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ASYMPTOTIC BEHAVIOR OF ORTHOGONAL POLYNOMIALS CORRESPONDING TO A MEASURE WITH INFINITE DISCRETE PART OFF AN ARC

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ABSTRACT. We study the asymptotic behavior of orthogonal polynomials. The measure is concentrated on a complex rectifiable arc and has an infinity of masses in the region exterior to the arc.

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1. Introduction. Kaliaguine has studied in [3] the asymptotic behavior of orthogonal polynomials associated to a measure of the type $\sigma_l = \alpha + \gamma_l$, where α is concentrated on a complex rectifiable arc E and is absolutely continuous with respect to the Lebesgue measure $|d\xi|$ on the arc, and γ_l is a finite discrete measure with masses A_k at the points $z_k \in \text{Ext}(E)$, $k = 1, 2, \dots, l$, that is, $\gamma_l = \sum_{k=1}^{l} A_k \delta_{z_k}$, $A_k > 0$, where δ_{z_k} being the Dirac measure at the points z_k. In this paper, we generalize the previous study, when $\sigma = \alpha + \gamma$, where α possess the same properties as in [3] and γ is concentrated on an infinite discrete part $\{z_k\}_{k=1}^{\infty} \in \text{Ext}(E)$, $\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k}$. The masses $\{A_k\}_{k=1}^{\infty}$ satisfy

$$A_k > 0, \quad \sum_{k=1}^{\infty} A_k < \infty. \tag{1.1}$$

We note that the cases of a closed curve and a circle studied in [4, 5] are different from the case of an arc with respect to the asymptotics of orthogonal polynomials.

2. The space $H^2(\Omega, \rho)$. Suppose that E is a rectifiable arc in the complex plane, $\Omega = \text{Ext}(E)$, $G = \{w \in C / |w| > 1\}$ ($\infty \in \Omega$, $\infty \in G$), and $1/C(E) = \lim_{z \to \infty} (\Phi(z)/z) > 0$, where $\Phi : \Omega \to G$ is the conformal mapping. We denote by Ψ the inverse of Φ.

Let $\rho(\xi)$ be an integrable nonnegative function on E. If the weight function $\rho(\xi)$ satisfies the Szegö condition

$$\int_E \log(\rho(\xi)) \, |\Phi'(\zeta)| \, |d\xi| > -\infty. \tag{2.1}$$

Then one can construct the so-called Szegö function $D(z)$ associated with the domain Ω and the weight function $\rho(\xi)$ with the following properties.

$D(z)$ is analytic in Ω; $D(z) \neq 0$ in Ω; $D(\infty) > 0$; $D(z)$ has boundary values on both sides of E (a.e.) and $|D_{\pm}|^{-2} |\Phi'_{\pm}| = \rho(\xi)$ (a.e. on E).

Let $f(z)$ be an analytic function in Ω, we say that $f(z) \in H^2(\Omega, \rho)$ if and only if $f(\Psi(w))/D(\Psi(w)) \in H^2(G)$, and for a function F analytic in G, $F \in H^2(G)$ if and only if

$F(1/w) \in H^2(D)$; $w \in D$; $D = \{z \in C/|z| < 1\}$. The space $H^2(D)$ is well known (see [6]). Any function from $H^2(\Omega,\rho)$ has boundary values f_+, f_- on both sides of E, $f_+, f_- \in L^2(\rho)$. We define the norm in Hardy space by

$$\|f\|_{H^2(\Omega,\rho)} = \oint_E |f(\xi)|^2 \rho(\xi)|d\xi|. \tag{2.2}$$

Here, we take the integral on both sides of E.

3. Extremal properties of the orthogonal polynomials. We denote by P_n the set of polynomials of degree almost n. Define $\mu(\rho)$, $\mu^*(\rho)$, $m_n(\rho)$, $m_n(\sigma_l)$, and $m_n(\sigma)$ as the extremal values of the following problems:

$$\mu(\rho) = \inf\left\{\|\varphi\|^2_{H^2(\Omega,\rho)} : \varphi \in H^2(\Omega,\rho),\ \varphi(\infty) = 1\right\}, \tag{3.1}$$

$$\mu^*(\rho) = \inf\left\{\|\varphi\|^2_{H^2(\Omega,\rho)} : \varphi \in H^2(\Omega,\rho),\ \varphi(\infty) = 1,\ \varphi(z_k) = 0,\ k = 1,2,\ldots\right\}, \tag{3.2}$$

$$m_n(\rho) = \min\left\{\int_E |Q_n(\xi)|^2 \rho(\xi)|d\xi|,\ Q_n(z) = z^n + \cdots\right\}, \tag{3.3}$$

$$m_n(\sigma_l) = \min\left\{\int_E |Q_n(\xi)|^2 \rho(\xi)|d\xi| + \sum_{k=1}^{l} A_k |Q_n(z_k)|^2,\ Q_n(z) = z^n + \cdots\right\}, \tag{3.4}$$

$$m_n(\sigma) = \min\left\{\int_E |Q_n(\xi)|^2 \rho(\xi)|d\xi| + \sum_{k=1}^{\infty} A_k |Q_n(z_k)|^2,\ Q_n(z) = z^n + \cdots\right\}. \tag{3.5}$$

We denote, respectively, by φ^* and ψ^* the extremal functions of the problems (3.1) and (3.2). We denote by $\{T_n^l(z)\}$ and $\{T_n(z)\}$ the systems of the monic orthogonal polynomials, respectively, associated to the measures σ_l and σ, that is,

$$\begin{gathered} T_n^l(z) = z^n + \cdots, \\ \int_E T_n^l(\xi)\bar{\xi}^p \rho(\xi)|d\xi| + \sum_{k=1}^{l} A_k T_n^l(z_k)\bar{\xi}_k^p = 0; \quad p = 0,1,2,\ldots,n-1, \\ T_n(z) = z^n + \cdots, \\ \int_E T_n(\xi)\bar{\xi}^p \rho(\xi)|d\xi| + \sum_{k=1}^{\infty} A_k T_n(z_k)\bar{\xi}_k^p = 0; \quad p = 0,1,2,\ldots,n-1. \end{gathered} \tag{3.6}$$

It is easy to see that the polynomials $\{T_n^l(z)\}$ and $\{T_n(z)\}$ are, respectively, the optimal solutions of the extremal problems (3.4) and (3.5).

Lemma 3.1. *Let $\varphi \in H^2(\Omega,\rho)$ such that $\varphi(\infty) = 1$ and $\varphi(z_k) = 0$, $k = 1,2,\ldots$, and let*

$$B(z) = \prod_{k=1}^{\infty} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \frac{|\Phi(z_k)|^2}{\Phi(z_k)} \tag{3.7}$$

be the Blashke product, then

(1) $B \in H^2(\Omega,\rho)$; $B(\infty) = 1$; $|B_\pm(\xi)| = \prod_{k=1}^{\infty} |\Phi(z_k)|$ *(a.e. on E),*
(2) $\varphi/B \in H^2(\Omega,\rho)$ *and* $(\varphi/B)(\infty) = 1$.

The proof is the same as that of Lemma 3.1 given in [1].

Lemma 3.2. *The extremal functions* φ^* *and* ψ^* *are connected by*

$$\psi^* = B(z)\cdot\varphi^*, \qquad \mu^*(\rho) = \left(\prod_{k=1}^{\infty} |\Phi(z_k)|\right)^2 \mu(\rho). \tag{3.8}$$

The proof is the same as that of a closed curve given in [2, Lemma 4.2]. We replace the finite Blashke product by the infinite product B and using its properties announced by Lemma 3.1.

4. Main results

Definition 4.1. The measure $\sigma = \alpha+\gamma$ belongs to the class A (and we write $\sigma \in A$), if the absolutely continuous part α and the discrete part of σ satisfy (in addition to conditions (1.1) and (2.1))

$$\left(\sum_{k=1}^{\infty} |\Phi(z_k)| - 1\right) < \infty. \tag{4.1}$$

An arc E is from $C^{\alpha+}$ class if E is rectifiable and its coordinates are α-times differentiable, with αth derivatives satisfying a Lipschitz condition positive exponent.

Theorem 4.2. *Let* σ *be a measure,* $\sigma = \alpha+\gamma$*, such that* $\sigma \in A$*. Then*

$$\lim_{l\to\infty} m_n(\sigma_l) = m_n(\sigma). \tag{4.2}$$

Proof. The extremal property of $T_n(z)$ implies that

$$m_n(\sigma_l) \le \int_E |T_n(\xi)|^2 \rho(\xi)|d\xi| + \sum_{k=1}^{l} A_k |T_n(z_k)|^2 \le m_n(\sigma), \tag{4.3}$$

then

$$m_n(\sigma_l) \le m_n(\sigma). \tag{4.4}$$

On the other hand, the extremal property of $T_n(z)$ implies that

$$\begin{aligned} m_n(\sigma) &\le \int_E |T_n^l(\xi)|^2 \rho(\xi)|d\xi| + \sum_{k=1}^{\infty} A_k |T_n^l(z_k)|^2 \\ &= m_n(\sigma_l) + \sum_{k=l+1}^{\infty} A_k |T_n^l(z_k)|^2. \end{aligned} \tag{4.5}$$

According to the reproducing property of the kernel function $K_n(\xi,z)$ (see [7]), and $T_n^l(z) \in P_n$, we have

$$T_n^l(z_k) = \int_E T_n^l(\zeta)\overline{K_n(\xi,z_k)}\rho(\xi)|d\xi|. \tag{4.6}$$

The Scharwz inequality and the fact that $|\Phi(\xi)| = 1$ for $\xi \in E$ and $K_n(z,z_k) \in P_n$ imply

$$\begin{aligned} |T_n^l(z_k)|^2 &\le \int_E |T_n^l(\xi)|^2 \rho(\xi)|d\xi| \cdot \int_E |K_n(\xi,z_k)|^2 \rho(\xi)|d\xi| \\ &\le m_n(\sigma_l) \cdot K_n(z_k,z_k). \end{aligned} \tag{4.7}$$

The inequalities (1.1), (4.5), and (4.7) imply

$$\begin{aligned} m_n(\sigma) &\le m_n(\sigma_l) + \sum_{k=l+1}^{\infty} A_k m_n(\sigma_l) K_n(z_k,z_k) \\ &\le m_n(\sigma_l)\left[1 + \sup_{k \ge l+1} K_n(z_k,z_k) \sum_{k=l+1}^{\infty} A_k\right], \end{aligned} \tag{4.8}$$

so we have

$$\frac{m_n(\sigma)}{m_n(\sigma_l)} \le 1 + \delta_l, \quad \text{where } \delta_l \longrightarrow 0,\ l \longrightarrow \infty. \tag{4.9}$$

Using (4.4) and (4.9), we obtain

$$m_n(\sigma) \le \liminf_{l \to \infty} m_n(\sigma_l) \le \limsup_{l \to \infty} m_n(\sigma_l) \le m_n(\sigma), \quad \forall n, \tag{4.10}$$

this implies that

$$\lim_{l \to \infty} m_n(\sigma_l) = m_n(\sigma), \quad \forall n. \tag{4.11}$$

□

Theorem 4.3. *Let σ be a measure, $\sigma = \alpha + \gamma$, such that $\sigma \in A$ and*

$$\frac{m_n(\sigma_l)}{m_n(\rho)} \le \left(\prod_{k=1}^{l} |\Phi(z_k)|\right)^2, \quad \forall n, \forall l. \tag{4.12}$$

Suppose that $E \in C^{2+}$. Then we have

$$\begin{aligned} &\lim_{n \to \infty} \frac{m_n(\sigma)}{C(E)^{2n}} = \mu^*(\rho), \\ &\int_E |C(E)^{-n} T_n(\xi) - H_n(\xi)|^2 \rho(\xi)|d\xi| \longrightarrow 0, \\ &T_n(z) = C(E)^n \Phi^n(z)[\psi^*(z) + \epsilon_n(z)], \end{aligned} \tag{4.13}$$

where $H_n(\xi) = \Phi_+^n(\xi)\psi_+^*(\xi) + \Phi_-^n(\xi)\psi_-^*(\xi)$, $\epsilon_n \to 0$ uniformly on the compact subsets of Ω.

Proof. By passing to the limit when l tends to infinity and using Theorem 4.2 and (4.12), we obtain

$$\frac{m_n(\sigma)}{C(E)^{2n}} \le \left(\prod_{k=1}^{\infty} |\Phi(z_k)|\right)^2 \frac{m_n(\rho)}{C(E)^{2n}}. \tag{4.14}$$

This implies that

$$\limsup_{n \to \infty} \frac{m_n(\sigma)}{C(E)^{2n}} \le \left(\prod_{k=1}^{\infty} |\Phi(z_k)|\right)^2 \mu(\rho) = \mu^*(\rho) \tag{4.15}$$

(see Lemma 3.2).

The extremal property of the polynomials $T_n(z)$ and the fact that $|\Phi(\xi)| = 1$, for $\xi \in E$ imply (see [2] for details)

$$\frac{2m_n(\sigma)}{C(E)^{2n}} = \left\| \frac{T_n}{[C(E)\Phi]^n} \right\|^2_{H^2(\Omega,\rho)} + 2\sum_{k=1}^{\infty} A_k \left| \frac{T_n(z_k)}{[C(E)\Phi(z_k)]^n} \right| |\Phi(z_k)|^{2n}, \tag{4.16}$$

so

$$\left\| \frac{T_n}{[C(E)\Phi]^n} \right\|^2_{H^2(\Omega,\rho)} \le \frac{2m_n(\sigma)}{C(E)^{2n}}. \tag{4.17}$$

Equations (4.15) and (4.17) imply that

$$\limsup_{n\to\infty} \left\| \frac{T_n}{[C(E)\Phi]^n} \right\|^2_{H^2(\Omega,\rho)} \le 2\mu^*(\rho). \tag{4.18}$$

Now we take the integral

$$\begin{aligned} I_n &= \int_E |C(E)^{-n}T_n(\xi) - H_n(\xi)|^2 \rho(\xi)|d\xi| \\ &= \int_E \left| \left(\frac{1}{2}C(E)^{-n}T_n(\xi) - \Phi_+^n(\xi)\psi_+^*(\xi) \right) \right. \\ &\qquad \left. + \left(\frac{1}{2}C(E)^{-n}T_n(\xi) - \Phi_-^n(\xi)\psi_-^*(\xi) \right) \right|^2 \rho(\xi)|d\xi|, \end{aligned} \tag{4.19}$$

by the triangular inequality, we have

$$\begin{aligned} I_n^{1/2} &\le \left(\int_E \left| \frac{1}{2}C(E)^{-n}T_n(\xi) - \Phi_+^n(\xi)\psi_+^*(\xi) \right|^2 \rho(\xi)|d\xi| \right)^{1/2} \\ &\quad + \left(\int_E \left| \frac{1}{2}C(E)^{-n}T_n(\xi) - \Phi_-^n(\xi)\psi_-^*(\xi) \right|^2 \rho(\xi)|d\xi| \right)^{1/2} \\ &\le 2\left(\oint_E \left| \frac{1}{2}C(E)^{-n}T_n(\xi) - \Phi^n(\xi)\psi^*(\xi) \right|^2 \rho(\xi)|d\xi| \right)^{1/2}. \end{aligned} \tag{4.20}$$

Then we deduce that

$$I_n \le 4\left\| \frac{1}{2}\frac{T_n}{[C(E)\Phi]^n} - \psi^* \right\|^2_{H^2(\Omega,\rho)}. \tag{4.21}$$

By using the parallelogram rule in $H^2(\Omega,\rho)$, we have

$$I_n \le 4\left[2\left\| \frac{1}{2}\frac{T_n}{C(E)^n} \right\|^2_{H^2(\Omega,\rho)} + 2\|\psi^*\|^2_{H^2(\Omega,\rho)} - \left\| \frac{1}{2}\frac{T_n}{C(E)^n} + \psi^* \right\|^2_{H^2(\Omega,\rho)} \right], \tag{4.22}$$

so

$$\limsup_{n\to\infty} I_n \le 4\left[\mu^*(\rho) + 2\mu^*(\rho) - \frac{9}{4}\frac{4}{3}\mu^*(\rho) \right] = 0, \tag{4.23}$$

where we have used the fact that $\liminf_{n\to\infty} \|g_n\|^2_{H^2(\Omega,\rho)} \ge 2\mu^*(\rho) \ge (4/3)\mu^*(\rho)$, since the function $g_n(z) = (2/3)((1/2)T_n(z)/C(E)^n + \psi^*(z)) \in H^2(\Omega,\rho)$, $g_n(\infty) = 1$, and $g_n(z_k) \to 0$, $n \to \infty$. This yields

$$0 \le \liminf_{n\to\infty} I_n \le \limsup_{n\to\infty} I_n \le 0, \tag{4.24}$$

finally,

$$\lim_{n\to\infty} I_n = 0. \tag{4.25}$$

For the asymptotics in the region exterior to the arc E we need the Szegö reproducing kernel function $K(\xi,z)$ (see [8, page 173]) and the fact that $T_n(z)/C(E)^n\Phi^n(z) \in H^2(\Omega,\rho)$ for all $z\in\Omega$, then

$$\begin{aligned}
\frac{T_n(z)}{C(E)^n\Phi^n(z)} &= \oint_E \frac{T_n(\xi)}{C(E)^n\Phi^n(\xi)}\overline{K(\xi,z)}\rho(\xi)|d\xi| \\
&= \int_E C^{-n}T_n(\xi)\Big\{\Phi_+^{-n}(\xi)\overline{K_+(\xi,z)}+\Phi_-^{-n}(\xi)\overline{K_-(\xi,z)}\Big\}\rho(\xi)|d\xi| \\
&= \int_E \{C^{-n}T_n(\xi)-H_n(\xi)\}\Big\{\Phi_+^{-n}(\xi)\overline{K_+(\xi,z)}+\Phi_-^{-n}(\xi)\overline{K_-(\xi,z)}\Big\}\rho(\xi)|d\xi| \\
&\quad + \int_E H_n(\xi)\Big\{\Phi_+^{-n}(\xi)\overline{K_+(\xi,z)}+\Phi_-^{-n}(\xi)\overline{K_-(\xi,z)}\Big\}\rho(\xi)|d\xi|.
\end{aligned} \tag{4.26}$$

The first integral approaches 0 as $n\to\infty$ (part 2 of Theorem 4.3), the second one may be transformed into the form

$$\begin{aligned}
&\int_E \{\Phi_+^n(\xi)\psi_+^*(\xi)+\Phi_-^n(\xi)\psi_-^*(\xi)\}\Big\{\Phi_+^{-n}(\xi)\overline{K_+(\xi,z)}+\Phi_-^{-n}(\xi)\overline{K_-(\xi,z)}\Big\}\rho(\xi)|d\xi| \\
&\quad = \oint_E \psi^*(\xi)\overline{K(\xi,z)}\rho(\xi)|d\xi| \\
&\qquad + \int_E \Big\{\Phi_+^n(\xi)\psi_+^*(\xi)\Phi_-^{-n}(\xi)\overline{K_-(\xi,z)}+\psi_-^*(\xi)\Phi_+^{-n}(\xi)\Phi_-^n(\xi)\overline{K_+(\xi,z)}\Big\}\rho(\xi)|d\xi| \\
&\quad = \psi^*(z)+\lambda_n,
\end{aligned} \tag{4.27}$$

where $\lambda_n\to 0$ (coefficients of an integrable function). This proves part 3. □

Remark 4.4. It is not difficult to find families of points $\{A_k\}_{k=1}^{\infty}$ and $\{z_k\}_{k=1}^{\infty}$ satisfying condition (4.12). For example if $E=[-1,+1]$, then

$$\Phi(z) = z+\sqrt{z^2-1} \quad \Big(\Big|z+\sqrt{z^2-1}\Big| > 1\Big). \tag{4.28}$$

We can take z_k such that

$$\Phi(z_k) = 1+\frac{1}{k^2}, \qquad A_k = \frac{1}{2^k}. \tag{4.29}$$

As weight function we take

$$\rho(\xi) = (1-\xi^2)^{-1/2}. \tag{4.30}$$

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