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NONAUTONOMOUS DIFFERENTIAL EQUATIONS OF ALTERNATELY RETARDED AND ADVANCED TYPE

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ABSTRACT. We obtain a solution formula of the differential equation $\dot{x}(t)+a(t)x(t)+b(t)x(g(t))=f(t)$. At the same time, we study its oscillation and asymptotic stability properties.

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1. Introduction and preliminary. In this paper, we investigate the global asymptotic behavior as well as oscillation of equations with piecewise constant argument

$$\dot{x}(t)+a(t)x(t)+b(t)x(g(t))=f(t) \quad \text{for } t>0 \tag{1.1}$$

subject to the initial condition

$$x(0)=x_0, \tag{1.2}$$

where $a(t), b(t)$, and $f(t)$ are locally integrable functions on $[0,\infty)$, $g(t)$ is a piecewise constant function defined by

$$g(t)=np \quad \text{for } t\in[np-l,(n+1)p-l)\ (n\in N), \tag{1.3}$$

where p and l are positive constants satisfying $p>l$.

Since the argument deviation of (1.1), namely

$$\tau(t)=t-g(t) \tag{1.4}$$

is negative in $[np-l,np)$ and positive in $[np,(n+1)p-l)$, equation (1.1) is said to be of alternately advanced and retarded type.

Equations with piecewise constant argument (EPCA) deviation were investigated in many papers (see [1, 2, 3, 4, 5, 6, 7, 8, 9]). Since EPCA combine the features of both differential and difference equations, their asymptotic behavior as $t\to\infty$ resembles in some cases the solution growth of differential equations, while in others it inherits the properties of difference equations. So this makes EPCA more interesting.

Definition 1.1. *A function $x:[0,\infty)\to R$ is a solution of (1.1) and (1.2) if the following conditions hold:*

(i) *x is continuous on $[0,\infty)$.*

(ii) *x is differentiable in $[0,\infty)$, except possibly at the points $t=np-l$, $n\in\{1,2,\dots\}$, where one-sided derivatives exist.*

(iii) $x(0) = x_0$ *and* x *satisfies (1.1) in* $(0, p-l)$ *and in every interval of the form* $[np-l,(n+1)p-l)$ *for* $n \in \{1,2,\ldots\}$.

A solution of (1.1) and (1.2) is oscillatory if it has no last zero. Let $[\cdot]$ denote the greatest integer function. This paper was motivated by [7] in which the equation

$$\dot{x}(t) + Ax(t) + Bx(g(t)) = f(t) \quad \text{for } t > 0 \tag{1.5}$$

was investigated, where A and B are $r \times r$ matrices, x is an r-vector and $f(t)$ is a locally integrable function on $[0,\infty)$.

2. The case $a(t) \equiv 0$. In this case, (1.1) becomes

$$\dot{x}(t) + b(t)x(g(t)) = f(t) \quad \text{for } t > 0. \tag{2.1}$$

To simplify the notation, define

$$\begin{gathered} B(a,b) = 1 - \int_a^b b(s)\,ds, \quad B(0,-l) = 1, \qquad x(np) = x_n, \\ I_n = [np-l,(n+1)p-l) \quad \text{for } n = 1,2,\ldots. \end{gathered} \tag{2.2}$$

Theorem 2.1. *Let* $b(t)$ *and* $f(t)$ *be locally integrable on* $[0,\infty)$. *Then (1.2), (1.4), and (2.1) has a unique solution on* $[0,\infty)$ *given by*

$$\begin{aligned} x(t) = B(g(t),t)&\left(\prod_{j=1}^{g(t)/p} \frac{B((j-1)p, jp-l)}{B(jp, jp-l)}\right) \\ &\times\left[x_0 + \sum_{j=1}^{g(t)/p}\left(\prod_{i=1}^{j} \frac{B((i-1)p,(i-1)p-l)}{B((i-1)p, ip-l)}\right)\int_{(j-1)p}^{jp} f(s)\,ds\right] \\ &+\int_{g(t)}^{t} f(s)\,ds, \end{aligned} \tag{2.3}$$

where $B(a,b)$ *is defined in (2.2).*

In addition, if $b(t)$ and $f(t)$ are integrable on $(-\infty,0]$, this solution can be continued backwards on $(-\infty,0]$ and is given by

$$\begin{aligned} x(t) = B(g(t),t)&\left(\prod_{j=1}^{-g(t)/p} \frac{B((-j-1)p, -jp-l)}{B(-jp, -jp-l)}\right) \\ &\times\left[x_0 + \sum_{j=1}^{-g(t)/p}\left(\prod_{i=1}^{j} \frac{B((-i-1)p,(-i-1)p-l)}{B((-i-1)p, -ip-l)}\right)\int_{(-j-1)p}^{-jp} f(s)\,ds\right] \\ &+\int_{g(t)}^{t} f(s)\,ds \end{aligned} \tag{2.4}$$

Proof. We use the notation given in (2.2).

In each interval of the type I_n, (2.1) becomes

$$\dot{x}(t) + b(t)x(np) = f(t) \tag{2.5}$$

which has a unique solution whenever a preassigned value for $x(np)$ is given. The solution of (2.1), with $x(np) = x_n$, is

$$x(t) = B(np,t)x_n + \int_{np}^{t} f(s)\,ds \quad \text{for } t \in I_n, \tag{2.6}$$

and with $x((n+1)p) = x_{n+1}$ is

$$x(t) = B((n+1)p,t)x_{n+1} + \int_{(n+1)p}^{t} f(s)\,ds \quad \text{for } t \in I_{n+1}. \tag{2.7}$$

Continuity of the solution at $t = (n+1)p - l$ requires

$$\begin{aligned} &B(np,(n+1)p-l)x_n + \int_{np}^{(n+1)p-l} f(s)\,ds \\ &\quad = B((n+1)p,(n+1)p-l)x_{n+1} + \int_{(n+1)p}^{(n+1)p-l} f(s)\,ds, \end{aligned} \tag{2.8}$$

so that

$$x_{n+1} = \frac{B(np,(n+1)p-l)}{B((n+1)p,(n+1)p-l)}x_n + \frac{1}{B((n+1)p,(n+1)p-l)}\int_{np}^{(n+1)p} f(s)\,ds, \tag{2.9}$$

from which it follows that

$$x_n = \left(\prod_{j=1}^{n} \frac{B((j-1)p,jp-l)}{B(jp,jp-l)}\right)\left[x_0 + \sum_{j=1}^{n}\left(\prod_{i=1}^{j}\frac{B((i-1)p,(i-1)p-l)}{B((i-1)p,ip-l)}\right)\int_{(j-1)p}^{jp} f(s)\,ds\right]. \tag{2.10}$$

Substituting (2.10) into (2.6) yields (2.3). The continuation of (2.3) on $(-\infty,0]$ is obtained in a similar way. This completes the proof. □

Theorem 2.2. *Let $b(t)$ be locally integrable on $[0,\infty)$. Assume that $|b(t)| < B_1$ ($B_1 > 0$) for $t \in [0,\infty)$ and*

$$\left|\frac{B((n-1)p,np-l)}{B(np,np-l)}\right| < \alpha < 1 \quad \text{for } n \in \{1,2,\dots\}. \tag{2.11}$$

(a) *If $f(t) \equiv 0$ then the trivial solution of (2.1) is globally asymptotically stable.*
(b) *If $\lim_{t\to\infty} f(t) = 0$ then every solution of (2.1) tends to zero as $t \to \infty$.*

Proof. (a) Note that for $t \in I_n$

$$\left|B(g(t),t)\left(\prod_{j=1}^{g(t)/p}\frac{B((j-1)p,jp-l)}{B(jp,jp-l)}\right)x_0\right| < B_2\alpha^n|x_0|, \tag{2.12}$$

where $B_2 = 1 + B_1\max\{l,p-l\}$. Therefore (a) is proved.

(b) We observe that the remaining term in (2.3) tends to zero as $t \to \infty$. For $t \in I_n$

$$\left|\int_{g(t)}^{t} f(s)\,ds\right| < \max\{l,p-l\}\max\{|f(t)| : t \in I_n\}. \tag{2.13}$$

Similarly, $F_j = \int_{(j-1)p}^{jp} f(s)\,ds \to 0$ as $j \to \infty$. Hence, given $\varepsilon > 0$, choose P_1 such that $|F_j| < K$ if $j < P_1$ and $|F_j| < \varepsilon(1-\alpha)B_3/2B_2$ for $j \geq P_1$, choose P_2 so that if $n > P_2$ then $\alpha^n < \varepsilon B_3/2KB_2P_1$, where $B_3 = 1/|1-B_1 l|$. If $n > \max\{P_1, P_2\}$, then

$$\begin{aligned}
&\left|\prod_{j=1}^{g(t)/p}\frac{B((j-1)p,jp-l)}{B(jp,jp-l)}\left[\sum_{j=1}^{g(t)/p}\left(\prod_{i=1}^{j}\frac{B((i-1)p,(i-1)p-l)}{B((i-1)p,ip-l)}\right)\int_{(j-1)p}^{jp} f(s)\,ds\right]\right| \\
&\quad\leq \sum_{j=1}^{n}\left[\left(\prod_{i=j+1}^{n}\left|\frac{B((i-1)p,ip-l)}{B(ip,ip-l)}\right|\right)\frac{1}{|B(jp,jp-l)|}|F_j|\right] \\
&\quad\leq \frac{1}{B_3}\sum_{j=1}^{P_1}\left[\left(\prod_{i=j+1}^{n}\alpha^i\right)|F_j|\right]+\frac{1}{B_3}\sum_{j=P_1+1}^{n}\left[\left(\prod_{i=j+1}^{n}\alpha^i\right)|F_j|\right] \leq \frac{\varepsilon}{B_2},
\end{aligned} \tag{2.14}$$

where we define $\prod_{i=n+1}^{n} B((i-1)p,ip-l)/B(ip,ip-l) = 1$. This completes the proof. □

Theorem 2.3. *Let $b(t)$ be locally integrable on $[0,\infty)$. Every solution of the equation*

$$\dot{x}(t)+b(t)x(g(t)) = 0, \quad x(0) = x_0, \tag{2.15}$$

is oscillatory if $B(np,(n+1)p-l)/B(np,np-l)$ is not eventually positive.

Proof. Let $x(t)$ be a solution of (2.15). The continuity of $x(t)$ at $t = (n+1)p-l$ in (2.6) gives

$$x((n+1)p-l) = B(np,(n+1)p-l)x_n. \tag{2.16}$$

Again using (2.6) with $t = np-l$ we have that

$$x(np-l) = B(np,np-l)x_n. \tag{2.17}$$

From (2.16) and (2.17) we obtain that

$$x((n+1)p-l) = \frac{B(np,(n+1)p-l)}{B(np,np-l)}x(np-l). \tag{2.18}$$

It is easy to see that the sequence $\{x(np-l)\}$ oscillates if $B(np,(n+1)p-l)/B(np,np-l)$ is not eventually positive. Therefore $x(t)$ oscillates if $\{x(np-l)\}$ oscillates. This completes the proof. □

Corollary 2.4. *If $b(t) \leq 0$, the sign of every solution of (2.15) is identical with the sign of its initial value.*

The proof of Corollary 2.4 is obvious from (2.3).

Remark 2.5. Corollary 2.4 can be recounted that if $b(t) \le 0$, then all solutions of (2.15) are nonoscillatory. The following example is to illustrate Theorem 2.3.

Example 2.6. Consider the equation

$$\dot{x}(t) + (2-t)x\left(2\left[\frac{t+1}{2}\right]\right) = 0, \quad t > 0 \tag{2.19}$$

with $x(0) = x_0$. Theorem 2.1 asserts that (2.19) subject to $x(0) = x_0$ has a unique solution on $[0,\infty)$. The solution of (2.19) is given by

$$x(t) = \left[1 - 2(t-n) + \frac{t^2 - n^2}{2}\right]\left(\prod_{j=1}^{n} \frac{4j-5}{7-4j}\right)x_0 \quad \text{for } t \in [2n-1, 2n+1). \tag{2.20}$$

From Theorem 2.3, all solutions of (2.19) are oscillatory if

$$\frac{B(2n, 2(n+1)-1)}{B(2n, 2n-1)} = -1 + \frac{6}{7-4n} \tag{2.21}$$

is not eventually positive.

3. The case (1.1).

To simplify the notation, define

$$\begin{aligned} B(a,b) &= 1 - \int_a^b b(s)\exp\left(\int_a^s a(u)\,du\right)ds, \\ F(a,b) &= \int_a^b f(s)\exp\left(\int_a^s a(u)\,du\right)ds, \\ \int_{-p}^{0} a(s)\,ds &= 0, \qquad x(np) = x_n, \\ I_n &= [np-l, (n+1)p-l) \quad \text{for } n = 1,2,\dots. \end{aligned} \tag{3.1}$$

We state some theorems for (1.1). The proofs of Theorems 3.1, 3.2, and 3.3 can be obtained by the techniques used in the proofs of Theorems 2.1, 2.2, and 2.3 of Section 2.

Theorem 3.1. *Let $a(t), b(t)$, and $f(t)$ be locally integrable on $[0,\infty)$. Then (1.1) and (1.2) has a unique solution on $[0,\infty)$ given by*

$$\begin{aligned} x(t) = {} & B(g(t),t)\exp\left(-\int_{g(t)}^{t} a(s)\,ds\right) \\ & \times \left(\prod_{j=1}^{g(t)/p} \exp\left(-\int_{(j-1)p}^{jp} a(s)\,ds\right)\frac{B((j-1)p, jp-l)}{B(jp, jp-l)}\right) \\ & \times \left[x_0 - \sum_{j=1}^{g(t)/p}\left(\prod_{i=1}^{j}\exp\left(-\int_{(i-1)p}^{ip} a(s)\,ds\right)\frac{B^{-1}((i-1)p, ip-l)}{B^{-1}(ip, ip-l)}\right)\frac{F(jp,(j-1)p)}{B(jp,jp-l)}\right] \\ & + \exp\left(-\int_{g(t)}^{t} a(s)\,ds\right)F(g(t),t), \end{aligned} \tag{3.2}$$

where $B(a,b)$ and $F(a,b)$ are defined in (3.1).

In addition, if $a(t), b(t)$, and $f(t)$ are integrable on $(-\infty, 0]$, this solution can be continued backwards on $(-\infty, 0]$ and is given by

$$\begin{aligned} x(t) = {} & B(g(t),t)\exp\left(-\int_{g(t)}^{t} a(s)\,ds\right) \\ & \times \left(\prod_{j=1}^{-g(t)/p} \exp\left(-\int_{(j-1)p}^{jp} a(s)\,ds\right) \frac{B((-j-1)p,-jp-l)}{B(-jp,-jp-l)} \right) \\ & \times \left[x_0 - \sum_{j=1}^{-g(t)/p} \left(\prod_{i=1}^{j} \exp\left(-\int_{(-i-1)p}^{-ip} a(s)\,ds\right) \frac{B^{-1}((-i-1)p,-ip-l)}{B^{-1}(-ip,-ip-l)} \right) \frac{F(-jp,-(j-1)p)}{B(-jp,-jp-l)} \right] \\ & + \exp\left(-\int_{g(t)}^{t} a(s)\,ds\right) F(g(t),t). \end{aligned} \tag{3.3}$$

Theorem 3.2. *Let $a(t)$ and $b(t)$ be locally integrable on $[0, \infty)$. Assume that $|a(t)| < A$, $|b(t)| < B_1$ $(A, B_1 > 0)$ for $t \in [0, \infty)$ and*

$$\left| \frac{B((n-1)p, np-l)}{B(np, np-l)} \right| < \alpha < 1 \quad \text{for } n \in \{1, 2, \ldots\}. \tag{3.4}$$

(a) *If $f(t) \equiv 0$ then the trivial solution of (1.1) is globally asymptotically stable.*
(b) *If $\lim_{t\to\infty} f(t) = 0$ then every solution of (1.1) tends to zero as $t \to \infty$.*

Theorem 3.3. *Let $a(t)$ and $b(t)$ be locally integrable on $[0, \infty)$. Every solution of the equation*

$$\dot{x}(t) + a(t)x(t) + b(t)x(g(t)) = 0, \quad x(0) = x_0 \tag{3.5}$$

is oscillatory if $B(np, (n+1)p-l)/B(np, np-l)$ is not eventually positive.

In summary, equations with piecewise constant argument are interesting in their own right, and have some curious and unpredictable properties. The systems of nonautonomous differential equations of alternately retarded and advanced type can be studied in similar ways.

References

[1] K. L. Cooke and J. Wiener, *Retarded differential equations with piecewise constant delays*, J. Math. Anal. Appl. **99** (1984), no. 1, 265–297. MR 85d:34072. Zbl 557.34059.

[2] ———, *An equation alternately of retarded and advanced type*, Proc. Amer. Math. Soc. **99** (1987), no. 4, 726–732. MR 88g:34119. Zbl 628.34074.

[3] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations: With Applications*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1991. MR 93m:34109. Zbl 780.34048.

[4] I. Györi, G. Ladas, and L. Pakula, *Conditions for oscillation of difference equations with applications to equations with piecewise constant arguments*, SIAM J. Math. Anal. **22** (1991), no. 3, 769–773. MR 92c:39004. Zbl 734.39003.

[5] K. N. Jayasree and S. G. Deo, *On piecewise constant delay differential equations*, J. Math. Anal. Appl. **169** (1992), no. 1, 55–69. MR 93f:34122. Zbl 913.34054.

[6] G. Papaschinopoulos, *On asymptotic behavior of the solutions of a class of perturbed differential equations with piecewise constant argument and variable coefficients*, J. Math. Anal. Appl. **185** (1994), no. 2, 490–500. MR 95f:34064. Zbl 810.34079.

[7] I. W. Rodrigues, *Systems of differential equations of alternately retarded and advanced type*, J. Math. Anal. Appl. **209** (1997), no. 1, 180–190. MR 98c:34103. Zbl 880.34071.

[8] J. Wiener, *Generalized Solutions of Functional-Differential Equations*, World Scientific Publishing, New Jersey, 1993. MR 94m:34174. Zbl 874.34054.

[9] J. Wiener and A. R. Aftabizadeh, *Differential equations alternately of retarded and advanced type*, J. Math. Anal. Appl. **129** (1988), no. 1, 243–255. MR 88j:34142. Zbl 671.34063.

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