AN ASCOLI THEOREM FOR SEQUENTIAL SPACES

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ABSTRACT. Ascoli theorems characterize "precompact" subsets of the set of morphisms between two objects of a category in terms of "equicontinuity" and "pointwise precompactness," with appropriate definitions of precompactness and equicontinuity in the studied category. An Ascoli theorem is presented for sets of continuous functions from a sequential space to a uniform space. In our development we make extensive use of the natural function space structure for sequential spaces induced by continuous convergence and define appropriate concepts of equicontinuity for sequential spaces. We apply our theorem in the context of C^*-algebras.

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1. Introduction. Ascoli theorems characterize "precompact" subsets of the set of morphisms between two objects of a category in terms of "equicontinuity" and "pointwise precompactness," with appropriate definitions of precompactness and equicontinuity in the studied category. Such general theorems are inspired by the classical Ascoli theorem, proved by G. Ascoli (and independently by C. Arzelà) in the 19th century (see [3, 4]). It characterizes compactness of sets of continuous real-valued functions on the interval [0, 1] with respect to the topology of uniform convergence. Since then, many related theorems have been proved, for example, characterizing compactness of sets of continuous functions from a topological to a uniform space (see [6]), of uniformly continuous functions from a merotopic to a uniform space (see [5]) of continuous functions between topological spaces (see [14, 17]). As was pointed out by Wyler in [24], it is clear that the setting for Ascoli theorems requires natural function space structures; the existence of nice function spaces is guaranteed by Cartesian closedness of the considered topological construct. Around 1980, Dubuc [8] and Gray [12] both proposed a general theory for Ascoli theorems in a categorical setting, but as neither of them seems to be entirely satisfactory, Wyler suggested that more examples should be constructed in order to guide the general theory. Wyler himself developed new examples of Ascoli theorems for sets of continuous functions between limit spaces and of uniformly continuous functions from a uniform convergence space to a pseudo-uniform space (see [24]).

In this paper, we present another setting for an Ascoli theorem: we choose the construct L of sequential spaces. Sequential spaces were already introduced at the beginning of the century by Fréchet (see [9, 10]) and Urysohn (see [2, 23]), even before topological spaces were axiomatized. Since then they have extensively been used as a tool in topology and analysis. Since the 60s sequential structures have been investigated from a categorical point of view (for categorical background, we refer to [1]);

in particular, it was shown that the construct L is Cartesian closed. The natural function spaces available in the setting of L are extensively used in our development of the theory. We define appropriate concepts of equicontinuity and even continuity in L, and study the relations between these concepts. Further, we investigate relations between the different sequential function space structures (pointwise convergence, continuous convergence, and uniform convergence) and look at the induced structures on equicontinuous and evenly continuous sets. In this way, we obtain two versions of an Ascoli theorem for sets of continuous functions from an L-space to a uniform space. Finally, we apply our theorem in an example in the context of $\mathscr{C}^*$-algebras.

The set $\mathbb{N}$ is the set of nonnegative integers and MON_s the set of all strictly increasing mappings from $\mathbb{N}$ to $\mathbb{N}$. If ξ is a sequence in a set X, we often write ξ_n for $\xi(n)$, and the sequence itself is denoted by $\langle \xi_n \rangle$. The Fréchet-filter of ξ on X is denoted by $\mathscr{F}(\xi)$, that is, the filter generated by the sets $\{\xi_n;\ n \geq m\}$ with $m \in \mathbb{N}$. A sequence of ξ is always of the form $\xi \circ s$, with $s \in \mathrm{MON}_s$. If $x \in X$, $\dot{x}$ is the constant sequence x in X.

A convergence on the set X is a set $\mathscr{L} \subset X^{\mathbb{N}} \times X$ satisfying

$$\{(\dot{x}, x);\ x \in X\} \subset \mathscr{L},$$
$$(\xi, x) \in \mathscr{L} \implies \forall s \in \mathrm{MON}_s : (\xi \circ s, x) \in \mathscr{L}. \tag{1.1}$$

Then $(X, \mathscr{L})$ is called an L-space or sequential (convergence) space. As usual, such a space will often be denoted by its underlying set only. If $(X, \mathscr{L})$ is an L-space, $\xi \in X^{\mathbb{N}}$ and $x \in X$, $\xi \xrightarrow{(X, \mathscr{L})} x$ (or $\xi \xrightarrow{X} x$, $\xi \xrightarrow{\mathscr{L}} x$ or simply $\xi \to x$) means $(\xi, x) \in \mathscr{L}$; we say that ξ $\mathscr{L}$-converges to x (or simply X-converges to x or converges to x) and that x is an $\mathscr{L}$-limit point of ξ. If X and Y are L-spaces, a function $f : X \to Y$ is called continuous in $x \in X$ if

$$\xi \xrightarrow{X} x \implies f \circ \xi \xrightarrow{Y} f(x), \tag{1.2}$$

and continuous if it is continuous at each point of X. The set of all continuous functions from X to Y is denoted by $C(X, Y)$; it is a subset of $F(X, Y)$, the set of all functions from the set X to the set Y. The construct of all L-spaces and continuous maps as morphisms is denoted by L. It is a well-fibred topological construct. A source

$$(f_i : (X, \mathscr{L}) \longrightarrow (X_i, \mathscr{L}_i))_{i \in I} \tag{1.3}$$

in L is initial if and only if

$$\xi \xrightarrow{\mathscr{L}} x \iff \forall i \in I : f_i \circ \xi \xrightarrow{\mathscr{L}_i} f_i(x). \tag{1.4}$$

Often, we will work in L^*, the bireflective subconstruct of L with as objects all L-spaces $(X, \mathscr{L})$ satisfying the Urysohn-axiom:

$$\forall \xi \in X^{\mathbb{N}}, \forall x \in X : \left(\forall s \in \mathrm{MON}_s, \exists t \in \mathrm{MON}_s, \xi \circ s \circ t \xrightarrow{X} x \right) \implies \xi \xrightarrow{X} x \tag{1.5}$$

($\mathscr{L}$ is then called an L^*-structure on X). Objects of L^* are called L^*-spaces or Urysohn sequential (convergence) spaces. For example, if $(X, \mathscr{U})$ is a uniform space, with $\mathscr{B}$ a base for $\mathscr{U}$, an L^*-structure on X is defined by

$$\xi \longrightarrow x \iff \forall B \in \mathscr{B},\ \exists k \in \mathbb{N}, \forall n \geq k, (x, \xi_n) \in B \tag{1.6}$$

(it is the convergence of sequences in the topology on X induced by $\mathcal{U}$, and it is independent of the choice of the base $\mathcal{B}$). In the following, if a uniform space $(X, \mathcal{U})$ is considered as an L^*-space, X will always be endowed with the above-mentioned L^*-structure.

If $(X, \mathcal{L})$ is an L-space, a pretopological structure $P(\mathcal{L})$ on X is defined (see [15]) by the closure-operator

$$\mathcal{P}(X) \longrightarrow \mathcal{P}(X) : A \longrightarrow \{x \in X; \, \exists \xi \in A^{\mathbb{N}}, \, \xi \longrightarrow x\}. \qquad (1.7)$$

For a subset A of an L-space $(X, \mathcal{L})$, $\mathrm{cl}_{\mathcal{L}} A$ (or simply $\mathrm{cl}\, A$) always means the closure of A in the pretopological space $(X, P(\mathcal{L}))$ and A is called closed in $(X, \mathcal{L})$ if A is closed in $(X, P(\mathcal{L}))$. For $x \in X, \mathcal{V}_{\mathcal{L}}(x)$ (or $\mathcal{V}(x)$) is the neighborhood filter of x in $(X, P(\mathcal{L}))$. A subset D of an L-space $(X, \mathcal{L})$ is called dense if it is dense in $(X, P(\mathcal{L}))$, that is, if each point in X is a limit point of a sequence in D. An L-space $(X, \mathcal{L})$ is called separable if there is a countable dense subset in $(X, \mathcal{L})$, that is, if $(X, P(\mathcal{L}))$ is separable. A neighborhood covering system (or shortly ncs) of an L-space $(X, \mathcal{L})$ is a neighborhood covering system of the pretopological space $(X, P(\mathcal{L}))$, that is, a set σ of subsets of X that contain a neighborhood of each point of X.

A bornology (see [13]) of a set X is a subset α of $\mathcal{P}(X)$ with

  (i) $A \in \alpha, A' \subset A \Rightarrow A' \in \alpha$,
 (ii) finite unions of sets in α are in α,
(iii) all finite subsets of X are in α.

A set with a bornology is called a bornological set. Bornological sets are objects of a topological construct Born. A morphism $f : (X, \alpha) \to (X, \beta)$ in Born is a mapping $f : X \to Y$ with $f(\alpha) \subset \beta$.

2. Compactness and precompactness for L-spaces.

An L-space X is called compact if each sequence in X has a convergent subsequence. If A is a subset of X, then A is called compact if the subspace A of X is a compact L-space, that is, if each sequence in A has a subsequence that converges in X to a point of A.

PROPOSITION 2.1. *If X is an L-space, ξ a sequence in X that converges to $x \in X$, then $\xi(\mathbb{N}) \cup \{x\}$ is compact in X.*

We now define the concept of precompactness in L which, according to Wyler (see [24]), should be a functor $L \to$ Born preserving the underlying sets and mappings.

DEFINITION 2.2. A subset A of an L-space is called precompact if each sequence in A has a subsequence that converges to a point of X.

Evidently, compact subsets of an L-space are precompact. It is easily seen that a subset of a precompact subset of an L-space still is precompact, that all finite subsets of an L-space are precompact and that the union of a finite number of precompact subsets again is precompact. Also we have that, if $f : X \to Y$ is a continuous function between L-spaces, the image $f(A)$ by f of a precompact subset A of X is precompact in Y. Thus precompact subsets define a functor $\mathrm{Pr} : L \to$ Born, which preserves underlying sets and mappings. We can use Example 5.3 in [22] to show that Pr does not preserve products.

3. Uniformizable, regular, and R_0 spaces. The following definitions are inspired by the concepts of R_0-limit spaces in [21] and uniformizable limit spaces in [24].

DEFINITION 3.1. An L-space X is called an R_0-space if it satisfies the condition

$$\xi \xrightarrow{X} x, \qquad \dot{x} \xrightarrow{X} y \Rightarrow \xi \xrightarrow{X} y, \tag{3.1}$$

and it is called uniformizable if it satisfies

$$\xi \xrightarrow{X} x, \quad \eta \xrightarrow{X} x, \quad \eta \xrightarrow{X} y \Rightarrow \xi \xrightarrow{X} y. \tag{3.2}$$

Uniformizable spaces clearly are R_0-spaces. The full subcategories of L with as objects all R_0-spaces (resp., all uniformizable spaces) are bireflective in L.

In [11] a notion of regularity for L-spaces is defined. We formulate this definition here in the context of L^*-spaces.

DEFINITION 3.2. Let X be an L^*-space. Take $\xi \in X^{\mathbb{N}}$, $x \in X$, and $\langle \Xi_n \rangle \in (X^{\mathbb{N}})^{\mathbb{N}}$. We say that $\langle \Xi_n \rangle$ links ξ and x if for each $k \in \mathbb{N}$, the sequence Ξ_k is X-converging to ξ_k and for each $f \in \mathbb{N}$, the sequence $\langle \Xi_n(f(n)) \rangle$ is X-converging to x; in this case ξ and x are said to be linked.

DEFINITION 3.3. An L^*-space X is called regular if ξ is X-converging to x if and only if ξ and x are linked, for all $\xi \in X^{\mathbb{N}}$ and $x \in X$.

4. Function spaces in L. We first introduce some L-structures on function spaces.

(1) If X is a set and Y is an L-space, the L-structure π of pointwise convergence on $F(X,Y)$ is the product structure in L on $F(X,Y)$, that is, for a sequence $\langle f_n \rangle$ in $F(X,Y)$ and $f \in F(X,Y)$, we have

$$\langle f_n \rangle \xrightarrow{\pi} f \Longleftrightarrow \forall x \in X : \langle f_n(x) \rangle \xrightarrow{\pi} f(x). \tag{4.1}$$

If Y is an L^*-space, then so is $(F(X,Y),\pi)$.

(2) If X and Y are L-spaces, the L-structure Γ of continuous convergence on $C(X,Y)$ is defined by

$$\begin{aligned}
\langle f_n \rangle \xrightarrow{\Gamma} f \Longleftrightarrow &\forall x \in X, \; \forall \xi \in X^{\mathbb{N}}, \\
&\forall s \in \mathrm{MON}_s : \left(\xi \xrightarrow{X} x \Rightarrow \langle f_{s(n)}(\xi_n) \rangle \xrightarrow{Y} f(x) \right).
\end{aligned} \tag{4.2}$$

If X is an L^*-space, then

$$\langle f_n \rangle \xrightarrow{\Gamma} f \Longleftrightarrow \forall x \in X, \; \forall \xi \in X^{\mathbb{N}} : \left(\xi \xrightarrow{X} x \Rightarrow \langle f_s(\xi_n) \rangle \xrightarrow{Y} f(x) \right). \tag{4.3}$$

Again, if Y is an L^*-space, then so is $(C(X,Y),\Gamma)$.

(3) If X is a set and $(Y,\mathcal{U})$ is a uniform space with $\mathcal{B}$ a base for $\mathcal{U}$, and if $A \subset X$, the sets

$$\{(f,g) \in F(X,Y) \times F(X,Y); \forall x \in A : (f(x),g(x)) \in B\} \tag{4.4}$$

for $B \in \mathcal{B}$ form a base for a uniformity on $F(X,Y)$, which induces an L^*-structure on $F(X,Y)$. It does not depend on the choice of the base $\mathcal{B}$. We call it the sequential

structure of uniform convergence on A, and denote it by $s_{\mathcal{U},A}^{X,Y}$, or simply $s_{\mathcal{U},A}$. For a sequence $\langle f_n \rangle$ in $F(X,Y)$ and $f \in F(X,Y)$, we have

$$\langle f_n \rangle \xrightarrow{s_{\mathcal{U},A}} f \iff \forall B \in \mathcal{B}, \; \exists k \in \mathbb{N}, \; \forall n \geq k, \; \forall x \in A, \; (f(x), f_n(x)) \in B. \qquad (4.5)$$

REMARK 4.1. Note that

$$A \subset A' \subset X \implies s_{\mathcal{U},A'} \subset s_{\mathcal{U},A}. \qquad (4.6)$$

If $A = X$, we write $s_{\mathcal{U}}$ instead of $s_{\mathcal{U},A}$, and call $s_{\mathcal{U}}$ the sequential structure of uniform convergence. If now $\sigma \subset \mathcal{P}(X)$, we define $s_\sigma^{X,Y}$ (or s_σ) as the supremum in L^* of all the structures $s_{\mathcal{U},A}$ on $F(X,Y)$, with $A \in \sigma$; s_σ is called the structure of uniform convergence on the sets of σ. For a sequence $\langle f_n \rangle$ in $F(X,Y)$ and $f \in F(X,Y)$, we have

$$\langle f_n \rangle \xrightarrow{s_\sigma} f \iff \forall A \in \sigma : \langle f_n \rangle \xrightarrow{s_{\mathcal{U},A}} f. \qquad (4.7)$$

If for $A \subset X$, r_A is the restriction map

$$F(X,Y) \longrightarrow F(A,Y) : f \longrightarrow f|_A, \qquad (4.8)$$

it is easily seen that

$$\left(r_A : \left(F(X,Y), s_\sigma^{X,Y} \right) \longrightarrow \left(F(A,Y), s_{\mathcal{U}}^{A,Y} \right) \right)_{A \in \sigma} \qquad (4.9)$$

is an initial source in L^*. Even more, each map

$$r_A : \left(F(X,Y), s_{\mathcal{U},A}^{X,Y} \right) \longrightarrow \left(F(A,Y), s_{\mathcal{U}}^{A,Y} \right) \qquad (4.10)$$

is initial. Finally, remark that, if $\sigma = \{A\}$ with $A \subset X$, then $s_\sigma = s_{\mathcal{U},A}$, and that, if $\sigma = \{\{x\}; \; x \in X\}$, then $s_\sigma = \pi$.

For the above-defined function convergences π, Γ, and so forth, we use the same symbols for their induced convergences on subsets of their definition sets, that is, if X and Y both carry L-structures, we use π for the subspace structure of the sequential structure of pointwise convergence on the subset $C(X,Y)$ of $F(X,Y)$.

We recall that L and L^* are Cartesian closed topological constructs and that, for L-spaces X and Y, $(C(X,Y),\Gamma)$ is the corresponding power-object (see [1]): $f : X \times Z \to Y$ is continuous if and only if $f^* : Z \to (C(X,Y),\Gamma)$ is continuous where $f^*(z)(x) = f(x,z)$.

We now investigate limits of sequences of continuous functions in the sequential structures of uniform convergence.

PROPOSITION 4.2. *Let X be an L-space, $(Y, \mathcal{U})$ a uniform space, $x \in X$ and $A \in \mathcal{V}(x)$. If $\langle f_n \rangle$ is a sequence in $F(X,Y)$ of continuous functions in x, $f \in F(X,Y)$, and $\langle f_n \rangle \xrightarrow{s_{\mathcal{U},A}} f$, then f is also continuous in x.*

COROLLARY 4.3. *Let X be an L-space and let $(Y, \mathcal{U})$ be a uniform space. If σ is an ncs of X, then $C(X,Y)$ is closed in $(F(X,Y), s_\sigma)$.*

In the following theorem we discuss some relations between the different function space structures on $F(X,Y)$.

Theorem 4.4. (a) *If X and Y are L-spaces, $\Gamma \subset \pi$ on $C(X,Y)$.*

(b) *If X is a set and $(Y,\mathcal{U})$ is a uniform space, $s_\sigma \subset \pi$ on $F(X,Y)$, for all covers σ of X.*

(c) *If X is a set and $(Y,\mathcal{U})$ is a uniform space, $s_\mathcal{U} \subset s_\sigma$ on $F(X,Y)$, for all $\sigma \subset \mathcal{P}(X)$.*

(d) *If X is a set and $(Y,\mathcal{U})$ is a uniform space, $s_\mathcal{U} \subset \pi$ on $F(X,Y)$.*

(e) *If X is an L-space and $(Y,\mathcal{U})$ is a uniform space, $s_\mathcal{U} \subset \Gamma$ on $C(X,Y)$.*

(f) *If X is an L-space and $(Y,\mathcal{U})$ is a uniform space, $s_\sigma \subset \Gamma$ on $C(X,Y)$, for all neighborhood covering systems σ of X.*

(g) *If X is a compact L-space and $(Y,\mathcal{U})$ is a uniform space, $s_\mathcal{U} = \Gamma$ on $C(X,Y)$.*

(h) *If X is an L-space and $(Y,\mathcal{U})$ is a uniform space, $s_\sigma = \Gamma$ on $C(X,Y)$, if σ is the collection of all compact subsets of X.*

5. Even continuity. In [14], Kelley defined even continuous mappings between topological spaces, and Poppe [19] generalized this notion to sets of continuous mappings between limit spaces. The following is straightforward transcription of these definitions to L-spaces.

Definition 5.1. If X and Y are L-spaces, a subset of $C(X,Y)$ is called evenly continuous if for all $x \in X$, $y \in Y$, $\xi \in X^\mathbb{N}$, and $\langle f_n \rangle \in H^\mathbb{N}$, the conditions $\xi \xrightarrow{X} x$ and $\langle f_n(x) \rangle \xrightarrow{Y} y$ imply $\langle f_n(\xi_n) \rangle \xrightarrow{Y} y$.

It is easily seen that a subset of an evenly continuous set in $C(X,Y)$ is still evenly continuous and that a finite union of evenly continuous subsets is still evenly continuous. This does not mean that the evenly continuous subsets of $C(X,Y)$ always form a bornology on $C(X,Y)$; the next proposition clarifies when they do.

Proposition 5.2. *The following statements are equivalent:*

(a) *The L-space Y is an R_0-space.*

(b) *For all L-spaces X the evenly continuous subsets of $C(X,Y)$ form a bornology on $C(X,Y)$.*

Proposition 5.3. *For a fixed R_0-space Y, even continuity defines a functor $EC_Y : L^{\mathrm{op}} \to \mathrm{Born}$, with $EC_Y(X) = (C(X,Y),\{\text{evenly continuous subsets}\})$.*

6. Equicontinuity. In this section, X is an L-space and $(Y,\mathcal{U})$ is a uniform space.

Definition 6.1. A subset H of $C(X,Y)$ is called equicontinuous at a point x of X if for all $U \in \mathcal{U}$ and all sequences ξ X-converging to x,

$$\exists k \in \mathbb{N}, \ \forall n \geq k, \ \forall f \in H : (f(x),f(\xi_n)) \in U. \tag{6.1}$$

The subset H is called equicontinuous in $A \subset X$ if H is equicontinuous at each point x of A, and H is called equicontinuous if H is equicontinuous in X.

Proposition 6.2. *The subsets of $C(X,Y)$ that are equicontinuous in x form a bornology on $C(X,Y)$.*

Proposition 6.3. *An equicontinuous subset of $C(X,Y)$ is evenly continuous.*

REMARK 6.4. To see that an evenly subset of $C(X,Y)$ need not be equicontinuous, consider the following example, based on an example of Poppe in [20]. Let X be the L^*-space with the underlying set $\mathbb{Q} \cap [0,1]$ and with the sequential structure induced by the usual metric on $\mathbb{Q}$. Further, let Y be the uniform subspace of $\mathbb{R}$ with as underlying set all strictly positive rational numbers. Then take an irrational number $i_0 \in [0,1]$ and a strictly decreasing sequence $\langle r_n \rangle$ of rationals in $[0,1]$, converging in $\mathbb{R}$ to i_0. Now for all $n \in \mathbb{N}$ define a continuous function

$$f_n : X \longrightarrow Y : x \longrightarrow r_n + nx. \tag{6.2}$$

Finally put $H = \{f_n;\ n \in \mathbb{N}\}$.

We first show that H is not equicontinuous at 0. If ξ is the sequence in X with $\xi_n = 1/n$ for $n \in \mathbb{N}_0$ and $\xi_0 = 0$, then $\xi \to x$. For $n \in \mathbb{N}_0$ we now have

$$\left| f_n(\xi_n) - f_n(0) \right| = 1. \tag{6.3}$$

To prove that H is evenly continuous, take $x \in X$, $y \in Y$, $\xi \in X^{\mathbb{N}}$, and a sequence $\langle f_{s(n)} \rangle$ in H (s is a function $\mathbb{N} \to \mathbb{N}$) with $\xi \overset{X}{\to} x$ and $\langle f_{s(n)}(x) \rangle \overset{Y}{\to} y$. If $x = 0$, the condition $\langle r_{s(n)} \rangle \overset{Y}{\to} y$ clearly implies that the set $\{r_{s(n)};\ n \in \mathbb{N}\}$ is finite, and so that also $\{f_{s(n)};\ n \in \mathbb{N}\}$ is finite. If $x > 0$, we have for all n such that $s(n) \geq (y + 1 - i_0)/x$

$$f_{s(n)}(x) = r_{s(n)} + s(n)x > i_0 + y + 1 - i_0 = y + 1, \tag{6.4}$$

and so the set $\{f_{s(n)};\ n \in \mathbb{N}\}$ is finite, otherwise the sequence $\langle f_{s(n)}(x) \rangle$ has an infinite number of terms greater than $y + 1$, which contradicts the convergence $\langle f_{s(n)}(x) \rangle \overset{Y}{\to} y$.

So in each case the set $\{f_{s(n)};\ n \in \mathbb{N}\}$ is finite, and so evenly continuous. This means that $\langle f_{s(n)}(\xi_n) \rangle \overset{Y}{\to} y$. So we have proved that H is evenly continuous.

Note that

$$H(0) = \{r_n;\ n \in \mathbb{N}\} \tag{6.5}$$

is not precompact in Y. This is not surprising, for the following proposition shows that H would be equicontinuous in 0 if $H(0)$ were to be precompact in Y, since H is evenly continuous.

PROPOSITION 6.5. *Let $H \subset C(X,Y)$ and $x \in X$. If H is evenly continuous and $H(x)$ is precompact in Y, then H is equicontinuous in x.*

PROOF. Suppose H is not equicontinuous in x. Then there exists $U \in \mathcal{U}$ and a sequence ξ with $\xi \overset{X}{\to} x$ such that

$$\forall k \in \mathbb{N},\ \exists n \geq k,\ \exists f \in H : (f(x), f(\xi_n)) \notin U. \tag{6.6}$$

This gives a subsequence $\xi \circ s\,(s \in \mathrm{MON}_s)$ and a sequence $\langle f_n \rangle$ in H with

$$\forall n \in \mathbb{N} : (f_n(x), f_n(\xi_{s(n)})) \notin U. \tag{6.7}$$

Because $H(x)$ is precompact there would exist a subsequence $\langle f_{t(n)} \rangle$ of $\langle f_n \rangle$ ($t \in \mathrm{MON}_s$) such that $\langle f_{t(n)}(x) \rangle$ converges in Y to a point y. Now H is evenly continuous and so

$$\langle f_{t(n)}(\xi_{s(t(n))}) \rangle \overset{Y}{\longrightarrow} y \tag{6.8}$$

because $\xi \circ s \circ t \xrightarrow{X} x$. By the last two convergences we can find $k \in \mathbb{N}$ with

$$(f_{t(k)}(x), f_{t(k)}(\xi_{s(t(k))})) \in U \tag{6.9}$$

yielding a contradiction. $\qquad\square$

The following proposition reduces equicontinuity of a set of functions to an ordinary continuity of a suitable function, as one can do in the context of topological spaces [14]. First we need a few notations and remarks.

For $x \in X$, $\hat{x}$ is the evaluation map in x

$$\hat{x} : F(X, Y) \longrightarrow Y : f \longrightarrow f(x). \tag{6.10}$$

If H is endowed with the structure π of pointwise convergence, it is easily seen that all functions $\hat{x}|_H : H \to Y$ are continuous. So we have a function

$$\lambda_H : X \longrightarrow C(H, Y) : x \longrightarrow \hat{x}|_H. \tag{6.11}$$

PROPOSITION 6.6. *If $H \subset C(X, Y)$ and $x \in X$, the following are equivalent:*
(a) *H is equicontinuous in x,*
(b) *$\lambda_H : X \to (C(H, Y), S_{\mathcal{U}}^{H,Y}) : x \to \hat{x}|_H$ is continuous in x.*

PROOF. First remark that for a sequence ξ in X, we have

$$\lambda_H \circ \xi \xrightarrow{S_{\mathcal{U}}^{H,Y}} \lambda_H(x) \Longleftrightarrow \langle \hat{\xi}_n|_H \rangle \xrightarrow{S_{\mathcal{U}}^{H,Y}} \hat{x}|_H$$

$$\Longleftrightarrow \forall U \in \mathcal{U}, \exists k \in \mathbb{N}, \forall n \geq k, \forall f \in H : (\hat{x}|_H(f), \hat{\xi}_n|_H(f)) \in U$$

$$\Longleftrightarrow \forall U \in \mathcal{U}, \exists k \in \mathbb{N}, \forall n \geq k, \forall f \in H : (f(x), f(\xi_n)) \in U. \tag{6.12}$$

Now λ_H is continuous in x if and only if, for all sequences ξ converging to x in X, we have

$$\lambda_H \circ \xi \xrightarrow{S_{\mathcal{U}}^{H,Y}} \lambda_H(x), \tag{6.13}$$

and with the equivalences above, this is precisely the condition for H to be equicontinuous in x. $\qquad\square$

7. Convergence on evenly continuous and equicontinuous sets

PROPOSITION 7.1. *Let X and Y be L-spaces. If $\langle f_n \rangle$ is a sequence in $C(X, Y)$ such that $\{f_n; n \in \mathbb{N}\}$ is evenly continuous, and if $f \in C(X, Y)$, then we have*

$$\langle f_n \rangle \xrightarrow{\pi} f \Longleftrightarrow \langle f_n \rangle \xrightarrow{\Gamma} f. \tag{7.1}$$

COROLLARY 7.2. *If X and Y are L-spaces, then the closures in $C(X, Y)$ of an evenly continuous subset of $C(X, Y)$ for the sequential structures π and Γ coincide.*

PROPOSITION 7.3. *Let X and Y be L-spaces with Y a regular L^*-space. If $H \subset C(X, Y)$ is evenly continuous and $f \in F(X, Y)$, then*

$$f \in \mathrm{cl}_\pi H \Longrightarrow f \in C(X, Y). \tag{7.2}$$

PROOF. Take $x \in X$, $\xi \in X^{\mathbb{N}}$ with $\xi \xrightarrow{X} x$ and let $\langle f_n \rangle \in H^{\mathbb{N}}$ be a sequence in H with $\langle f_n \rangle \xrightarrow{\pi} f$. For all $n \in \mathbb{N}$, we put

$$\Xi_n : \mathbb{N} \longrightarrow Y : k \longrightarrow f_{k+n+1}(\xi_n). \tag{7.3}$$

We now prove that $\langle \Xi \rangle$ links $f \circ \xi$ and $f(x)$ in Y.

For all $n \in \mathbb{N}$, Ξ_n is a subsequence of $\langle f_k(\xi_n) \rangle_k$, and

$$\langle f_k(\xi_n) \rangle_k \xrightarrow{Y} f(\xi_n), \tag{7.4}$$

so we have $\Xi_n \xrightarrow{Y} f(\xi_n)$.

Take a function $F \in \mathbb{N}^{\mathbb{N}}$. Put, for all $n \in \mathbb{N}$, $T(n) = \Xi_n(F(n))$, so that T is a sequence in Y. If $s \in \mathrm{MON}_s$, we inductively define $t : \mathbb{N} \to \mathbb{N}$ as follows:

$$t(0) = 0, \qquad \forall n \geq 1 : t(n) = F\big(s(t(n-1))\big) + s(t(n-1)) + 1. \tag{7.5}$$

Then clearly t also is a strictly increasing function $\mathbb{N} \to \mathbb{N}$. Now $\langle f_{t(n+1)} \rangle$ is a subsequence of $\langle f_n \rangle$ and $\langle f_{t(n+1)} \rangle \xrightarrow{\pi} f$, and

$$\langle f_{t(n+1)}(x) \rangle \xrightarrow{Y} f(x). \tag{7.6}$$

Further, $\xi \circ s \circ t \xrightarrow{X} x$ and so

$$\langle f_{t(n+1)}(\xi_{s(t(n))}) \rangle \xrightarrow{Y} f(x) \tag{7.7}$$

is evenly continuous. Now

$$\begin{aligned}
T \circ s \circ t(n) &= \Xi_{s(t(n))}\big(F(s(t(n)))\big) \\
&= f_{F(s(t(n)))+s(t(n))+1}(\xi_{s(t(n))}) \\
&= f_{t(n+1)}(\xi_{s(t(n))}),
\end{aligned} \tag{7.8}$$

and so $T \circ s \circ t \xrightarrow{Y} f(x)$. This proves $T \xrightarrow{Y} f(x)$. Regularity of Y now gives $f \circ \xi \xrightarrow{Y} f(x)$, and so f is continuous. $\square$

PROPOSITION 7.4. *Let X be a compact L-space, and $(Y, \mathcal{U})$ a uniform space. If $\{f_n;$ $n \in \mathbb{N}\}$ is an evenly continuous subset of $C(X, Y)$, $f \in F(X, Y)$, then*

$$\langle f_n \rangle \xrightarrow{\pi} f \Longleftrightarrow f \in C(X, Y), \qquad \langle f_n \rangle \xrightarrow{s_{\mathcal{U}}} f. \tag{7.9}$$

PROOF. If $\langle f_n \rangle \xrightarrow{\pi} f$, then f is continuous by Proposition 7.3. Further, $\langle f_n \rangle \xrightarrow{\Gamma} f$ by Proposition 7.1 and finally $\langle f_n \rangle \xrightarrow{s_{\mathcal{U}}} f$ by Theorem 4.4. $\square$

COROLLARY 7.5. *If X is a compact L-space, and $(Y, \mathcal{U})$ is a uniform space, then the closures $F(X, Y)$ of an evenly continuous subset of $C(X, Y)$ in the sequential structures π and $s_{\mathcal{U}}$ coincide.*

PROPOSITION 7.6. *Let X be an L-space, $x \in X$, and let $(Y, \mathcal{U})$ be a uniform space. If H is a subset of $C(X, Y)$, then H is equicontinuous in x if and only if $\mathrm{cl}_\pi H$ is equicontinuous in x.*

8. An extension theorem. In this section, X is an L-space and $(Y, \mathcal{U})$ is a uniform space. Recall that $\xi \in Y^{\mathbb{N}}$ is a Cauchy-sequence if and only if

$$\forall U \in \mathcal{U}, \ \exists n_0 \in \mathbb{N} : p, q \geq n_0 \Longrightarrow (\xi_p, \xi_q) \in U. \tag{8.1}$$

PROPOSITION 8.1. *Let $\xi \xrightarrow{X} x$ and let $\langle f_n \rangle$ be a sequence in $C(X,Y)$ with $\{f_n; \ n \in \mathbb{N}\}$ equicontinuous in x. If for all $k \in \mathbb{N}$, $\langle f_n(\xi_k)\rangle_n$ is a Cauchy-sequence in $(Y, \mathcal{U})$, then $\langle f_n(x)\rangle$ also is a Cauchy-sequence in $(Y, \mathcal{U})$.*

PROOF. Take $U \in \mathcal{U}$. Choose a symmetric $V \in \mathcal{U}$ with $V^3 \subset U$. Equicontinuity in x gives a $k_0 \in \mathbb{N}$ with

$$\forall j \geq k_0, \forall n \in \mathbb{N} : (f_n(x), f_n(\xi_j)) \in V. \tag{8.2}$$

Because now $\langle f_n(\xi_{k_0})\rangle_n$ is a Cauchy-sequence, we have $k \in \mathbb{N}$ with

$$\forall m, n \geq k : (f_m(\xi_{k_0})) \in V. \tag{8.3}$$

Then we have for $m, n \geq k$

$$(f_m(x), f_n(x)) \in V^3 \subset U. \tag{8.4}$$

This proves the proposition. $\qquad\square$

COROLLARY 8.2. *Let D be a dense subset of X, $\langle f_n \rangle$ a sequence in $C(X,Y)$ with $H = \{f_n; \ n \in \mathbb{N}\}$ equicontinuous. Suppose that $H(x)$ is precompact in Y for all $x \in X$. If, for all $d \in D$, $\langle f_n(d)\rangle$ is a Cauchy-sequence in Y, then $\langle f_n(x)\rangle$ is a convergent sequence in Y for all $x \in X$.*

THEOREM 8.3. *Suppose D is a dense subset of X, $\langle f_n \rangle$ a sequence in $C(X,Y)$ with $H = \{f_n; \ n \in \mathbb{N}\}$ equicontinuous and $H(x)$ precompact in Y for all $x \in X$. If $f : D \to Y$ is a function with*

$$\forall d \in D : \langle f_n(d)\rangle \longrightarrow f(d), \tag{8.5}$$

then f has a continuous extension $\tilde{f} : X \to Y$ with $\langle f_n \rangle \xrightarrow{\Gamma} \tilde{f}$.

PROOF. This theorem is an immediate consequence of the previous proposition. $\qquad\square$

9. An Ascoli theorem

PROPOSITION 9.1. *Let X and Y be L-spaces with Y an L^*-space, which is R_0. If H is precompact in $(C(X,Y),\Gamma)$, then H is evenly continuous.*

PROOF. Take $x \in X$, $y \in Y$, $\xi \in X^{\mathbb{N}}$, and $\langle f_n \rangle \in H^{\mathbb{N}}$ with $\xi \xrightarrow{X} x$ and $\langle f_n(X)\rangle \xrightarrow{Y} y$. If $s \in \mathrm{MON}_s$, then $\langle f_{s(n)} \rangle$ has a Γ-convergent subsequence $\langle f_{s(t(n))} \rangle \xrightarrow{\Gamma} f$, with $f \in C(X,Y)$ and $t \in \mathrm{MON}_s$. Then, because $\xi \circ s \circ t \xrightarrow{X} x$, we have

$$\langle f_{s(t(n))}(\xi_{s(t(n))}) \rangle \xrightarrow{Y} f(x). \tag{9.1}$$

Furthermore,

$$\langle f_{s(t(n))}(x) \rangle \xrightarrow{Y} y, \qquad \langle f_{s(t(n))}(x) \rangle \xrightarrow{Y} f(x). \tag{9.2}$$

So, because Y is R_0,

$$\langle f_{s(t(n))}(\xi_{s(t(n))}) \rangle \xrightarrow{Y} y. \tag{9.3}$$

This proves

$$\langle f_n(\xi_n) \rangle \xrightarrow{Y} y \tag{9.4}$$

because Y is an L^*-space. $\qquad\square$

THEOREM 9.2. *Let X and Y be L-spaces with Y an L^*-space, which is R_0. If H is precompact in $(C(X,Y),\Gamma)$, then H is evenly continuous and for all $x \in X$, $H(x)$ is precompact in Y.*

PROOF. The proof follows from Proposition 9.1 and the continuity of

$$\hat{x} : (C(X,Y),\Gamma) \longrightarrow Y, \tag{9.5}$$

and $H(x) = \hat{x}(H)$. □

THEOREM 9.3. *Let X be an L-space and $(Y,\mathcal{U})$ a uniform space. If H is precompact in $(C(X,Y),\Gamma)$, then H is equicontinuous and for all $x \in X$, $H(x)$ is precompact in Y. Further, if X is a separable L-space, these two conditions are also sufficient for H to be precompact in $(C(X,Y)\Gamma)$.*

PROOF. The first part of the theorem follows from Theorem 9.2 and Proposition 6.5. For the second part, suppose H is equicontinuous and $H(x)$ is precompact in Y for all $x \in X$, and let $D = \{d_n;\ n \in \mathbb{N}\}$ be a dense subset of X. Take a sequence $\langle f_n \rangle$ in H. Because $\langle f_n(d_0) \rangle$ is a sequence in $H(d_0)$, it has a convergent subsequence $\langle f_{s_0(n)}(d_0) \rangle$ ($s_0 \in \mathrm{MON}_s$); denote the limit point of this subsequence by $f(d_0)$. Now suppose, by induction, that $s_k \in \mathrm{MON}_s$ and

$$\langle f_{s_k(n)}(d_k) \rangle_n \xrightarrow{Y} f(d_k). \tag{9.6}$$

Then $\langle f_{s_k(n)}(d_{k+1}) \rangle$ has a convergent subsequence $\langle f_{s_k \circ s(n)}(d_{k+1}) \rangle$. Put $s_{k+1} = s_k \circ s$ and denote by $f(d_{k+1})$ a limit point of $\langle f_{s_k \circ s(n)}(d_{k+1}) \rangle$. This defines a function $f : D \to Y$. Now for all $k \in \mathbb{N}$ there is a $t \in \mathrm{MON}_s$ such that for all $n \geq k$,

$$f_{s_n(n)}(d_k) = f_{s_k \circ t(n)}(d_k). \tag{9.7}$$

This means that $\langle f_{s_n(n)}(d_k) \rangle_{n \geq k}$ is a subsequence of $\langle f_{s_k(n)}(d_k) \rangle$, and thus

$$\langle f_{s_n(n)}(d_k) \rangle \xrightarrow{Y} f(d_k). \tag{9.8}$$

So $\langle f_{s_n(n)} \rangle$ is a subsequence of $\langle f_n \rangle$ with $\langle f_{s_n(n)}(d) \rangle \xrightarrow{Y} f(d)$ for all $d \in D$. The extension theorem (Theorem 8.3) then gives a continuous extension $\tilde{f} : X \to Y$ of f with $\langle f_{s_n(n)} \rangle \xrightarrow{\Gamma} \tilde{f}$. This proves that H is precompact in $(C(X,Y),\Gamma)$. □

COROLLARY 9.4. *Let X be a compact and separable L-space and $(Y,\mathcal{U})$ a uniform space. Then a subset H of $C(X,Y)$ is precompact in $(C(X,Y),s_{\mathcal{U}})$ if and only if H is equicontinuous and for all $x \in X$, $H(x)$ is precompact in Y.*

PROOF. The proof follows from Theorem 9.3 and Theorem 4.4(g). □

We can easily rewrite Theorem 9.3 in another form. Therefore, if F is a set, X and Y are L-spaces, a map $\Phi : X \times F \to Y$ is called a dual map if each function $\Phi_f : X \to Y : x \to \Phi(x,f)$ for $f \in F$ belongs to $C(X,Y)$. For such a dual map, we put Γ_Φ, the initial L-structure, on F for the map $F \to C(X,Y) : f \to \Phi_f$. If Y is an L^*-space, Γ^*-space, Γ_Φ is an L^*-structure on F. Finally, for $H \subset F$, we set $\Phi_H = \{\Phi_f;\ f \in H\}$.

THEOREM 9.5. *Let F be a set, X an L-space, $(Y,\mathcal{U})$ a uniform space, and $\Phi : X \times F \to Y$ a dual map. If H is precompact in (F,Γ_Φ), then Φ_H is equicontinuous in $C(X,Y)$ and for*

all $x \in X$, $\Phi_H(x)$ is precompact in Y. Further, if X is separable and $\mathrm{cl}_{\Gamma_\Phi}\Phi_H \subset \Phi_F$, then these two conditions are also sufficient for H to be precompact in (F, Γ_Φ).

10. An example. We give an example of an application of our Ascoli theorem in the context of $\mathscr{C}^*$-algebras. Some experience with the fundamental parts of the theory of $\mathscr{C}^*$-algebras is needed and can be found in [7, 18]. The example is based on an application of another Ascoli theorem for "pseudotopological classes" by McKennon in [16].

Let A be a $\mathscr{C}^*$-algebra, which we may and will regard as a subalgebra of its enveloping von Neumann algebra A^ν. Let $Q_A = \{x \in A^\nu;\ \forall a, b \in A : a^*xb \in A\}$ be the set of quasi-multipliers of A and S_A the set of all states of A (i.e., the set of all positive linear functionals on A with norm 1). Each linear functional $\varphi : A \to \mathbb{C}$ has a unique linear extension $\varphi^\nu : A^\nu \to \mathbb{C}$ that is continuous for the weak topology on A^ν, and for $x \in Q_A$ the map $\hat{x} : A' \to \mathbb{C} : \varphi \to \varphi^\nu(x)$ belongs to the bidual of A. On S_A we place the L^*-structure introduced by the weak*-topology on A' relativized to S_A. In this case, we can show that $\Phi : S_A \times Q_A \to \mathbb{C}$ is a dual map. Furthermore, norm-bounded sequences in Q_A converge in Γ_Φ if and only if they converge in the quasi-strict topology on Q_A, that is, the topology on Q_A induced by all semi-norms $Q_A \to \mathbb{C} : x \to \|a^*xa\|$ for $a \in A$. If the C^*-algebra A is separable, then S_A is also separable. And finally, for a norm-bounded subset H of Q_A, the condition $\mathrm{cl}_{\Gamma_\Phi}\Phi_H \subset \Phi_F$ is satisfied. So we can apply Theorem 9.5 and get the following theorem.

THEOREM 10.1. *If A is a separable $\mathscr{C}^*$-algebra, then for each norm-bounded subset H of Q_A the following are equivalent:*

(1) *H is precompact in the sequential structure induced by the quasi-strict topology on Q_A.*

(2) *$\{\hat{x}|_{S_A};\ x \in H\}$ is equicontinuous in $C(S_A, \mathbb{C})$ and for all $\varphi \in S_A$ the set $\{\varphi^\nu(x);\ x \in H\}$ is precompact in $\mathbb{C}$.*

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