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COUNTABLY I-COMPACT SPACES

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Abstract. We introduce the class of countably I-compact spaces as a proper subclass of countably S-closed spaces. A topological space (X,T) is called countably I-compact if every countable cover of X by regular closed subsets contains a finite subfamily whose interiors cover X. It is shown that a space is countably I-compact if and only if it is extremally disconnected and countably S-closed. Other characterizations are given in terms of covers by semiopen subsets and other types of subsets. We also show that countable I-compactness is invariant under almost open semi-continuous surjections.

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1. Introduction. A topological space (X,T) is called S-closed by Thompson [8] if every cover of X by semiopen subsets contains a finite subfamily whose union is dense in (X,T). Cameron [1] showed that (X,T) is S-closed if and only if every cover of X by regular closed sets has a finite subcover (accordingly, S-closed spaces are called rc-compact). A topological space (X,T) is called I-compact by Cameron [2] if every cover of X by regular closed sets contains a finite subfamily whose interiors cover X. I-compact spaces were further studied by Sivaraj in [7].

In [3] the class of countably S-closed spaces was introduced and studied. A space (X,T) is called countably S-closed if every countable cover of X by regular closed subsets has a finite subcover for X. It was studied further in [5], under the name countably rc-compact.

In the present paper, after the preliminaries in Section 2, we define in Section 3 the class of countably I-compact spaces, where a space (X,T) is called countably I-compact if every countable cover of X by regular closed subsets contains a countable subfamily whose interiors cover X. Then we provide characterizations of countably I-compact spaces in terms of semiopen covers or semipreopen covers. Also it is shown that (X,T) is countably I-compact if and only if it is countably S-compact and extremally disconnected.

In Section 4, we include main properties of countably I-compact spaces, while we deal in Section 5 with mappings of countably I-spaces.

Throughout this paper, a space will mean a topological space with no separation axiom assumed. We always use (X,T) and (Y,M) to denote topological spaces, and $\mathbb{N}$ denotes the set of natural numbers.

2. Preliminaries. Let (X,T) be a space and let $A \subseteq X$. Then $\operatorname{int}_T(A)$ and $\operatorname{cl}_T(A)$ (or simply $\operatorname{int}(A)$ and $\operatorname{cl}(A)$) denote the interior of A and the closure of A in (X,T), respectively. The subset $A \subseteq X$ is *regular open* (*regular closed*) if $A = \operatorname{int}\operatorname{cl}(A)$ $(A = \operatorname{cl}\operatorname{int}(A))$.

It is clear that A is regular open if and only if its complement is regular closed. Also we include the following easy facts.

Remark 2.1. Let A be a subset of a space (X,T). Then

(a) A is regular open if and only if $A = \mathrm{int}(F)$ for some closed subset F of (X,T).

(b) A is regular closed if and only if $A = \mathrm{cl}(U)$ for some $U \subseteq T$.

We use $\mathrm{RO}(X,T)$ and $\mathrm{RC}(X,T)$ to denote the family of all regular open subsets and the family of all regular closed subsets of (X,T), respectively.

A subset A of a space (X,T) is called *semiopen* (resp., *preopen*, *α-open*) if $A \subseteq \mathrm{cl}\,\mathrm{int}(A)$ (resp., $A \subseteq \mathrm{int}\,\mathrm{cl}(A)$, $A \subseteq \mathrm{int}\,\mathrm{cl}\,\mathrm{int}(A)$). We let $\mathrm{SO}(X,T)$ (resp., $\mathrm{PO}(X,T)$, T^α) denote the family of semiopen (resp., preopen, α-open) subsets of a space (X,T). We point here to the fact that T^α is a topology on X with $T \subseteq T^\alpha$.

Remark 2.2. For a space (X,T) it is well known that

(a) $\mathrm{RO}(X,T) \bigcup \mathrm{RC}(X,T) \subseteq \mathrm{SO}(X,T)$,

(b) $T^\alpha = \mathrm{SO}(X,T) \cap \mathrm{PO}(X,T)$.

A space (X,T) is called *extremally disconnected* (abbreviated e.d.) if $\mathrm{cl}(U)$ is open for any open subset U of X. We include for later use the following well-known facts.

Lemma 2.3. *A space (X,T) is e.d. if and only if whenever $U,V \in T$ and $U \cap V = \varnothing$ then $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \varnothing$.*

Lemma 2.4 (see [3]). *If P is a preopen (≡ locally dense) subset of a space (X,T), then*

$$\mathrm{RC}(P, T|_P) = \{F \cap P : F \in \mathrm{RC}(X,T)\}. \tag{2.1}$$

3. Countably I-compact spaces. In [3], a space (X,T) is called *countably S-closed* if every countable cover of X by regular closed subsets has a finite subcover (such a space is also studied in [5] where it was called *countably rc-compact*). A space (X,T) is called *feebly compact* if every countable open cover of X contains a finite subfamily whose union is dense in (X,T). It is known that every countably S-closed space is feebly compact (see [3, Proposition 2.1]) while [3, Example 4.3] provides several feebly compact spaces which are not countably S-closed.

We define now the class of countably I-compact spaces.

Definition 3.1. A space (X,T) is called *countably I-compact* if every countable cover $\{F_n : n \in \mathbb{N}\}$ of X by regular closed subsets contains a finite subfamily

$$\{F_k : k = 1,\dots,m\} \tag{3.1}$$

such that

$$X = \bigcup_{k=1}^{m} \mathrm{int}(F_k). \tag{3.2}$$

We start with the following characterization of countably I-compact spaces.

Theorem 3.2. *A space (X,T) is countably I-compact if and only if it is countably S-compact and e.d.*

Proof

Necessity. Let (X,T) be countably I-compact. It is immediate from the definition that (X,T) is countably S-closed. Now, suppose that (X,T) is not e.d. We find an open set U such that $\mathrm{cl}(U)$ is not open and therefore $\mathrm{cl}(U)-\mathrm{int}\,\mathrm{cl}(U)\neq\emptyset$. We put $V=X-\mathrm{cl}(U)$. Then $\{\mathrm{cl}(U),\mathrm{cl}(V)\}$ is a countable cover of X by regular closed subsets but $X\neq\mathrm{int}\,\mathrm{cl}(U)\bigcup\mathrm{int}\,\mathrm{cl}(V)$, a contradiction.

Sufficiency. Let $\{F_n : n\in\mathbb{N}\}$ be a countable cover of X by regular closed subsets. Since (X,T) is countably S-closed then there exists $m\in\mathbb{N}$ such that $X=\bigcup_{k=1}^{m}F_k$. For each $k=1,\dots,m$, we pick $U_k\in T$ such that $F_k=\mathrm{cl}(U_k)$. Since (X,T) is e.d., then $\mathrm{cl}(U_k)$ is open for each $k=1,\dots,m$. Thus $X=\bigcup_{k=1}^{m}F_k=\bigcup_{k=1}^{m}\mathrm{int}(F_k)$ and (X,T) is countably I-compact.

We point here that [3, Example 4.2] provides a countably S-closed space which is not e.d. Thus the class of countably I-compact spaces is a proper subclass of the countably S-closed spaces. □

Definition 3.3 (see [3]). A space (X,T) is called *km-perfect* if for every $G\in\mathrm{RO}(X,T)$ and for every point $x\in X-G$ there exists a sequence $\{U_n : n\in\mathbb{N}\}$ of open subsets of X such that $\bigcup_{n\in\mathbb{N}}U_n\subseteq G\subseteq\bigcup_{n\in\mathbb{N}}\mathrm{cl}(U_n)$ and $x\notin\bigcup_{n\in\mathbb{N}}\mathrm{cl}(U_n)$.

We include the following fact from [3].

Proposition 3.4 (see [3, Theorem 3.2]). *Let (X,T) be countably S-closed and km-perfect. Then (X,T) is e.d.*

We now have the following result.

Theorem 3.5. *A space (X,T) is countably I-compact if and only if (X,T) is countably S-closed and km-perfect.*

Proof

Necessity. Follows easily from the fact that every e.d. space is km-perfect (see [3, Theorem 3.1(i)]).

Sufficiency. Follows from Theorem 3.2 and Proposition 3.4. □

Theorem 3.6. *The following conditions are equivalent for an e.d. space (X,T):*

(a) (X,T) *is countably I-compact,*

(b) (X,T) *is countably S-closed,*

(c) (X,T) *is feebly compact.*

Proof. (a)⇒(b). Is clear.

(b)⇒(c). Is clear.

(c)⇒(a). Since (X,T) is feebly compact and e.d. then it is countably S-closed (see [3, Corollary 3.3(i)]). Thus (X,T) is countably I-compact by Theorem 3.2. □

To state our final characterization of countably I-compact spaces we recall that a subset A of a space (X,T) is called *regular semiopen* if there exists $G\in\mathrm{RO}(X,T)$ such that $G\subseteq A\subseteq\mathrm{cl}(G)$. The subset A is called *semipreopen* (see [4]) if $A\subseteq\mathrm{cl}\,\mathrm{int}\,\mathrm{cl}(A)$.

Let $\mathrm{RSO}(X,T)$ and $\mathrm{SPO}(X,T)$ denote, respectively, the family of all regular semiopen subsets of (X,T) and the family of all semipreopen subsets of (X,T). It is easy to check the following inclusions for a space (X,T):

$$\mathrm{RC}(X,T) \subseteq \mathrm{RSO}(X,T) \subseteq \mathrm{SO}(X,T) \subseteq \mathrm{SPO}(X,T). \tag{3.3}$$

Also, the following is an easy fact whose proof is omitted.

Proposition 3.7. *A subset A of a space (X,T) is semipreopen if and only if* $\mathrm{cl}(A)$ *is regular closed.*

We have now the following characterization.

Theorem 3.8. *The following conditions are equivalent for a space (X,T):*

(a) *(X,T) is countably I-compact.*

(b) *For every countable cover $\{A_n : n \in \mathbb{N}\}$ of X by semipreopen subsets there exists $m \in n$ such that $X = \bigcup_{k=1}^{m} \mathrm{int\,cl}(A_k)$.*

(c) *For every countable cover $\{S_n : n \in \mathbb{N}\}$ of X by semiopen subsets there exists $m \in n$ such that $X = \bigcup_{k=1}^{m} \mathrm{int\,cl}(S_k)$.*

(d) *For every countable cover $\{R_n : n \in \mathbb{N}\}$ of X by regular semiopen subsets there exists $m \in n$ such that $X = \bigcup_{k=1}^{m} \mathrm{int\,cl}(R_k)$.*

Proof. Follows easily from Proposition 3.7 and the remark preceding it involving the stated inclusions. □

4. Properties of countably I-compact spaces. To begin with, we point to the fact that, given a space (X,T), the family $\mathrm{RO}(X,T)$ is a base for a topology $T_s \subseteq T$ on X called the *semiregularization* of (X,T). A property P of topological spaces is called a *semiregular property* if a space (X,T) has property P if and only if (X,T_S) has property P. Countable S-closedness is a semiregular property (see [3, Proposition 2.6]). Also, it is a well-known fact that extremal disconnectedness is a semiregular property. Now, these remarks, together with Theorem 3.2, form the proof of the following result.

Theorem 4.1. *The property of being a countably I-compact space is a semiregular property.*

The remaining results of this section deal with subsets of countably I-compact spaces, or with those subsets which are countably I-compact.

Proposition 4.2. *Let (X,T) be a countably I-compact space and let S be a regular semiopen subset of (X,T). Then $(S,T \mid S)$ is countably I-compact.*

Proof. Since extremal disconnectedness is semiopen hereditary (see [6, Corollary 4.3]), it follows that $(S,T \mid S)$ is e.d.

We show that $(S,T \mid S)$ is countably S-closed. Choose $U \in \mathrm{RO}(X,T)$ such that $U \subseteq S \subseteq \mathrm{cl}(U)$. We have that U is, by [3, Proposition 2.9(i)], countably S-closed. It follows that $(S,T \mid S)$ is, by [3, Proposition 2.9(ii)], countably S-closed. We conclude that $(S,T \mid S)$ is, by Theorem 3.2, countably I-compact. □

Corollary 4.3. *Let (X,T) be a countably I-compact space.*

(a) *If $G \in \mathrm{RO}(X,T)$ then $(G,T \mid G)$ is countably I-compact.*

(b) *If $F \in \mathrm{RC}(X,T)$ then $(F,T \mid F)$ is countably I-compact.*

Proof. Follows easily from the obvious facts that $\mathrm{RO}(X,T) \subseteq \mathrm{RSO}(X,T)$ and $\mathrm{RC}(X,T) \subseteq \mathrm{RSO}(X,T)$. □

Proposition 4.4. *If a space (X,T) is a finite union of regular open countably I-compact subspaces G_k, $k = 1,\dots,n$, then (X,T) is countably I-compact.*

Proof. Let $\{F_n : n \in \mathbb{N}\}$ be a countable cover of X by regular closed subsets of the space (X,T). For $1 \le k \le m$, we let $\mathcal{F}_k = \{G_k \cap F_n : n \in \mathbb{N}\}$. Again, by Lemma 2.4, $\mathcal{F}_k$ is a cover of G_k by regular closed subsets of the countably I-compact subspace $(G_k, T|_{G_k})$. Thus there exists $\ell_k \in \mathbb{N}$ such that

$$G_k = \bigcup_{j=1}^{\ell_k} \mathrm{int}_{G_k}(G_k I F_j) \subseteq \bigcup_{j=1}^{\ell_k} \mathrm{int}_T(F_j). \tag{4.1}$$

We let $\ell = \max\{\ell_k : k = 1,\dots,n\}$. Then $X = \bigcup_{k=1}^{m} G_k = \bigcup_{j=1}^{\ell} \mathrm{int} F_j$, and the proof is complete. □

5. Mappings of countably I-compact spaces. A function $f : (X,T) \to (Y,M)$ is called *irresolute* (resp., *semi-continuous*) if $f^{-1}(V)$ is a semiopen subset of (X,T) for each semiopen (resp., open) subset V of (Y,M). The function f is called *almost open* if $f^{-1}(\mathrm{cl}(B)) \subseteq \mathrm{cl}(f^{-1}(B))$ for every $B \in M$.

It is well known (see [3, Proposition 2.7(i)]) that the irresolute image of a countably S-closed space is countably S-closed. So the next result follows easily from Theorem 3.2.

Theorem 5.1. *Let f be an irresolute function from a countably I-compact space (X,T) onto an e.d. space (Y,M). Then (Y,M) is countably I-compact.*

Recall that a subset A of (X,T) is called *semiclosed* if $X - A$ is semiopen. The *semiclosure* of a subset A of a space (X,T), written $\mathrm{scl}(A)$, is the intersection of all semiclosed subsets of (X,T) that contain A.

Proposition 5.2 (see [9, Theorem 3.1]). *A function $f : (X,T) \to (Y,M)$ is semi-continuous if and only if $f(\mathrm{scl}(A)) \subseteq \mathrm{cl}(f(A))$ for every $A \subseteq X$.*

Proposition 5.3 (see [7, Corollary 2.3]). *Let (X,T) be e.d. If $A \in \mathrm{SO}(X,T)$ then* $\mathrm{scl}(A) = \mathrm{cl}(A)$

Now, we state our main result of this section.

Theorem 5.4. *Let $f : (X,T) \to (Y,M)$ be a semi-continuous almost open surjection. If (X,T) is countably I-compact then so is (Y,M).*

Proof. First, we show that (Y,M) is countably S-closed. Let $\{S_n : n \in \mathbb{N}\}$ be a countable semiopen cover of the space (Y,M). For each $n \in \mathbb{N}$ we choose $V_n \in M$ such that $V_n \subseteq S_n \subseteq \mathrm{cl}(V_n)$. Note that $\mathrm{cl}(S_n) = \mathrm{cl}(V_n)$ for each $n \in \mathbb{N}$. Since f is semi-continuous, $f^{-1}(V_n) \in \mathrm{SO}(X,T)$ and hence $\mathrm{cl}(f^{-1}(V_n)) \in \mathrm{RC}(X,T)$ for each $n \in \mathbb{N}$.

Note that

$$X = f^{-1}\left(\bigcup_{n\in\mathbb{N}} S_n\right) \subseteq f^{-1}\left(\bigcup_{n\in\mathbb{N}} \mathrm{cl}\,(V_n)\right) = \bigcup_{n\in\mathbb{N}} f^{-1}(\mathrm{cl}\,(V_n)) \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{cl}\,(f^{-1}(V_n)) \quad (5.1)$$

(since f is almost open).

So the family $\{\mathrm{cl}(f^{-1}(V_n)) : n \in \mathbb{N}\}$ is a countable cover of X by regular closed subsets. So there exists $m \in \mathbb{N}$ such that $X = \bigcup_{k=1}^{m} \mathrm{cl}(f^{-1}(V_k))$. It follows that

$$\begin{aligned} Y &= f\left(\bigcup_{k=1}^{m} \mathrm{cl}\,(f^{-1}(V_k))\right) = (\text{by Proposition 5.3})\, f\left(\bigcup_{k=1}^{m} \mathrm{scl}\,(f^{-1}(V_k))\right) \\ &= \bigcup_{k=1}^{m} f(\mathrm{scl}\,(f^{-1}(V_k))) \subseteq (\text{by Proposition 5.2}) \bigcup_{k=1}^{m} \mathrm{cl}\,(f(f^{-1}(V_k))) \\ &= \bigcup_{k=1}^{m} \mathrm{cl}\,(V_k) = \bigcup_{k=1}^{m} \mathrm{cl}\,(S_k). \end{aligned} \quad (5.2)$$

This proves that (Y,M) is countably S-closed.

Next, we prove that (Y,M) is e.d. Let $G,H \in M$ with $G \cap H = \varnothing$. It is enough, by Lemma 2.3, to show that $\mathrm{cl}(G) \cap \mathrm{cl}(H) = \varnothing$. Now, we have

$$\begin{aligned} f^{-1}(\mathrm{cl}(G) \cap \mathrm{cl}(H)) &= f^{-1}(\mathrm{cl}(G)) \cap f^{-1}(\mathrm{cl}(H)) \\ &\subseteq (\text{as } f \text{ is almost open})\, \mathrm{cl}\,(f^{-1}(G)) \cap \mathrm{cl}\,(f^{-1}(H)). \end{aligned} \quad (5.3)$$

But f is semi-continuous, so $f^{-1}(G),\ f^{-1}(H) \in \mathrm{SO}(X,T)$. We choose $U,V \in T$ such that $U \subseteq f^{-1}(G) \subseteq \mathrm{cl}(U)$ while $V \subseteq f^{-1}(H) \subseteq \mathrm{cl}(V)$. We note that $U \cap V = \varnothing$ (as $f^{-1}(G) \cap f^{-1}(H) = \varnothing$) and, since (X,T) is e.d., then (by Lemma 2.3) we have $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \varnothing$. It is clear that $\mathrm{cl}(U) = \mathrm{cl}(f^{-1}(G))$ and $\mathrm{cl}(V) = \mathrm{cl}(f^{-1}(H))$. We conclude that $f^{-1}(\mathrm{cl}(G) \cap \mathrm{cl}(H)) \subseteq \mathrm{cl}(f^{-1}(G)) \cap \mathrm{cl}(f^{-1}(H)) = \varnothing$ and therefore $\mathrm{cl}(G) \cap \mathrm{cl}(H) = \varnothing$, as required. The proof of the theorem is now complete. □

Corollary 5.5. *Let $f : (X,T) \to (Y,M)$ be an open continuous surjection. If (X,T) is countably I-compact then so is the space (Y,M).*

Corollary 5.6. *If a product space $\prod_{\alpha\in\Delta} X_\alpha$ is countably I-compact then (X_α, T_α) is countably I-compact, for each $\alpha \in \Delta$.*

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