg\{ \frac{8\pi^2\varphi^2(4-\varphi^2-3\varphi^4)(\theta^{(2)} - 1 - \varphi^2/2)^3}{5(1-\varphi^2)^{11/2}}$$

$$+ \frac{64\pi^2\varphi^2(\theta^{(2)} - 1 - \varphi^2/2)^2}{15(1-\varphi^2)^{5/2}}$$

$$- \frac{16\pi^2}{15} \left(1 - \frac{1}{(1-\varphi^2)^{1/2}} \right) \left(\theta^{(2)} - 1 - \frac{\varphi^2}{2} \right) \Bigg\}$$

$$+ \operatorname{Re}^2 \Bigg\{ - \frac{22\pi^2\varphi^2(\theta^{(2)} - 1 - \varphi^2/2)^2}{135(1-\varphi^2)^{5/2}} \tag{5.10}$$

$$+ \frac{22\pi^2}{15} \left(1 - \frac{1}{(1-\varphi^2)^{1/2}} \right) \left(\theta^{(2)} - 1 - \frac{\varphi^2}{2} \right)$$

$$+ \frac{\pi^2\varphi^2}{135} \left(\theta^{(2)} - 1 - \frac{\varphi^2}{2} \right) + \frac{\pi^2\varphi^2(\varphi^2+4)}{1080} \Bigg\}$$

$$- \operatorname{Wi}^2 \Bigg\{ \frac{16\pi^2\varphi^2(5\varphi^4 + 20\varphi^2 + 8)}{(1-\varphi^2)^{13/2}} \left(\frac{\theta^{(2)} - 1 - \varphi^2}{2} \right)^3$$

$$+ \frac{16\pi^2\varphi^2(3\varphi^4 + \varphi^2 - 4)}{(1-\varphi^2)^{11/2}} \left(\theta^{(2)} - 1 - \frac{\varphi^2}{2} \right)^2$$

$$- \frac{192\pi^2\varphi^2}{(1-\varphi^2)^{5/2}} \left(\theta^{(2)} - 1 - \frac{\varphi^2}{2} \right) + 128\pi^2 \left(1 - \frac{1}{(1-\varphi^2)^{1/2}} \right) \Bigg\} \Bigg],$$

where

$$\theta^{(2)} = F^{(2)} + 1 + \frac{\varphi^2}{2}. \tag{5.11}$$

6. Results and conclusion. It is clear that our results calculate the velocity, the pressure rise and the friction force without restrictions on the amplitude ratio, the Reynolds number and the Weissenberg number but we used a small wave number. Further, the results extend the work of Shapiro et al. [8] as well as it include the effect of Weissenberg number Wi.

In Figure 6.1, the dimensionless pressure rise $(\Delta p_\lambda)^{(2)}$ is graphed versus the dimensionless flow rate $\theta^{(2)}$ for different values of Weissenberg number (Wi $= 0, 0.04, 0.08$) at wave number $\delta = 0.156$ and Reynolds number Re $= 0.1$, for both cases ($\varphi = 0.35$ and $\varphi = 0.6$). As shown, for $\varphi = 0.35$, the effect of Weissenberg number is very small and the three curves coincide. But for $\varphi = 0.6$, the effect of Weissenberg number is
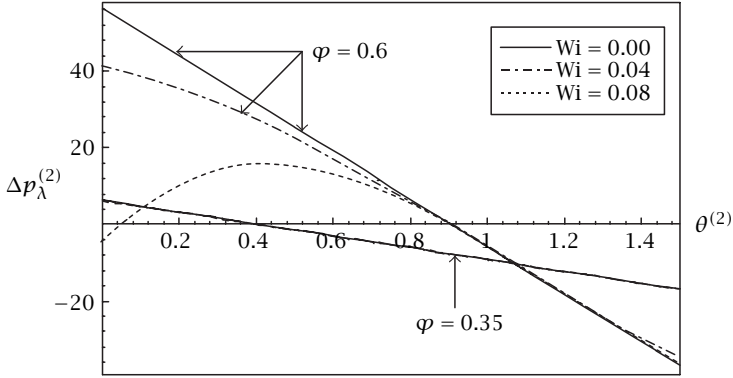
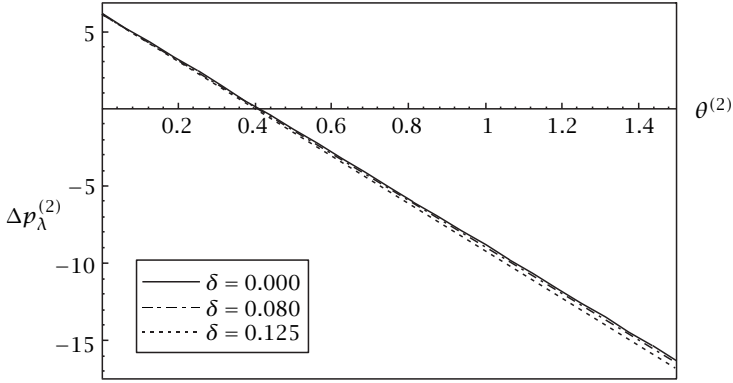FIGURE 6.1. The pressure rise versus flow rate at Re = 0.1, $\delta = 0.156$, and $\varphi = 0.35, 0.6$.

FIGURE 6.2. The pressure rise versus flow rate at Re = 0.1, Wi = 0.08, and $\varphi = 0.35$.

very clear and show that the pumping rate of Oldroyd fluid is less than that for a Newtonian having a shear viscosity the same as Oldroyd fluid and the pressure rise decreases with increasing Weissenberg number. Further, it is clear that the pressure rise is independent on Weissenberg number at a certain value of flow rate and the peristaltic pumping, where $(\Delta p_\lambda)^{(2)} > 0$ and $\theta^{(2)} > 0$, occur at $0 \leq \theta^{(2)} \leq 0.9$ and the augmented pumping, where $(\Delta p_\lambda)^{(2)} < 0$ and $\theta^{(2)} > 0$, occur at $0.9 \leq \theta^{(2)} \leq 1.5$, for $\varphi = 0.6$. The linear relation for a Newtonian fluid is obvious in (5.9), with Wi = 0 and $\delta = 0$. Figures 6.2 and 6.3 show the effect of the wave number δ on the pressure rise at Re = 0.1, Wi = 0.08, and $\varphi = 0.35, 0.6$, respectively. Figure 6.2 reveals that, for $\varphi = 0.35$, an increase in the wave number yields a slight increase in the magnitude of the pressure rise but for $\varphi = 0.6$, the effect of the wave number δ is very clear and the pressure rise decreases as wave number increases as shown in Figure 6.3.

Shown in Figure 6.4 the dimensionless pressure rise versus flow rate at Wi = 0, $\delta = 0.02$, $\varphi = 0.6$, and Re = 0, 50, 100. The results reveal that the magnitude of the pressure rise increases with increasing Reynolds number.
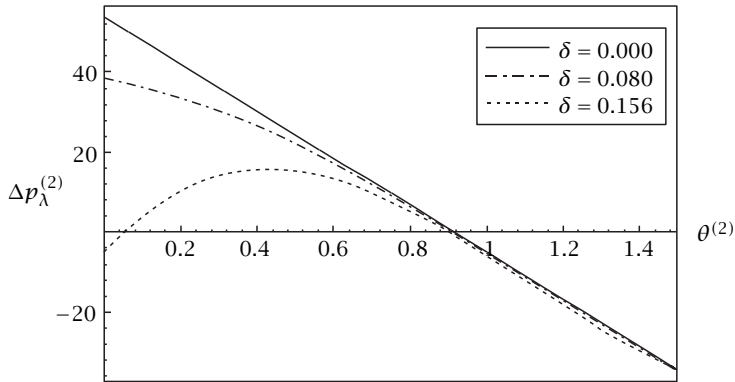
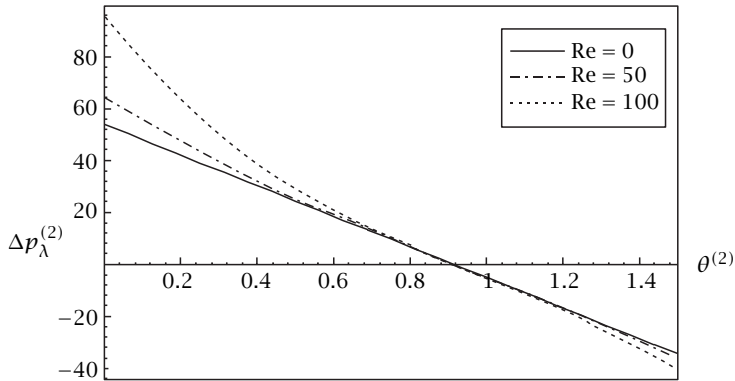FIGURE 6.3. The pressure rise versus flow rate at Re = 0.1, Wi = 0.08, and $\varphi = 0.6$.

FIGURE 6.4. The pressure rise versus flow rate at Wi = 0, $\delta = 0.02$, and $\varphi = 0.6$.

The dimensionless friction force is plotted versus flow rate in Figures 6.5, 6.6, 6.7, and 6.8. Figure 6.5 shows the friction force versus flow rate at $\delta = 0.156$, Re = 0.1, and Wi = 0, 0.04, 0.08 in both cases $\varphi = 0.35$ and $\varphi = 0.6$. It is shown that the friction force is independent on Weissenberg number at $\varphi = 0.35$ but its magnitude decreases with increasing Weissenberg number at $\varphi = 0.6$ and it does not depend on Weissenberg number at a certain value of flow rate in this case. Shown in Figures 6.6 and 6.7 the effect of wave number δ on the friction force at $\varphi = 0.35$ and $\varphi = 0.6$, respectively. We notice from Figure 6.6 that the magnitude of the friction force increases with increasing the wave number and it is independent on wave number at a certain value of flow rate. This result is different at $\varphi = 0.6$ as shown in Figure 6.7. Finally, the friction force is displayed versus flow rate in Figure 6.8 at Wi = 0, $\delta = 0.02$, and $\varphi = 0.6$, for various values of Reynolds number (Re = 0, 50, 100). It is clear that the magnitude of friction force decreases with increasing Reynolds number and it does not depend on Reynolds number at a certain value of flow rate.
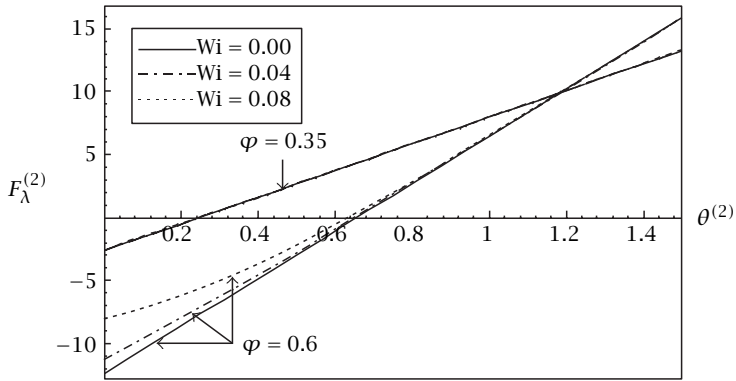
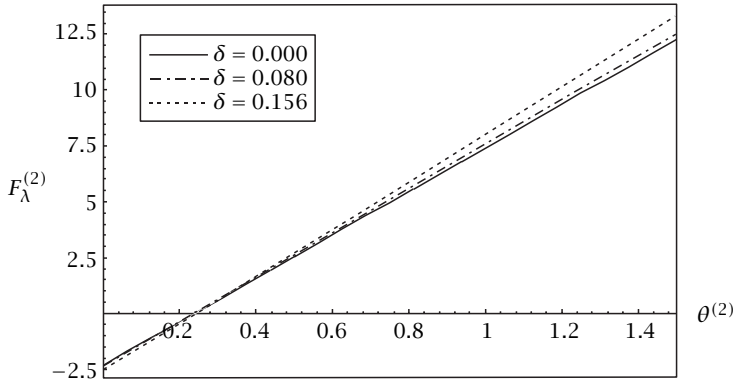FIGURE 6.5. The friction force versus flow rate at Re = 0.1, δ = 0.156, and $\varphi = 0.35, 0.6$.

FIGURE 6.6. The friction force versus flow rate at Re = 0.1, Wi = 0.08, and $\varphi = 0.35$.
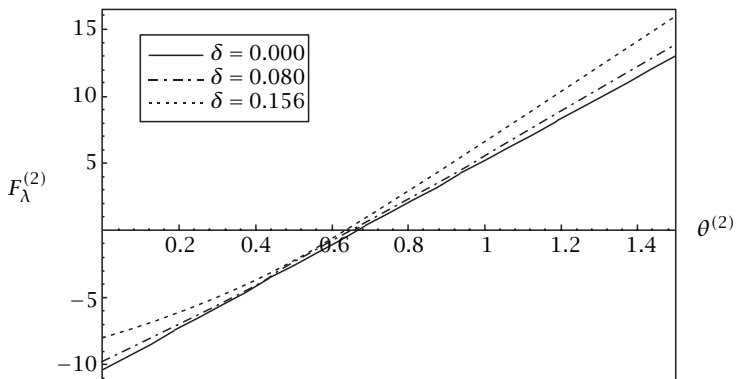
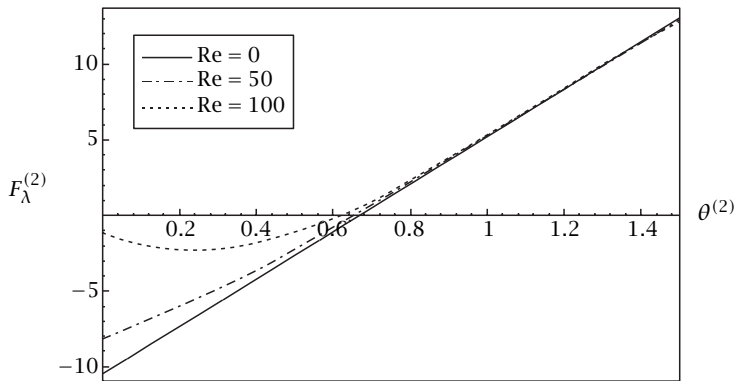FIGURE 6.7. The friction force versus flow rate at Re = 0.1, Wi = 0.08, and $\varphi = 0.6$.

FIGURE 6.8. The friction force versus flow rate at Wi = 0, $\delta = 0.02$, and $\varphi = 0.6$.

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