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ANALYTIC FUNCTIONS OF NON-BAZILEVIČ TYPE AND STARLIKENESS

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ABSTRACT. Two classes $\bar{\mathcal{B}}_n(\mu,\alpha,\lambda)$ and $\bar{\mathcal{P}}_n(\mu,\alpha,\lambda)$ of analytic functions which are not Bazilevič type in the open unit disk $\mathbb{U}$ are introduced. The object of the present paper is to consider the starlikeness of functions belonging to the classes $\bar{\mathcal{B}}_n(\mu,\alpha,\lambda)$ and $\bar{\mathcal{P}}_n(\mu,\alpha,\lambda)$.

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1. Introduction. Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$. For $n \geqq 1$, we define

$$\mathcal{A}_n = \left\{ f : f(z) = z + \sum_{n+1}^{\infty} a_j z^j \text{ analytic in } \mathbb{U} \right\}. \tag{1.1}$$

Also, we need the following notations and definitions. Let

$$\mathcal{S}^* = \left\{ f \in \mathcal{A}_1 : \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > 0,\ z \in \mathbb{U} \right\} \tag{1.2}$$

be the class of starlike functions (with respect to the origin) in $\mathbb{U}$, and let

$$\mathcal{T}_\lambda = \left\{ f \in \mathcal{A}_1 : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \lambda,\ 0 < \lambda \leqq 1,\ z \in \mathbb{U} \right\} \tag{1.3}$$

be the subclass of $\mathcal{S}^*$. Further we define

$$\mathcal{B}(\mu,\lambda) = \left\{ f \in \mathcal{A}_1 : \left| f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - 1 \right| < \lambda,\ \mu > 0,\ 0 < \lambda \leqq 1,\ z \in \mathbb{U} \right\} \tag{1.4}$$

which is the subclass of Bazilevič class of univalent functions (cf. [1]). Ponnusamy [8] has considered the starlikeness and other properties of functions $f(z)$ in the class $\mathcal{B}(\mu,\lambda)$. For negative μ, that is, for $-1 < \mu < 0$, which is better to write (with $0 < \mu < 1$) in the form

$$\bar{\mathcal{B}}(\mu,\lambda) = \left\{ f \in \mathcal{A}_1 : \left| f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} - 1 \right| < \lambda,\ 0 < \mu < 1,\ 0 < \lambda < 1,\ z \in \mathbb{U} \right\}, \tag{1.5}$$

we obtain the class which was considered earlier by Obradović [3, 4], Obradović and Owa [5], and Obradović and Tuneski [6].

For the limit case $\mu = 0$, this class becomes the class $\mathcal{T}_\lambda$. When $\mu = 1$, this class becomes the class of univalent functions $f(z)$ satisfying

$$\left|\frac{z^2 f'(z)}{f(z)^2}-1\right|<\lambda, \tag{1.6}$$

which was studied by Ozaki and Nunokawa [7].

Next, for $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_k)\in\mathbb{R}^k$ with $\alpha_j\in\mathbb{R}$, we define the operator D_α by

$$D_\alpha=1+\alpha_1 z\frac{d}{dz}+\alpha_2 z^2\frac{d^2}{dz^2}+\cdots+\alpha_k z^k\frac{d^k}{dz^k}, \tag{1.7}$$

and, by virtue of the operator D_α, the subclasses

$$\bar{\mathcal{B}}_n(\mu,\alpha,\lambda)=\left\{f\in\mathcal{A}_n:\left|D_\alpha\left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\right)-1\right|<\lambda,\ 0<\mu<1,\ 0<\lambda<1,\ z\in\mathbb{U}\right\}, \tag{1.8}$$

$$\bar{\mathcal{P}}_n(\mu,\alpha,\lambda)=\left\{f\in\mathcal{A}_n:\left|D_\alpha\left[\left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\right)'\right]\right|<\lambda,\ 0<\mu<1,\ 0<\lambda<1,\ z\in\mathbb{U}\right\} \tag{1.9}$$

of $\mathcal{A}_n$.

Samaris [9] has investigated the appropriate classes for the case (1.4), and has obtained results which are stronger than those given earlier and in several cases sharp ones. By using the method by Samaris [9], we will generalize some results given in [3, 4], and we will obtain some new results. We also note that we cannot directly apply some nice estimates given by Samaris [9].

For $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_k)\in\mathbb{R}^k$, we define the polynomial $P_\alpha(x)$ by

$$P_\alpha(x)=1+\alpha_1 x+\alpha_2 x(x-1)+\cdots+\alpha_k x(x-1)\cdots(x-k+1). \tag{1.10}$$

In this paper, we will use the classes such that $\alpha=0(P_\alpha(x)=1)$ or $P_\alpha(x)$ has non-positive real zeros given by ρ_j $(j=1,2,3,\dots,k)$. In this case, we can write

$$P_\alpha(x)=\alpha_k\prod_{j=1}^{k}(x-\rho_j) \tag{1.11}$$

with

$$\alpha_k\prod_{j=1}^{k}(-\rho_j)=1. \tag{1.12}$$

Further, for $t=(t_1,t_2,\dots,t_k)\in(0,1)^k$ such that $t_j\in(0,1)$, we denote t_α by

$$t_\alpha=t_1^{-1/\rho_1}t_2^{-1/\rho_2}\cdots t_k^{-1/\rho_k}=\prod_{j=1}^{k}t_j^{-1/\rho_j}. \tag{1.13}$$

If $P_\alpha(x)=1$, then we define $t_\alpha=1$. Also if $n=1,2,3,\dots$, then we denote by $\mathcal{W}_\alpha$ the class of analytic functions $w(z)$ in $\mathbb{U}$ for which $|w(z)|\leqq|z|^n$ $(z\in\mathbb{U})$.

2. Starlikeness of the classes $\bar{\mathcal{B}}_n(\mu,\alpha,\lambda)$ and $\bar{\mathcal{P}}_n(\mu,\alpha,\lambda)$. Our first result is contained in the following theorem.

Theorem 2.1. *Let $\bar{\mathcal{B}}_n(\mu,\alpha,\lambda)$ be the class defined by (1.8) for which $(n-\mu)P_\alpha(n)-\lambda n>0$ $(n=1,2,3,\ldots)$, $0<\mu<1$.*

(i) *If $\lambda n/((n-\mu)P_\alpha(n)-\lambda\mu)\leqq r$, then $\bar{\mathcal{B}}_n(\mu,\alpha,\lambda)\subset\mathcal{T}_r$,*

(ii) *if $\lambda\leqq P_\alpha(n)((n-\mu)/(n+\mu))$, then $\bar{\mathcal{B}}_n(\mu,\alpha,\lambda)\subset\mathcal{T}_1$.*

Proof. From

$$D_\alpha\left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\right)=1+\lambda w(z)=1+\lambda\sum_{n=1}^{\infty}w_nz^n\quad(w\in\mathcal{W}_n),\tag{2.1}$$

we obtain

$$\begin{aligned}f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}&=1+\lambda\sum_{n=1}^{\infty}w_n\frac{z^n}{P_\alpha(n)},\\ \left(\frac{z}{f(z)}\right)^{\mu}&=1-\lambda\sum_{n=1}^{\infty}w_n\left(\frac{\mu}{n-\mu}\right)\frac{z^n}{P_\alpha(n)}.\end{aligned}\tag{2.2}$$

Since

$$\frac{1}{P_\alpha(n)}=\int_{[0,1]^k}t_\alpha^n,\qquad\frac{\mu}{P_\alpha(n)(n-\mu)}=\int_{[0,1]^{k+1}}t_\alpha^nt_{k+1}^{n/\mu-2},\tag{2.3}$$

using (2.2), we have

$$\begin{aligned}f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}&=1+\lambda\int_{[0,1]^k}w(t_\alpha z),\\ \left(\frac{z}{f(z)}\right)^{\mu}&=1-\lambda\int_{[0,1]^{k+1}}w(t_\alpha t_{k+1}^{1/\mu}z)t_{k+1}^{-2}.\end{aligned}\tag{2.4}$$

From (2.4), we easily obtain that

$$\frac{zf'(z)}{f(z)}=\frac{1+\lambda\int_{[0,1]^k}w(t_\alpha z)}{1-\lambda\int_{[0,1]^{k+1}}w(t_\alpha t_{k+1}^{1/\mu}z)t_{k+1}^{-2}},\tag{2.5}$$

and from here

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}-1\right|&\leqq\lambda\frac{\int_{[0,1]^k}|w(t_\alpha z)|+\int_{[0,1]^{k+1}}|w(t_\alpha t_{k+1}^{1/\mu}z)|t_{k+1}^{-2}}{1-\lambda\int_{[0,1]^{k+1}}|w(t_\alpha t_{k+1}^{1/\mu}z)|t_{k+1}^{-2}}\\ &<\lambda\frac{\int_{[0,1]^k}t_\alpha^n+\int_{[0,1]^{k+1}}t_\alpha^nt_{k+1}^{n/\mu-2}}{1-\lambda\int_{[0,1]^{k+1}}t_\alpha^nt_{k+1}^{n/\mu-2}}=\frac{\lambda n}{(n-\mu)P_\alpha(n)-\lambda\mu},\end{aligned}\tag{2.6}$$

from which the conclusion of the theorem easily follows. □

Remark 2.2. For $n=1$ and $P_\alpha=1$ from Theorem 2.1, we have the result given by Obradović [3].

Remark 2.3. From (2.5), similar to the proof of Theorem 2.1, for the class $\bar{\mathcal{B}}_n(\mu,\alpha,\lambda)$ we have

$$\left|\frac{zf'(z)}{f(z)}-t\right|\leqq|1-t|+\frac{\lambda n}{(n-\mu)P_\alpha(n)-\lambda\mu}\quad(t>0).\tag{2.7}$$

If $\lambda_1 = \lambda n/((n-\mu)P_\alpha(n)-\lambda\mu) < 1$ and $t \geqq (1+\lambda_1)/2$, then we have

$$\bar{\mathcal{B}}_n(\mu,\alpha,\lambda) \subset (\mathcal{S}^*)_t, \tag{2.8}$$

where

$$(\mathcal{S}^*)_t = \left\{ f \in \mathcal{A}_1 : \left| \frac{zf'(z)}{f(z)} - t \right| < t,\ z \in \mathbb{U} \right\}. \tag{2.9}$$

Especially, for $n = 1$ and $\alpha = 0$, we have

$$\bar{\mathcal{B}}(\mu,\lambda) \subset (\mathcal{S}^*)_t \tag{2.10}$$

for $\lambda/(1-\mu-\lambda) < 1$ and $t \geqq 1/2+\lambda/(2(1-\mu-\lambda))$.

In the next theorem for the class $\bar{\mathcal{B}}(\mu,\lambda)\cap\mathcal{A}_n = \bar{\mathcal{B}}_n(\mu,0,\lambda)$, we will prove that the appropriate results are the best possible.

Theorem 2.4. *Let $\bar{\mathcal{B}}(\mu,\lambda)\cap\mathcal{A}_n$ be the class for which $n-\mu-\lambda\mu > 0$. Then*
(i) $\bar{\mathcal{B}}(\mu,\lambda)\cap\mathcal{A}_n \subset \mathcal{T}_r$ *if and only if* $\lambda n/(n-\mu-\lambda\mu) \leqq r$ *and*
(ii) $\bar{\mathcal{B}}(\mu,\lambda)\cap\mathcal{A}_n \subset \mathcal{S}^*$ *if and only if* $\lambda \leqq (n-\mu)/\sqrt{(n-\mu)^2+\mu^2}$.

Proof. For $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$, with the lemma by Fournier [2], there exists a sequence of functions $\phi_k(z)$ analytic in the closed unit disk $\bar{\mathbb{U}}$ such that $|\phi_k(z)| \leqq |z|$, $\lim_{k\to\infty}\phi_k(z) = ze^{\mathrm{i}t_1}$, $\lim_{k\to\infty}\phi_k(1) = e^{\mathrm{i}t_2}$ uniformly on compact subsets of $\mathbb{U}$. If $w_k(z) = z^{n-1}\phi_k(z)$, then we consider the sequence $f_k(z) \in \bar{\mathcal{B}}(\mu,\lambda)\cap\mathcal{A}_n$ which is given by

$$f_k'(z)\left(\frac{z}{f_k(z)}\right)^{1+\mu} = 1+\lambda w_k(z), \tag{2.11}$$

or

$$\frac{zf_k'(z)}{f_k(z)} = \frac{1+\lambda w_k(z)}{1-\lambda\int_0^1 w_k(t^{1/\mu}z)t^{-2}dt}. \tag{2.12}$$

Since $\lim_{k\to\infty} w_k(z) = z^n e^{\mathrm{i}t_1}$ and $\lim_{k\to\infty} w_k(1) = e^{\mathrm{i}t_2}$, we see that

$$\lim_{k\to\infty}\lim_{z\to 1}\frac{zf_k'(z)}{f_k(z)} = \frac{1+\lambda e^{\mathrm{i}t_1}}{1-\lambda(\mu/(n-\mu))e^{\mathrm{i}t_2}}. \tag{2.13}$$

For $t_1 = 0$ and $t_2 = 0$, we get

$$\lim_{k\to\infty}\lim_{z\to 1}\left|\frac{zf_k'(z)}{f_k(z)} - 1\right| = \frac{\lambda n}{n-\mu-\lambda\mu}. \tag{2.14}$$

Let $0 < \lambda < 1$ and $q_1 \in [0,\pi/2]$, $q_2 \in [0,\pi/2]$ such that $\sin q_1 = \lambda$, $\sin q_2 = \lambda\mu/(n-\mu)$. If we choose $t_1 = q_1+\pi/2$, $t_2 = \pi/2-q_2$, then we obtain

$$\arg\left(1+\lambda e^{\mathrm{i}t_1}\right) = q_1, \qquad \arg\left(1-\frac{\lambda\mu}{n-\mu}e^{\mathrm{i}t_2}\right) = -q_2, \tag{2.15}$$

and therefore

$$\lim_{k\to\infty}\lim_{z\to 1}\left|\arg\left(\frac{zf_k'(z)}{f_k(z)}\right)\right| = q_1+q_2. \tag{2.16}$$

From $\cos(q_1+q_2) \geqq 0$ or $\sin^2 q_1 + \sin^2 q_2 - 1 \leqq 0$, we have the statement (ii) (one part) of the theorem. We note that the "if" part of the theorem follows from the result of Theorem 2.1 and from the result given by Obradović and Owa [5]. □

Theorem 2.5. *Let $\bar{\mathcal{P}}_n(\mu,\alpha,\lambda)$ be the class defined by (1.9) such that $n(n-\mu)P_\alpha(n-1) - \lambda n > 0$ $(n=1,2,3,\dots)$, $0<\mu<1$. If*

$$\frac{\lambda n}{n(n-\mu)P_\alpha(n-1)-\lambda\mu} \leqq r, \tag{2.17}$$

then

$$\bar{\mathcal{P}}_n(\mu,\alpha,\lambda) \subset \mathcal{T}_r. \tag{2.18}$$

Proof. The proof of this theorem is similar to that of Theorem 2.1. By virtue of

$$D_\alpha\left\{\left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\right)'\right\} = \lambda w(z) \quad (w \in \mathcal{W}_{n-1}), \tag{2.19}$$

we have

$$\begin{gathered}\left\{f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\right\}' = \lambda\int_{[0,1]^k} w(t_\alpha z),\\ f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} = 1+\lambda\int_{[0,1]^{k+1}} w(t_\alpha t_{k+1}z)z.\end{gathered} \tag{2.20}$$

From the last relation, we see that

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1-\lambda\int_{[0,1]^{k+2}} w\left(t_\alpha t_{k+1}t_{k+2}^{1/\mu}z\right)t_{k+2}^{-2}z, \tag{2.21}$$

and so

$$\frac{zf'(z)}{f(z)} = \frac{1+\lambda\int_{[0,1]^{k+1}} w(t_\alpha t_{k+1}z)z}{1-\lambda\int_{[0,1]^{k+2}} w\left(t_\alpha t_{k+1}t_{k+2}^{1/\mu}z\right)t_{k+2}^{-2}z}. \tag{2.22}$$

By using (2.22), as in the proof of Theorem 2.1, we easily obtain that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leqq \frac{\lambda n}{n(n-\mu)P_\alpha(n-1)-\lambda\mu}. \tag{2.23}$$

The statement of this theorem follows from the above inequality. □

Finally we give the following example of the theorem.

Example 2.6. For $P_\alpha = 1+t$ and $0<\mu<1$, $0<\lambda<1$, if $f(z)\in\mathcal{A}_n$ satisfies

$$\left|\left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\right)' + z\left(f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\right)''\right| < \lambda \tag{2.24}$$

implies that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leqq \frac{\lambda n}{n^2(n-\mu)-\lambda\mu}. \tag{2.25}$$

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