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A NOTE ON MINIMAL ENVELOPES OF DOUGLAS ALGEBRAS, MINIMAL SUPPORT SETS, AND RESTRICTED DOUGLAS ALGEBRAS

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Abstract. We characterize the interpolating Blaschke products of finite type in terms of their support sets. We also give a sufficient condition on the restricted Douglas algebra of a support set that is invariant under the Bourgain map, and its minimal envelope is singly generated.

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1. Introduction. Let H^∞ be the Banach algebra of bounded analytic functions on the open unit disk D. We denote by $M(H^\infty)$ the set of nonzero complex valued homomorphism of H^∞. With the weak*-topology, $M(H^\infty)$ is a compact Hausdorff space. We identify a function in H^∞ with the Gelfand transform and consider H^∞ the supremum norm closed subalgebra of the space of continuous functions on $M(H^\infty)$. By Carleson's corona theorem, D is dense in $M(H^\infty)$ in the weak*-topology. For $f \in H^\infty$, put

$$Z(f) = \{x \in M(H^\infty)\backslash D : f(x) = 0\},$$
$$\{|f| < 1\} = \{x \in M(H^\infty)\backslash D : |f(x)| < 1\}. \tag{1.1}$$

For two points x, y in $M(H^\infty)$, the pseudohyperbolic distance is given by

$$\rho(x,y) = \sup\{|f(y)| : f \in H^\infty, \|f\|_\infty \le 1, f(x) = 0\}. \tag{1.2}$$

Then, $0 \le \rho(x,y) \le 1$ and put

$$P(x) = \{m \in M(H^\infty) : \rho(x,m) < 1\}. \tag{1.3}$$

The set $P(x)$ is called the Gleason part containing x. For $z, x \in D$, $\rho(z,w) = |(z-w)/(1-\bar{w}z)|$, and $P(z) = D$. When $P(x) \neq \{x\}$, both x and $P(x)$ are called nontrivial. We denote by G the set of nontrivial points in $M(H^\infty)$.

For an infinite sequence $\{z_n\}_n$ in D with $\sum_{n=1}^{\infty}(1-|z_n|) < \infty$, the corresponding Blaschke product is defined by

$$b(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|}\frac{z-z_n}{1-\bar{z}_n z}, \quad z \in D. \tag{1.4}$$

In addition, we have

$$\inf_n (1-|z_n|^2)|b'(z_n)| > 0, \tag{1.5}$$

both b and $\{z_n\}_n$ are called interpolating. When b is interpolating and

$$\lim_{n\to\infty}(1-|z_n|^2)\,|b'(z_n)|=1, \tag{1.6}$$

both b and $\{z_n\}_n$ are called sparse. An interpolating Blaschke product b is said to be unimodular on trivial points if $\{x : |b(x)| < 1\} \subset G$. In [4], Hoffman proved that for $x \in M(H^\infty)$, $x \in G$ if and only if $x \in Z(b)$ for some interpolating Blaschke product b. He also proved that for a point $x \in G$, there exists a one-to-one continuous onto map $L_x : D \to P(x)$ such that $L_x(0) = x$ and $f \circ L_x \in H^\infty$ for every $f \in H^\infty$. The map L_x, which is called the Hoffman map for the point x, is given by

$$L_x(z)=\lim_{\alpha}\frac{z+z_0}{1+\bar{z}_\alpha z},\quad z\in D, \tag{1.7}$$

where $\{Z_\alpha\}_\alpha$ is a net in D which converges to x. A part $P(x)$ is called sparse if there is a sparse Blaschke product b such that $b(x) = 0$. In this case we have $|(b \circ L_x)'(0)| = 1$. Therefore, b is a sparse Blaschke product if and only if $|(b \circ L_x)'(0)| = 1$ for every $x \in Z(b)$. A part is called locally sparse if there is an interpolating Blaschke product b such that $b(x) = 0$ and $|(b \circ L_x)'(0)| = 1$.

For an interpolating Blaschke product b with zeros $\{z_n\}_n$, let

$$\delta_0(b)=\liminf_{n\to\infty}\min_{k\neq n}\rho(z_n,z_k). \tag{1.8}$$

An interpolating Blaschke product b is called spreading if $\delta_0(b) = 1$. By considering boundary function, we may consider H^∞, as a closed subalgebra of L_∞, the Banach algebra of essentially bounded Lebesgue measurable functions on the unit circle T. It is known that $M(L^\infty) \subset M(H^\infty)$ and $M(L^\infty)$ is the Shilov boundary for H^∞. Any uniformly closed subalgebra B with $H^\infty \subset B \subset L^\infty$ is called a Douglas algebra. For a point $x \in M(H^\infty)$, there exists a probability measure μ_x on $M(L^\infty)$ such that

$$f(x)=\int_{M(L^\infty)} f\,d\mu_x \quad \forall f\in H^\infty. \tag{1.9}$$

We denote by $\operatorname{supp}\mu_x$ the closed support set of μ_x. Since $\operatorname{supp}\mu_x$ is a weak peak set of $M(L^\infty)$ for H^∞, we have $H^\infty_{\operatorname{supp}\mu_x} = \{f \in L^\infty : f_{|\operatorname{supp}\mu_x} \in H^\infty_{|\operatorname{supp}\mu_x}\}$ is a Douglas algebra.

For $E \subset M(H^\infty)$, a point $x \in E$ is called a minimal support point for E if

$$\operatorname{supp}\mu_x\subset\operatorname{supp}\mu_y \quad\text{or}\quad \operatorname{supp}\mu_x\cap\operatorname{supp}\mu_y=\phi \quad \forall y\in E. \tag{1.10}$$

If x is a minimal support point for E, $\operatorname{supp}\mu_x$ is called a minimal support set for E. For an interpolating Blaschke product b, we denote by $m(Z(b))$ the set of minimal support points for the set $\{x : |b(x)| < 1\}$. Let X be a Banach algebra with identity and let B be a closed subalgebra of X. The Bourgain algebra B_b of B relative to X is defined by the set of f in X such that $\|ff_n + B\| \to 0$ for every sequence $\{f_n\}_n$ in B with $f_n \to 0$ weakly. If A and B are Douglas algebras with $A \subseteq B$ and properly contained, then B is a minimal superalgebra of A if and only if $\operatorname{supp}\mu_x = \operatorname{supp}\mu_y$ for every $x, y \in M(A)\backslash M(B)$. We denote by B_m the smallest Douglas algebra which contains all minimal superalgebras of B. We note that $B_b \subset B_m$. An interpolating Blaschke product b such that $\{x : |b(x)| < 1\} \subset G$, with $Z(b) \cap P(x)$ being a finite set for every $x \in Z(b)$, is said to be of finite type.

2. Proofs of the theorems

Theorem 2.1. *An interpolating Blaschke product b that is unimodular on trivial parts is of finite type if and only if $m(Z(b)) = \{z : |b(z)| < 1\}$.*

Proof. Suppose b is an interpolating Blaschke product that is unimodular on the trivial points and of finite type. Let $z \in M(H^\infty + C)$ such that $|b(z)| < 1$. By [1, Theorems 1 and 2], there is an $x \in m(Z(b))$ such that $\operatorname{supp}\mu_x \subset \operatorname{supp}\mu_z$. By [3, Theorem 3.1], the set $\operatorname{supp}\mu_x$ is a maximal support set. Hence $\operatorname{supp}\mu_x = \operatorname{supp}\mu_z$. This implies that z is a minimal support point for b, that is, $z \in m(Z(b))$. So $\{z : |b(z)| < 1\} \subset m(Z(b))$. Since $m(Z(b)) \subset \{z : |b(z)| < 1\}$, we have $\{z : |b(z)| < 1\} = m(Z(b))$. Conversely, suppose $m(Z(b)) = \{z : |b(z)| < 1\}$ and assume that b is unimodular on trivial points but not of finite type. Then there is a $y \in Z(b)$ such that the set $Z(b) \cap P(y)$ is an infinite set. By [2, Theorems 1 and 2], there is an $x \in M(H^\infty + C)$ such that $|b(x)| < 1$, an uncountable index set I such that for $\alpha, \beta \in I$, $\alpha \neq \beta$, $\operatorname{supp}\mu_{x_\alpha} \cap \operatorname{supp}\mu_{x_\beta} = \phi$, $x_\alpha, x_\beta \in m(Z(b))$, and $\operatorname{supp}\mu_{x_\alpha} \subset \operatorname{supp}\mu_x$ for all $\alpha \in I$. Since $\operatorname{supp}\mu_{x_\alpha}$ is properly contained in $\operatorname{supp}\mu_x$, this implies that $x \notin m(Z(b))$ but $|b(x)| < 1$. This contradicts our assumption that $\{z : |b(z)| < 1\} = m(Z(b))$. Thus, b is of finite type. □

Theorem 2.2. *Suppose that b is a spreading nonsparse Blaschke product, and $x \in m(Z(b))$ such that $|(b \circ L_x)'(0)| \neq 1$. Then*

(i) $(H^\infty_{\operatorname{supp}\mu_x})_b = H_{\operatorname{supp}\mu_x}$,

(ii) $(H^\infty_{\operatorname{supp}\mu_x})_m = H^\infty_{\operatorname{supp}\mu_x}[\bar{b}]$.

Proof. By [5, Lemma 2.1], we have that P_x is a nonlocally sparse part. Hence, by [6, Theorem 5] we have that (i) holds.

Since b is spreading and $x \in m(Z(b))$,

$$M\left(H^\infty_{\operatorname{supp}\mu_x}\right) = M\left(H^\infty_{\operatorname{supp}\mu_x}[\bar{b}]\right) \cup E_x, \tag{2.1}$$

where $E_x = \{y \in M(H^\infty + C) : \operatorname{supp}\mu_x = \operatorname{supp}\mu_y\}$. This implies that $H^\infty_{\operatorname{supp}\mu_x}$ is properly contained in $(H^\infty_{\operatorname{supp}\mu_x})_m$. Since $H^\infty_{\operatorname{supp}\mu_x}$ is a maximal subalgebra of $H^\infty_{\operatorname{supp}\mu_x}[\bar{b}]$, $H^\infty_{\operatorname{supp}\mu_x}[\bar{b}]$ is contained in $(H^\infty_{\operatorname{supp}\mu_x})_m$. Since

$$M\left(H^\infty_{\operatorname{supp}\mu_x}\right) = M(L^\infty) \cup \{y \in M(H^\infty + C) : \operatorname{supp}\mu_y \subseteq \operatorname{supp}\mu_x\}, \tag{2.2}$$

we show that if q is an interpolating Blaschke product such that $\bar{q} \in (H^\infty_{\operatorname{supp}\mu_x})_m$, then $H^\infty_{\operatorname{supp}\mu_x}[\bar{q}] = H^\infty_{\operatorname{supp}\mu_x}[\bar{b}]$. This proves (ii). Suppose that we have $H^\infty_{\operatorname{supp}\mu_x}[\bar{b}]$ properly contained in $(H^\infty_{\operatorname{supp}\mu_x})_m$, then we have $M((H^\infty_{\operatorname{supp}\mu_x})_m)$ properly contained in $M(H^\infty_{\operatorname{supp}\mu_x}[\bar{b}])$. So there is a $y \in M(H^\infty_{\operatorname{supp}\mu_x}[\bar{b}])$, an interpolating Blaschke product q with $\bar{q} \in (H^\infty_{\operatorname{supp}\mu_x})_m$ and $q(y) = 0$. By (2.2) we have $y \in M(H^\infty_{\operatorname{supp}\mu_x})$ but $y \notin E_x$. Again, by (2.2), this implies that $\operatorname{supp}\mu_y$ is properly contained in the $\operatorname{supp}\mu_x$. By [2, Theorems 1 and 2], there is an uncountable index set I such that if $\alpha, \beta \in I$, $\alpha \neq \beta$, there are $x_\alpha, x_\beta \in Z(q)$ with $\operatorname{supp}\mu_\alpha \cap \operatorname{supp}\mu_{x_\beta} = \phi$ and $\operatorname{supp}\mu_\alpha$, $\operatorname{supp}\mu_{x_\beta}$ are both properly contained in $\operatorname{supp}\mu_x$. This implies that

$$\cup_{\alpha\in I} E_{x_\alpha} \subset \left\{m \in M\left(H^\infty_{\operatorname{supp}\mu_x}\right) : |q(m)| < 1\right\}. \tag{2.3}$$

But this contradicts [2, Theorem 3] since $\alpha \neq \beta$ implies that $E_{x_\alpha} \cap E_{x_\beta} = \phi$. Thus, no such y exists and we have $H^\infty_{\operatorname{supp}\mu_x}[\bar{b}] = H^\infty_{\operatorname{supp}\mu_x}[\bar{q}]$. So (ii) holds. □

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