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MIXED PROBLEM WITH NONLOCAL BOUNDARY CONDITIONS FOR A THIRD-ORDER PARTIAL DIFFERENTIAL EQUATION OF MIXED TYPE

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Abstract. We study a mixed problem with integral boundary conditions for a third-order partial differential equation of mixed type. We prove the existence and uniqueness of the solution. The proof is based on two-sided a priori estimates and on the density of the range of the operator generated by the considered problem.

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1. Introduction. In the rectangle $\Omega = (0,\ell)\times(0,T)$, we consider the equation

$$\mathscr{L}u = \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x\partial t}\right) = f(x,t), \tag{1.1}$$

where $a(x,t)$ is bounded with $0 < a_0 < a(x,t) \le a_1$ and has bounded partial derivatives such that $0 < a_2 \le \partial a(x,t)/\partial t \le a_3$ and $0 < a_4 \le \partial a(x,t)/\partial x \le a_5$ for $(x,t) \in \overline{\Omega}$.

To (1.1) we add the initial conditions

$$l_1 u = u(x,0) = \varphi(x), \quad l_2 u = \frac{\partial u}{\partial t}(x,0) = \psi(x), \quad x \in (0,\ell), \tag{1.2}$$

the Dirichlet condition

$$u(0,t) = 0, \quad t \in (0,T), \tag{1.3}$$

and the integral condition

$$\int_0^\ell u(\xi,t)\,d\xi = 0, \quad t \in (0,T), \tag{1.4}$$

where φ and ψ are known functions which satisfy the compatibility conditions given by (1.3) and (1.4), that is,

$$\varphi(0) = 0, \qquad \int_0^\ell \varphi(x)\,dx = 0, \qquad \psi(0) = 0, \qquad \int_0^\ell \psi(x)\,dx = 0. \tag{1.5}$$

Boundary-value problems for parabolic equations with integral boundary conditions are investigated by Batten [1], Bouziani and Benouar [2], Cannon [3, 4], Perez Esteva and van der Hoeck [5], Ionkin [8], Kamynin [9], Kartynnik [10], Shi [11], Yurchuk [13], and many references therein. We remark that integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics; see for example, [6, 7, 11, 12].

The present paper is devoted to the study of a mixed problem with boundary integral conditions for a third-order partial differential equation of mixed type.

We associate to problem (1.1), (1.2), (1.3), and (1.4) the operator $L = (\mathcal{L}, l_1, l_2)$, defined from E into F, where E is the Banach space of functions $u \in L_2(\Omega)$, satisfying (1.3) and (1.4), with the finite norm

$$\|u\|_E^2 = \int_\Omega (\ell - x)^2 \left[\left| \frac{\partial^2 u}{\partial t^2} \right|^2 + \left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^2 \right] dx\,dt + \sup_{0 \le t \le T} \int_0^\ell (\ell - x)^2 \left[\left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 \right] dx + \sup_{0 \le t \le T} \int_0^\ell \left[\left| \frac{\partial u}{\partial t} \right|^2 + |u|^2 \right] dx, \tag{1.6}$$

and F is the Hilbert space of vector-valued functions $\mathcal{F} = (f, \varphi, \psi)$ obtained by completion of the space $L_2(\Omega) \times W_2^2(0, \ell) \times W_2^2(0, \ell)$ with respect to the norm

$$\|\mathcal{F}\|_F^2 = \|(f, \varphi, \psi)\|_F^2 = \int_\Omega (\ell - x)^2 |f|^2\,dx\,dt + \int_0^\ell (\ell - x)^2 \left[\left| \frac{d\varphi}{dx} \right|^2 + \left| \frac{d\psi}{dx} \right|^2 \right] dx + \int_0^\ell \left[|\varphi|^2 + |\psi|^2 \right] dx. \tag{1.7}$$

Using the energy inequalities method proposed in [13], we establish two-sided a priori estimates. Then, we prove that the operator L is a linear homeomorphism between the spaces E and F.

2. Two-sided a priori estimates

Theorem 2.1. *For any function $u \in E$, there is the a priori estimate*

$$\|Lu\|_F \le c\|u\|_E, \tag{2.1}$$

where the constant c is independent of u.

Proof. Using (1.1) and the initial conditions (1.2), we obtain

$$\begin{gathered} \int_\Omega (\ell - x)^2 |\mathcal{L}u|^2\,dx\,dt \le 3\int_\Omega (\ell - x)^2 \left[\left| \frac{\partial^2 u}{\partial t^2} \right|^2 + a_5^2 \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 + a_1^2 \left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^2 \right] dx\,dt, \\ \int_0^\ell (\ell - x)^2 \left[\left| \frac{d\psi}{dx} \right|^2 + \left| \frac{d\varphi}{dx} \right|^2 \right] dx \le \sup_{0 \le t \le T} \int_0^\ell (\ell - x)^2 \left[\left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 \right] dx, \\ \int_0^\ell \left[|\psi|^2 + |\varphi|^2 \right] dx \le \sup_{0 \le t \le T} \int_0^\ell \left[\left| \frac{\partial u}{\partial t} \right|^2 + |u|^2 \right] dx. \end{gathered} \tag{2.2}$$

Combining the inequalities (2.2), we obtain (2.1) for $u \in E$. □

Theorem 2.2. *For any function $u \in E$, there is the a priori estimate*

$$\|u\|_E \le \alpha \|Lu\|_F, \tag{2.3}$$

with the constant

$$\alpha = \frac{\max(167/10, a_1)}{\min\left(\exp(-cT)/20, \exp(-cT)a_0^2/15\right)}, \tag{2.4}$$

and c is such that

$$c \geq 1, \quad ca_0 - 1 \geq a_3 + 2a_5^2. \tag{2.5}$$

Before proving this theorem, we first give the following two lemmas.

Lemma 2.3. *For $u \in E$ satisfying the first condition in (1.2),*

$$\frac{1}{2}\int_0^\tau \int_0^\ell (\ell - x)^2 \exp(-ct)\left|\frac{\partial^2 u}{\partial x \partial t}\right|^2 dx\,dt + \frac{c-1}{2}\int_0^\tau \int_0^\ell (\ell - x)^2 \exp(-ct)\left|\frac{\partial u}{\partial x}\right|^2 dx\,dt \geq \frac{1}{2}\int_0^\ell (\ell - x)^2 \exp(-c\tau)\left|\frac{\partial u}{\partial x}(x,\tau)\right|^2 dx - \frac{1}{2}\int_0^\ell (\ell - x)^2 \left|\frac{d\varphi}{dx}\right|^2 dx. \tag{2.6}$$

Proof. Starting from

$$\int_0^\tau \int_0^\ell (\ell - x)^2 \exp(-ct)\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \overline{u}}{\partial x}\,dx\,dt, \tag{2.7}$$

then integrating by parts and using elementary inequalities, we obtain (2.6). □

Lemma 2.4. *For $u \in E$ satisfying the initial conditions (1.2),*

$$\int_0^\ell \exp(-c\tau)\left|u(x,\tau)\right|^2 dx \leq \int_0^\tau \int_0^\ell \exp(-ct)\left|\frac{\partial u}{\partial t}\right|^2 dx\,dt + \int_0^\ell |\varphi|^2 dx, \tag{2.8}$$

with $c \geq 1$.

Proof. Integrating by parts the expression

$$\int_0^\tau \int_0^\ell \exp(-ct)u\frac{\partial \overline{u}}{\partial t}\,dx\,dt \tag{2.9}$$

and using elementary inequalities yield (2.8). □

Remark 2.5. We note that Lemmas 2.3 and 2.4 hold for weaker conditions on u.

Proof of Theorem 2.2. First, define

$$D(L) = \left\{u \in E \mid \frac{\partial^5 u}{\partial x^2 \partial t^3} \in L_2(\Omega)\right\}, \qquad Mu = (\ell - x)^2\frac{\partial^2 u}{\partial t^2} + 2(\ell - x)J\frac{\partial^2 u}{\partial t^2}, \tag{2.10}$$

where

$$Ju = \int_0^x u(\xi,t)\,d\xi. \tag{2.11}$$

We consider for $u \in D(L)$ the quadratic formula

$$\operatorname{Re}\int_0^\tau \int_0^\ell \exp(-ct)\mathcal{L}u\overline{Mu}\,dx\,dt, \tag{2.12}$$

with the constant c satisfying (2.5), obtained by multiplying (1.1) by $\exp(-ct)Mu$, by integrating over Ω^τ, where $\Omega^\tau = (0,\ell)\times(0,\tau)$, with $0 \le \tau \le T$, and by taking the real part. Integrating by parts (2.12) with the use of boundary conditions (1.3) and (1.4), we obtain

$$\begin{aligned}
\operatorname{Re}&\int_0^\tau\int_0^\ell \exp(-ct)\mathcal{L}uM\overline{u}\,dx\,dt\\
&=\int_0^\tau\int_0^\ell(\ell-x)^2\exp(-ct)\left|\frac{\partial^2 u}{\partial t^2}\right|^2dx\,dt+\frac{1}{2}\int_0^\tau\int_0^\ell\exp(-ct)\left|J\frac{\partial^2 u}{\partial t^2}\right|^2dx\,dt\\
&\quad+\operatorname{Re}\int_0^\tau\int_0^\ell(\ell-x)^2\exp(-ct)a\frac{\partial^2 u}{\partial x\partial t}\frac{\partial}{\partial t}\left(\frac{\partial^2\overline{u}}{\partial x\partial t}\right)dx\,dt\\
&\quad+2\operatorname{Re}\int_0^\tau\int_0^\ell\exp(-ct)\frac{\partial u}{\partial t}a\frac{\partial^2\overline{u}}{\partial t^2}dx\,dt\\
&\quad+2\operatorname{Re}\int_0^\tau\int_0^\ell\exp(-ct)\frac{\partial a}{\partial x}\frac{\partial u}{\partial t}J\frac{\partial^2\overline{u}}{\partial t^2}dx\,dt.
\end{aligned}\tag{2.13}$$

On the other hand, by using the elementary inequalities we get

$$\begin{aligned}
\operatorname{Re}&\int_0^\tau\int_0^\ell \exp(-ct)\mathcal{L}uM\overline{u}\,dx\,dt\\
&\ge\int_0^\tau\int_0^\ell(\ell-x)^2\exp(-ct)\left|\frac{\partial^2 u}{\partial t^2}\right|^2dx\,dt\\
&\quad+\operatorname{Re}\int_0^\tau\int_0^\ell(\ell-x)^2\exp(-ct)a\frac{\partial^2 u}{\partial x\partial t}\frac{\partial}{\partial t}\left(\frac{\partial^2\overline{u}}{\partial x\partial t}\right)dx\,dt\\
&\quad+2\operatorname{Re}\int_0^\tau\int_0^\ell\exp(-ct)\frac{\partial u}{\partial t}a\frac{\partial^2\overline{u}}{\partial t^2}dx\,dt\\
&\quad-2\operatorname{Re}\int_0^\tau\int_0^\ell\exp(-ct)\left|\frac{\partial a}{\partial x}\right|^2\left|\frac{\partial u}{\partial t}\right|^2dx\,dt.
\end{aligned}\tag{2.14}$$

Again, integrating by parts the second and third terms of the right-hand side of the inequality (2.14) and taking into account the initial conditions (1.2) give

$$\begin{aligned}
\operatorname{Re}&\int_0^\tau\int_0^\ell \exp(-ct)\mathcal{L}uM\overline{u}\,dx\,dt+\int_0^\ell a(x,0)|\psi|^2dx+\frac{1}{2}\int_0^\ell a(x,0)(\ell-x)^2\left|\frac{d\psi}{dx}\right|^2dx\,dt\\
&\ge\int_0^\tau\int_0^\ell\exp(-ct)(\ell-x)^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2dx\,dt-2\int_0^\tau\int_0^\ell\exp(-ct)\left|\frac{\partial a}{\partial x}\right|^2\left|\frac{\partial u}{\partial t}\right|^2dx\,dt\\
&\quad+\frac{1}{2}\int_0^\ell a(x,\tau)\exp(-c\tau)(\ell-x)\left|\frac{\partial^2 u}{\partial x\partial t}(x,\tau)\right|^2dx\\
&\quad-\frac{1}{2}\int_0^\tau\int_0^\ell\exp(-ct)\frac{\partial a}{\partial t}(\ell-x)^2\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2dx\,dt\\
&\quad+\frac{c}{2}\int_0^\tau\int_0^\ell\exp(-ct)(\ell-x)^2a\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2dx\,dt+\int_0^\ell\exp(-c\tau)a(x,\tau)\left|\frac{\partial u}{\partial t}(x,\tau)\right|^2dx\\
&\quad-\int_0^\tau\int_0^\ell\exp(-ct)\frac{\partial a}{\partial t}\left|\frac{\partial u}{\partial t}\right|^2dx\,dt+c\int_0^\tau\int_0^\ell\exp(-ct)a\left|\frac{\partial u}{\partial t}\right|^2dx\,dt.
\end{aligned}\tag{2.15}$$

By using the elementary inequalities on the first integral in the left-hand side of (2.15), we obtain

$$\begin{aligned}
&\frac{33}{2}\int_0^\tau\int_0^\ell \exp(-ct)(\ell-x)^2|\mathcal{L}u|^2\,dx\,dt+\frac{3}{4}\int_0^\tau\int_0^\ell \exp(-ct)(\ell-x)^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dx\,dt\\
&\quad+\int_0^\ell a(x,0)|\psi|^2\,dx+\frac{1}{2}\int_0^\ell a(x,0)(\ell-x)^2\left|\frac{d\psi}{dx}\right|^2 dx\\
&\geq\int_0^\tau\int_0^\ell \exp(-ct)(\ell-x)^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dx\,dt-2\int_0^\tau\int_0^\ell \exp(-ct)\left|\frac{\partial a}{\partial x}\right|^2\left|\frac{\partial u}{\partial t}\right|^2 dx\,dt\\
&\quad+\frac{1}{2}\int_0^\ell \exp(-c\tau)(\ell-x)^2\left|\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right|^2 dx-\frac{1}{2}\int_0^\tau\int_0^\ell \exp(-ct)(\ell-x)^2\frac{\partial a}{\partial t}\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dx\,dt\\
&\quad+\frac{c}{2}\int_0^\tau\int_0^\ell \exp(-ct)(\ell-x)^2 a\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dx\,dt+\int_0^\ell \exp(-c\tau)a(x,\tau)\left|\frac{\partial u(x,\tau)}{\partial t}\right|^2 dx\\
&\quad-\int_0^\tau\int_0^\ell \exp(-ct)\frac{\partial a}{\partial t}\left|\frac{\partial u}{\partial t}\right|^2 dx\,dt+c\int_0^\tau\int_0^\ell \exp(-ct)a\left|\frac{\partial u}{\partial t}\right|^2 dx\,dt.
\end{aligned}\tag{2.16}$$

Now, from (1.1) we have

$$\begin{aligned}
&\frac{1}{5}\int_0^\tau\int_0^\ell \exp(-ct)(\ell-x)^2|\mathcal{L}u|^2\,dx\,dt\\
&\qquad+\frac{1}{5}\int_0^\tau\int_0^\ell \exp(-ct)(\ell-x)^2\left|\frac{\partial a}{\partial x}\right|^2\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dx\,dt\\
&\qquad+\frac{1}{5}\int_0^\tau\int_0^\ell \exp(-ct)(\ell-x)^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dx\,dt\\
&\qquad\geq\frac{1}{15}\int_0^\tau\int_0^\ell \exp(-ct)(\ell-x)^2 a^2\left|\frac{\partial^3 u}{\partial x^2\partial t}\right|^2 dx\,dt.
\end{aligned}\tag{2.17}$$

Combining inequalities (2.16), (2.17), and Lemmas 2.3 and 2.4, we get

$$\begin{aligned}
&\frac{167}{10}\int_\Omega(\ell-x)^2|\mathcal{L}u|^2\,dx\,dt+\frac{a_1}{2}\int_0^\ell(\ell-x)^2\left|\frac{d\psi}{dx}\right|^2 dx\\
&\quad+a_1\int_0^\ell|\psi|^2\,dx+\frac{1}{2}\int_0^\ell(\ell-x)^2\left|\frac{d\varphi}{dx}\right|^2 dx+\int_0^\ell|\varphi|^2\,dx\\
&\geq\exp(-cT)\Bigg(\frac{1}{20}\int_0^\tau\int_0^\ell(\ell-x)^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dx\,dt+\frac{1}{2}\int_0^\ell(\ell-x)^2\left|\frac{\partial^2 u}{\partial x\partial t}(x,\tau)\right|^2 dx\,dt\\
&\qquad+\int_0^\ell|u(x,\tau)|^2\,dx+a_0\int_0^\ell\left|\frac{\partial u}{\partial t}(x,\tau)\right|^2 dx+\frac{1}{2}\int_0^\ell(\ell-x)^2\left|\frac{\partial u}{\partial x}(x,\tau)\right|^2 dx\\
&\qquad+\frac{a_0^2}{15}\int_0^\tau\int_0^\ell(\ell-x)^2\left|\frac{\partial^3 u}{\partial x^2\partial t}\right|^2 dx\,dt\Bigg).
\end{aligned}\tag{2.18}$$

As the left-hand side of (2.18) is independent of τ, by replacing the right-hand side by its upper bound with respect to τ in the interval $[0,T]$, we obtain the desired inequality. □

3. Solvability of the problem. From estimates (2.1) and (2.3) it follows that the operator $L : E \to F$ is continuous and its range is closed in F. Therefore, the inverse operator L^{-1} exists and is continuous from the closed subspace $R(L)$ onto E, which means that L is a homomorphism from E onto $R(L)$. To obtain the uniqueness of solution, it remains to show that $R(L) = F$. The proof is based on the following lemma.

Lemma 3.1. *Suppose that $\partial^3 a/\partial x^2 \partial t$ is also bounded. Let $D_0(L) = \{u \in D(L) : l_1 u = 0, l_2 u = 0\}$. If for $u \in D_0(L)$ and some $\omega \in L_2(\Omega)$,*

$$\int_\Omega (\ell - x)\mathcal{L}u\varpi \, dx\, dt = 0, \tag{3.1}$$

then $\omega = 0$.

Proof. From (3.1) we have

$$\int_\Omega (\ell - x)\frac{\partial^2 u}{\partial t^2}\varpi \, dx\, dt = \int_\Omega (\ell - x)\frac{\partial}{\partial x}\left(a\frac{\partial^2 u}{\partial x \partial t}\right)\varpi \, dx\, dt. \tag{3.2}$$

If we introduce the smoothing operators with respect to t (see [13]) $J_\xi^{-1} = (I + \xi(\partial/\partial t))^{-1}$ and $(J_\xi^{-1})^*$, then these operators provide the solutions of the respective problems

$$\xi \frac{dg_\xi(t)}{dt} + g_\xi(t) = g(t), \quad g_\xi(t)|_{t=0} = 0, \tag{3.3}$$

$$-\xi \frac{dg_\xi^*(t)}{dt} + g_\xi^*(t) = g(t), \quad g_\xi^*(t)|_{t=T} = 0, \tag{3.4}$$

and also have the following properties: for any $g \in L_2(0,T)$, the functions $g_\xi = (J_\xi^{-1})g$ and $g_\xi^* = (J_\xi^{-1})^* g$ are in $W_2^1(0,T)$ such that $g_\xi|_{t=0} = 0$ and $g_\xi^*|_{t=T} = 0$. Moreover, J_ξ^{-1} commutes with $\partial/\partial t$, so $\int_0^T |g_\xi - g|^2\, dt \to 0$ and $\int_0^T |g_\xi^* - g|^2\, dt \to 0$ for $\xi \to 0$.

Now, for given $\omega(x,t)$, we introduce the function

$$v(x,t) = \omega(x,t) - \int_0^x \frac{\omega(\xi,t)}{\ell - \xi}\, d\xi. \tag{3.5}$$

Integrating by parts with respect to ξ, we obtain

$$\begin{aligned}\int_0^x v(\xi,t)\, d\xi &= \int_0^x \omega(\xi,t)\, d\xi + \int_0^x \frac{\partial}{\partial \xi}(\ell - \xi)\int_0^\xi \frac{\omega(\eta,t)}{\ell - \eta}\, d\eta\, d\xi \\ &= (\ell - x)\big(\omega(x,t) - v(x,t)\big),\end{aligned} \tag{3.6}$$

which implies that

$$(\ell - x)v + Jv = (\ell - x)w, \qquad \int_0^\ell v(x,t)\, dx = 0. \tag{3.7}$$

Then, from equality (3.2) we obtain

$$-\int_\Omega \frac{\partial^2 u}{\partial t^2}\overline{Nv}\, dx\, dt = \int_\Omega A(t)\frac{\partial u}{\partial t}\overline{v}\, dx\, dt, \tag{3.8}$$

where

$$Nv = (\ell - x)v + Jv, \qquad A(t)u = -\frac{\partial}{\partial x}\left((\ell - x)a(x,t)\frac{\partial u}{\partial x}\right). \tag{3.9}$$

Replace $\partial u/\partial t$ by the smoothed function $J_\xi^{-1}(\partial u/\partial t)$ in (3.8) and use the relation

$$A(t)J_\xi^{-1} = J_\xi^{-1}A(\tau) + \xi J_\xi^{-1}\frac{\partial A(\tau)}{\partial \tau}J_\xi^{-1}. \tag{3.10}$$

Then, by taking the adjoint of the operator J_ξ^{-1}, and by integrating by parts with respect to t in the left-hand side, we obtain

$$\int_\Omega \frac{\partial u}{\partial t}N\overline{\frac{\partial v_\xi^*}{\partial t}}\,dx\,dt = \int_\Omega A(t)\frac{\partial u}{\partial t}\overline{v_\xi^*}\,dx\,dt + \xi\int_\Omega \frac{\partial A}{\partial t}\left(\frac{\partial u}{\partial t}\right)_\xi \overline{v_\xi^*}\,dx\,dt. \tag{3.11}$$

The operator $A(t)$ has a continuous inverse on $L_2(0,\ell)$ defined by the relation

$$A^{-1}(t)g = -\int_0^x \frac{d\xi}{a(\xi,t)(\ell-\xi)}\int_0^\xi g(\eta)\,d\eta + c\int_0^x \frac{d\xi}{a(\xi,t)(\ell-\xi)}, \tag{3.12}$$

where

$$c = \frac{\int_0^\ell (dx/a(x,t))\int_0^x g(\xi)\,d\xi}{\int_0^\ell (dx/a(x,t))}, \qquad \int_0^\ell A^{-1}(t)g\,dx = 0. \tag{3.13}$$

Hence, the function $(\partial u/\partial t)_\xi$ can be represented in the form

$$\left(\frac{\partial u}{\partial t}\right)_\xi = J_\xi^{-1}A^{-1}(t)A(t)\frac{\partial u}{\partial t}. \tag{3.14}$$

Then, $(\partial A/\partial t)(\partial u/\partial t)_\xi = A_\xi(t)A(t)(\partial u/\partial t)$, where

$$A_\xi(t) = \left(\frac{\partial^2 a}{\partial x\partial t}J_\xi^{-1} - \frac{\partial a}{\partial t}J_\xi^{-1}\frac{\partial a}{\partial x}\frac{1}{a}\right)\frac{1}{a}\left(\int_0^x g(\eta,t)\,d\eta - c\right) + \frac{\partial a}{\partial t}J_\xi^{-1}\frac{1}{a}g, \tag{3.15}$$

where the constant c is given by (3.13).

Consequently, equation (3.11) becomes

$$\int_\Omega \frac{\partial u}{\partial t}N\overline{\frac{\partial v_\xi^*}{\partial t}}\,dx\,dt = \int_\Omega A(t)\frac{\partial u}{\partial t}(v_\xi^* + \xi A_\xi^* v_\xi^*)\,dx\,dt, \tag{3.16}$$

in which the conjugate operator $A_\xi^*(t)$ of $A_\xi(t)$ is defined by

$$A_\xi^* v_\xi^* = \frac{1}{a}(J_\xi^{-1})^*\frac{\partial a}{\partial \tau}v_\xi^* + (Bv_\xi^*)(x) - (Bv_\xi^*)(0)\frac{\int_x^\ell (d\xi/a(\xi,t))}{\int_0^\ell (d\xi/a(\xi,t))}, \tag{3.17}$$

where

$$(Bv_\xi^*)(x) = \int_x^\ell \frac{1}{a(\xi,t)}\left[(J_\xi^{-1})^*\frac{\partial^2 a}{\partial\xi\partial\tau} - \frac{1}{a(\xi,t)}\frac{\partial a}{\partial \xi}(J_\xi^{-1})^*\frac{\partial a}{\partial \tau}\right]v_\xi^*(\xi,\tau)\,d\xi. \tag{3.18}$$

The left-hand side of (3.16) is a continuous linear functional of $\partial u/\partial t$. Hence, the function $h_\xi = v_\xi^* + \xi A_\xi^* v_\xi^*$ has the derivatives $(\ell - x)(\partial h_\xi/\partial x) \in L_2(\Omega)$, $(\partial/\partial x)((\ell - x)(\partial h_\xi/\partial x)) \in L_2(\Omega)$, and the following conditions are satisfied

$$h_\xi|_{x=0} = 0, \quad h_\xi|_{x=\ell} = 0, \quad (\ell - x)\frac{\partial h_\xi}{\partial x}\Big|_{x=\ell} = 0. \tag{3.19}$$

From (3.17) we have

$$(\ell - x)\frac{\partial h_\xi}{\partial x} = \left(I + \xi\frac{1}{a}(J_\xi^{-1})^*\frac{\partial a}{\partial \tau}\right)\frac{\partial v_\xi^*}{\partial x}, \tag{3.20}$$

$$\begin{aligned}\frac{\partial}{\partial x}\left((\ell - x)\frac{\partial h_\xi}{\partial x}\right) &= \left(I + \xi\frac{1}{a}(J_\xi^{-1})^*\frac{\partial a}{\partial \tau}\right)\frac{\partial}{\partial x}\left((\ell - x)\frac{\partial v_\xi^*}{\partial x}\right)\\ &\quad + \xi\left[-\frac{(\partial a/\partial x)(J_\xi^{-1})^*(\partial a/\partial \tau)}{a^2} + \frac{1}{a}(J_\xi^{-1})^*\frac{\partial^2 a}{\partial x\partial \tau}\right](\ell - x)\frac{\partial v_\xi^*}{\partial x},\end{aligned} \tag{3.21}$$

$$\left[\left(I + \xi\frac{1}{a}(J_\xi^{-1})^*\frac{\partial a}{\partial \tau}\right)v_\xi^*\right]_{x=0} = 0, \tag{3.22}$$

$$\left[\left(I + \xi\frac{1}{a}(J_\xi^{-1})^*\frac{\partial a}{\partial \tau}\right)v_\xi^*\right]_{x=\ell} = 0, \tag{3.23}$$

$$\left[\left(I + \xi\frac{1}{a}(J_\xi^{-1})^*\frac{\partial a}{\partial \tau}\right)(\ell - x)\frac{\partial v_\xi^*}{\partial x}\right]_{x=\ell} = 0. \tag{3.24}$$

Since $\|\xi(1/a)(J_\xi^{-1})^*(\partial a/\partial \tau)\|_{L_2(\Omega)} < 1$ for sufficiently small ξ, the operator $I + \xi(1/a)(J_\xi^{-1})^*(\partial a/\partial \tau)$ has a continuous inverse on $L_2(\Omega)$. In addition, the derivative of the above operator with respect to x is a bounded operator in $L_2(\Omega)$. Therefore, from (3.20) and (3.21), the function v_ξ^* has derivatives $(\ell - x)(\partial v_\xi^*/\partial x) \in L_2(\Omega)$ and $(\partial/\partial x)((\ell - x)(\partial v_\xi^*/\partial x)) \in L_2(\Omega)$.

In a similar way, we show that for each fixed $x \in [0,\ell]$ and sufficiently small ξ, the operator $I + \xi(1/a)(J_\xi^{-1})^*(\partial a/\partial \tau)$ has a continuous inverse on $L_2(0,T)$; hence, (3.22), and (3.23), and (3.24) imply that

$$v_\xi^*|_{x=0} = 0, \quad v_\xi^*|_{x=\ell} = 0, \quad (\ell - x)\frac{\partial v_\xi^*}{\partial x}\Big|_{x=\ell} = 0. \tag{3.25}$$

So, for ξ sufficiently small, the function v_ξ^* has the same properties as h_ξ. In addition, v_ξ^* satisfies the integral condition in (3.7).

Putting $u = \int_0^t\int_0^\tau \exp(c\eta)v_\xi^*(\eta,\tau)\,d\eta\,d\tau$ in (3.8), where the constant c satisfies $ca_0 - a_3 - a_3^2/a_0 \geq 0$, and using (3.4), we obtain

$$\int_\Omega \exp(ct)v_\xi^*\overline{Nv}\,dx\,dt = -\int_\Omega A(t)\frac{\partial u}{\partial t}\exp(-ct)\frac{\partial^2 \overline{u}}{\partial t^2}\,dx\,dt + \xi\int_\Omega A(t)\frac{\partial u}{\partial t}\frac{\partial \overline{v_\xi^*}}{\partial t}\,dx\,dt. \tag{3.26}$$

Integrating by parts each term in the left-hand side of (3.26) and taking the real parts yield

$$\begin{aligned} \operatorname{Re}\int_\Omega A(t)\frac{\partial u}{\partial t}\exp(-ct)\frac{\partial^2\overline{u}}{\partial t^2}\,dx\,dt & \\ \geq \frac{c}{2}\int_\Omega(\ell-x)a(x,t)\exp(-ct)\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dx\,dt & \\ -\frac{1}{2}\int_\Omega(\ell-x)\frac{\partial a}{\partial t}\exp(-ct)\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dx\,dt, & \end{aligned} \tag{3.27}$$

$$\operatorname{Re}\left(-\xi\int_\Omega A(t)\frac{\partial u}{\partial t}\frac{\partial\overline{v_\xi^*}}{\partial t}\,dx\,dt\right)\geq\frac{-\xi a_3^2}{2a_0}\int_\Omega(\ell-x)\exp(-ct)\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dx\,dt.$$

Now, using (3.27) in (3.26) with the choice of c indicated above we have

$$2\operatorname{Re}\int_\Omega \exp(ct)v_\xi^*\overline{Nv}\,dx\,dt\leq 0. \tag{3.28}$$

Then, for $\xi\to 0$ we obtain $2\operatorname{Re}\int_\Omega\exp(ct)v\overline{Nv}\,dx\,dt\leq 0$, that is,

$$2\operatorname{Re}\int_\Omega\exp(ct)(\ell-x)|v|^2\,dx\,dt+2\operatorname{Re}\int_\Omega\exp(ct)vJ\overline{v}\,dx\,dt\leq 0. \tag{3.29}$$

Since $\operatorname{Re}\int_\Omega\exp(ct)vJ\overline{v}\,dx\,dt=0$, we conclude that $v=0$; hence, $\omega=0$, which ends the proof of the lemma. □

Theorem 3.2. *The range $R(L)$ of L coincides with F.*

Proof. Since F is a Hilbert space, we have $R(L)=F$ if and only if the relation

$$\int_\Omega(\ell-x)^2\mathcal{L}u\overline{f}\,dx\,dt+\int_0^\ell\left[(\ell-x)^2\left(\frac{dl_1u}{dx}\frac{d\overline{\varphi}}{dx}+\frac{dl_2u}{dx}\frac{d\overline{\psi}}{dx}\right)\right]dx+\int_0^\ell(l_1u\overline{\varphi}+l_2u\overline{\psi})\,dx=0, \tag{3.30}$$

for arbitrary $u\in E$ and $(f,\varphi,\psi)\in F$, implies that $f=0$, $\varphi=0$ and $\psi=0$. Putting $u\in D_0(L)$ in (3.30), we conclude from Lemma 3.1 that $(\ell-x)f=0$. Hence,

$$\int_0^\ell\left[(\ell-x)^2\left(\frac{dl_1u}{dx}\frac{d\overline{\varphi}}{dx}+\frac{dl_2u}{dx}\frac{d\overline{\psi}}{dx}\right)+l_1u\overline{\varphi}+l_2u\overline{\psi}\right]dx=0\quad\forall u\in D(L). \tag{3.31}$$

Setting

$$D_{0k}(L)=\left\{u\in D(L):u^{(k)}|_{t=0}=0,\ k=0,1\right\}, \tag{3.32}$$

and taking $u\in D_{01}(L)$ in (3.31) yield

$$\int_0^\ell\left[(\ell-x)^2\frac{dl_1u}{dx}\frac{d\overline{\varphi}}{dx}+l_1u\overline{\varphi}\right]dx=0. \tag{3.33}$$

The range of the trace operator l_1 is everywhere dense in Hilbert space with the norm $[\int_0^\ell((\ell-x)^2|d\varphi/dx|^2+|\varphi|^2)\,dx]^{1/2}$; hence, $\varphi=0$. Likewise, for $u\in D_{00}(L)$, we get $\psi=0$. □

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