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COMMON FIXED POINT THEOREMS IN 2 NON-ARCHIMEDEAN MENGER PM-SPACE

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ABSTRACT. We introduce the concept of a 2 non-Archimedean Menger PM-space and prove a common fixed point theorem for weak compatible mappings of type (A).

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1. Introduction. Cho et al. [1] proved a common fixed point theorem for compatible mappings of type (A) in non-Archimedean (NA) Menger PM-space. The aim of this paper is to generalize the results of Cho et al. [1] for weak compatible mappings of type (A) in a 2 NA Menger PM-space.

We first give some definitions and notations.

Definition 1.1. Let X be any nonempty set and D the set of all left-continuous distribution functions. An ordered pair (X,F) is said to be a 2 non-Archimedean probabilistic metric space (briefly 2 NA PM-space) if F is a mapping from $X\times X\times X$ into D satisfying the following conditions where the value of F at $x,y,z\in X\times X\times X$ is represented by $F_{x,y,z}$ or $F(x,y,z)$ for all $x,y,z\in X$ such that

(i) $F_{x,y,z}(t)=1$ for all $t>0$ if and only if at least two of the three points are equal.

(ii) $F_{x,y,z}=F_{x,z,y}=F_{z,y,x}$.

(iii) $F_{x,y,z}(0)=0$.

(iv) If $F_{x,y,s}(t_1)=F_{x,s,z}(t_2)=F_{s,y,z}(t_3)=1$, then $F_{x,y,z}(\max\{t_1,t_2,t_3\})=1$.

Definition 1.2. A t-norm is a function $\Delta:[0,1]\times[0,1]\times[0,1]\to[0,1]$ which is associative, commutative, nondecreasing in each coordinate, and $\Delta(a,1,1)=a$ for every $a\in[0,1]$.

Definition 1.3. A 2 NA Menger PM-space is an ordered triplet (X,F,Δ), where Δ is a t-norm and (X,F) is a 2 NA PM-space satisfying the following condition

$$\begin{aligned}&F_{x,y,z}(\max\{t_1,t_2,t_3\})\\&\quad\geq\Delta(F_{x,y,s}(t_1),F_{x,s,z}(t_2),F_{s,y,z}(t_3))\quad\forall x,y,z\in X,\ t_1,t_2,t_3\geq0.\end{aligned}\tag{1.1}$$

Definition 1.4. Let (X,F,t) be a 2 NA Menger PM-space and t a continuous t-norm, then (X,F,t) is Hausdorff in the topology induced by the family of neighborhoods

$$\{U_x(\epsilon,\lambda,a_1,a_2,\dots,a_n);\ x,a_i\in X,\ \epsilon>0,\ i=1,2,\dots,n,\ n\in\mathbb{Z}^+\},\tag{1.2}$$

where $\mathbb{Z}^+$ is the set of all positive integers and

$$\begin{aligned}U_x(\epsilon,\lambda,a_1,a_2,\dots,a_n)&=\{y\in X;\ F_{x,y,a_i}(\epsilon)>1-\lambda,\ 1\leq i\leq n\}\\&=\cap_{i=1}^n\{y\in X;\ F_{x,y,a_i}(\epsilon)>1-\lambda,\ 1\leq i\leq n\}.\end{aligned}\tag{1.3}$$

Definition 1.5. A 2 NA Menger PM-space (X,F) is said to be of type $(C)_g$ if there exists a $g \in \Omega$ such that

$$g(F_{x,y,z}(t)) \le g(F_{x,y,a}(t)) + g(F_{x,a,z}(t)) + g(F_{a,y,z}(t)) \quad \forall x,y,z,a \in X,\ t \ge 0, \tag{1.4}$$

where $\Omega = \{g \mid g : [0,1] \to [0,\infty)$ is continuous, strictly decreasing, $g(1) = 0$, and $g(0) < \infty\}$.

Definition 1.6. A 2 NA Menger PM-space (X,F,Δ) is said to be of type $(D)_g$ if there exists a $g \in \Omega$ such that

$$g(\Delta(t_1,t_2,t_3)) \le g(t_1) + g(t_2) + g(t_3) \quad \forall t_1,t_2,t_3 \in [0,1]. \tag{1.5}$$

Remark 1.7. If 2 NA Menger PM-space (X,F,Δ) is of type $(D)_g$, then (X,F,Δ) is of type $(C)_g$.

Throughout this paper, let (X,F,Δ) be a complete 2 NA Menger PM-space with a continuous strictly increasing t-norm Δ. Let $\phi : [0,\infty) \to [0,\infty)$ be a function satisfying the condition (Φ) ϕ is upper semi-continuous from right and $\phi(t) < t$ for all $t > 0$.

Lemma 1.8. *If a function $\phi : [0,\infty) \to [0,\infty)$ satisfies the condition (Φ), then we get*
(1) *For all $t \ge 0$, $\lim_{n\to\infty} \phi^n(t) = 0$, where $\phi^n(t)$ is the nth iteration of $\phi(t)$.*
(2) *If $\{t_n\}$ is a nondecreasing sequence of real numbers and $t_{n+1} \le \phi(t_n)$, $n = 1,2,\ldots$. Then $\lim_{n\to\infty} t_n = 0$. In particular, if $t \le \phi(t)$ for all $t \ge 0$, then $t = 0$.*

Lemma 1.9. *Let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} F_{y_n,y_{n+1},a}(t) = 1$ for all $t > 0$. If the sequence $\{y_n\}$ is not a Cauchy sequence in X, then there exist $\epsilon_0 > 0$, $t_0 > 0$, and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that*
(i) *$m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$.*
(ii) *$F_{y_{m_i},y_{n_i},a}(t_0) < 1-\epsilon_0$ and $F_{y_{m_i-1},y_{n_i},a}(t_0) \ge 1-\epsilon_0$, $i = 1,2,\ldots$.*

Definition 1.10. Let $A,S : X \to X$ be mappings, A and S are said to be compatible if

$$\lim_{n\to\infty} g(F_{ASx_n,SAx_n,a}(t)) = 0 \quad \forall t > 0,\ a \in X, \tag{1.6}$$

when $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = z = \lim_{n\to\infty} Sx_n \quad \text{for some } z \in X. \tag{1.7}$$

Definition 1.11. Let $A,S : X \to X$ be mappings, A and S are said to be compatible of type (A) if

$$\lim_{n\to\infty} g(F_{ASx_n,SSx_n,a}(t)) = 0 = \lim_{n\to\infty} g(F_{SAx_n,AAx_n,a}(t)) \quad \forall t > 0,\ a \in X, \tag{1.8}$$

when $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = z = \lim_{n\to\infty} Sx_n \quad \text{for some } z \in X. \tag{1.9}$$

Definition 1.12. Let $A, S : X \to X$ be mappings, A and S are said to be weak compatible of type (A) if

$$\lim_{n\to\infty} g(F_{ASx_n,SSx_n,a}(t)) \geq \lim_{n\to\infty} g(F_{SAx_n,SSx_n,a}(t)),$$
$$\lim_{n\to\infty} g(F_{SAx_n,AAx_n,a}(t)) \geq \lim_{n\to\infty} g(F_{ASx_n,AAx_n,a}(t)) \quad \forall t > 0,\ a \in X, \tag{1.10}$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = z = \lim_{n\to\infty} Sx_n \quad \text{for some } z \in X. \tag{1.11}$$

Proposition 1.13. *Let $A, S : X \to X$ be continuous mappings. If A and S are compatible of type (A), then they are weak compatible of type (A).*

Proof. Suppose that A and S are compatible of type (A). Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} Ax_n = z = \lim_{n\to\infty} Sx_n \quad \text{for some } z \in X, \tag{1.12}$$

then

$$\begin{aligned} \lim_{n\to\infty} g(F_{SAx_n,SSx_n,a}(t)) &= 0 \\ &\leq \lim_{n\to\infty} g(F_{ASx_n,SSx_n,a}(t)) \\ &\Longrightarrow \lim_{n\to\infty} g(F_{ASx_n,SSx_n,a}(t)) \geq \lim_{n\to\infty} g(F_{SAx_n,SSx_n,a}(t)). \end{aligned} \tag{1.13}$$

Similarly, we can show that

$$\begin{aligned} \lim_{n\to\infty} g(F_{SAx_n,AAx_n,a}(t)) &= 0 \\ &\geq \lim_{n\to\infty} g(F_{ASx_n,AAx_n,a}(t)). \end{aligned} \tag{1.14}$$

Therefore, A and S are weak compatible of type (A). □

Proposition 1.14. *Let $A, S : X \to X$ be weak compatible mappings of type (A). If one of A and S is continuous, then A and S are compatible of type (A).*

Proof. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} Ax_n = z = \lim_{n\to\infty} Sx_n \quad \text{for some } z \in X. \tag{1.15}$$

Suppose S is continuous so $SSx_n, SAx_n \to Sz$ as $n \to \infty$. Since A and S are weak compatible of type (A), so we have

$$\begin{aligned} \lim_{n\to\infty} g(F_{ASx_n,SSx_n,a}(t)) &\geq \lim_{n\to\infty} g(F_{SAx_n,SSx_n,a}(t)) \\ &= \lim_{n\to\infty} g(F_{Sz,Sz,a}(t)) = 0. \end{aligned} \tag{1.16}$$

Thus

$$\lim_{n\to\infty} g(F_{ASx_n,SSx_n,a}(t)) = 0. \tag{1.17}$$

Similarly,

$$\lim_{n\to\infty} g(F_{SAx_n,AAx_n,a}(t)) = 0. \tag{1.18}$$

Hence A and S are compatible of type (A). □

Proposition 1.15. *Let $A, S : X \to X$ be continuous mappings. Then A and S are compatible of type (A) if and only if A and S are weak compatible of type (A).*

Note that Proposition 1.15 is a direct consequence of Propositions 1.13 and 1.14.

Proposition 1.16. *Let $A, S : X \to X$ be mappings. If A and S are weak compatible of type (A) and $Az = Sz$ for some $z \in X$. Then $SAz = AAz = ASz = SSz$.*

Proof. Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = z$, $n = 1, 2, \dots$, and $Az = Sz$ for some $z \in X$. Then we have $Ax_n, Sx_n \to Sz$ as $n \to \infty$. Since A and S are weak compatible of type (A) so

$$\begin{aligned} \lim_{n\to\infty} g\left(F_{ASx_n,SSx_n,a}(t)\right) &\geq \lim_{n\to\infty} g\left(F_{SAx_n,SSx_n,a}(t)\right), \\ \lim_{n\to\infty} g\left(F_{SAx_n,AAx_n,a}(t)\right) &\geq \lim_{n\to\infty} g\left(F_{ASx_n,AAx_n,a}(t)\right). \end{aligned} \tag{1.19}$$

Now

$$\begin{aligned} \lim_{n\to\infty} g\left(F_{SAz,AAz,a}(t)\right) &= \lim_{n\to\infty} g\left(F_{SAx_n,AAx_n,a}(t)\right) \geq \lim_{n\to\infty} g\left(F_{ASx_n,AAx_n,a}(t)\right) \\ &= g\left(F_{ASz,SSz,a}(t)\right). \end{aligned} \tag{1.20}$$

Since $Sz = Az$, then $SAz = AAz$. Similarly, we have $ASz = SSz$. But $Az = Sz$ for $z \in X$ implies that $AAz = ASz = SAz = SSz$. □

Proposition 1.17. *Let $A, S : X \to X$ be weak compatible mappings of type (A) and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Ax_n = z = \lim_{n\to\infty} Sx_n$ for some $z \in X$, then*

(1) *$\lim_{n\to\infty} ASx_n = Sz$ if S is continuous at z.*

(2) *$SAz = ASz$ and $Az = Sz$ if A and S are continuous at z.*

Proof. Suppose that S is continuous and $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = z = \lim_{n\to\infty} Sx_n \quad \text{for some } z \in X, \tag{1.21}$$

so

$$SSx_n \longrightarrow Sz \quad \text{as } n \longrightarrow \infty. \tag{1.22}$$

Since A and S are weak compatible of type (A), we have

$$\begin{aligned} g\left(F_{ASx_n,Sz,a}(t)\right) &= \lim_{n\to\infty} g\left(F_{ASx_n,SSx_n,a}(t)\right) \\ &\geq \lim_{n\to\infty} g\left(F_{SAx_n,SSx_n,a}(t)\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \end{aligned} \tag{1.23}$$

for all $t > 0$ which implies that $ASx_n \to Sz$ as $n \to \infty$.

(2) Suppose that A and S are continuous at z. Since $Ax_n \to z$ as $n \to \infty$ and S is continuous at z, by Proposition 1.17(1) $ASx_n \to Sz$ as $n \to \infty$. On the other hand, since $Sx_n \to z$ as $n \to \infty$ and A is also continuous at z, $ASx_n \to Az$ as $n \to \infty$. Thus $Az = Sz$ by the uniqueness of the limit and so by Proposition 1.16, $SAz = AAz = ASz = SSz$. Therefore, we have $ASz = SAz$. □

Theorem 1.18. *Let $A, B, S, T : X \to X$ be mappings satisfying*

(i) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(ii) *the pairs* A, S *and* B, T *are weak compatible of type* (A),
(iii) S *and* T *is continuous,*
(iv) $g(F_{Ax,By,a}(t)) \le \phi(\max\{g(F_{Sx,Ty,a}(t)), g(F_{Sx,Ax,a}(t)), g(F_{Ty,By,a}(t)), (1/2)(g(F_{Sx,By,a}(t)) + g(F_{Ty,Ax,a}(t)))\})$,

for all $t > 0$, $a \in X$ *where a function* $\phi : [0,\infty) \to [0,\infty)$ *satisfies the condition* (Φ). *Then by (i) since* $A(X) \subset T(X)$, *for any* $x_0 \in X$, *there exists a point* $x_1 \in X$ *such that* $Ax_0 = Tx_1$. *Since* $B(X) \subset S(X)$, *for this* x_1, *we can choose a point* $x_2 \in X$ *such that* $Bx_1 = Sx_2$ *and so on, inductively, we can define a sequence* $\{y_n\}$ *in* X *such that*

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, \quad \text{for } n = 0, 1, 2, \ldots. \tag{1.24}$$

First we prove the following lemma.

Lemma 1.19. *Let* $A, S : X \to X$ *be mappings satisfying conditions (i) and (iv), then the sequence* $\{y_n\}$ *defined by (1.24), such that*

$$\lim_{n\to\infty} g(F_{y_n,y_{n+1}}(t)) = 0 \quad \forall t > 0,\ a \in X, \tag{1.25}$$

is a Cauchy sequence in X.

Proof. Since $g \in \Omega$, it follows that $\lim_{n\to\infty}(F_{y_n,y_{n+1},a}(t)) = 0$ for all $a > 0$, $a \in X$ if and only if $\lim_{n\to\infty} g(F_{y_n,y_{n+1},a}(t)) = 0$ for all $t > 0$. By Lemma 1.9, if $\{y_n\}$ is not a Cauchy sequence in X, there exist $\epsilon_0 > 0$, $t_0 > 0$, and two sequences $\{m_i\}, \{n_i\}$ of positive integers such that

(A) $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$,

(B) $g(F_{y_{m_i},y_{n_i},a}(t_0)) > g(1-\epsilon_0)$ and $g(F_{y_{m_i-1},y_{n_i},a}(t_0)) \le g(1-\epsilon_0)$, $i = 1, 2, \ldots$, since $g(t) = 1 - t$. Thus we have

$$\begin{aligned} g(1-\epsilon_0) &< g(F_{y_{m_i},y_{n_i},a}(t_0)) \\ &\le g(F_{y_{m_i},y_{n_i},y_{m_i-1}}(t_0)) + g(F_{y_{m_i},y_{m_i-1}},a(t_0)) + g(F_{y_{m_i-1},y_{n_i},a}(t_0)) \\ &\le g(F_{y_{m_i},y_{n_i},y_{m_i-1}}(t_0)) + g(F_{y_{m_i},y_{m_i-1},a}(t_0)) + g(1-\epsilon_0). \end{aligned} \tag{1.26}$$

As $i \to \infty$ in (1.26), we have

$$\lim_{n\to\infty} g(F_{y_{m_i},y_{n_i},a}(t_0)) = g(1-\epsilon_0). \tag{1.27}$$

On the other hand, we have

$$\begin{aligned} g(1-\epsilon_0) &< g(F_{y_{m_i},y_{n_i},a}(t_0)) \\ &\le g(F_{y_{m_i},y_{n_i},y_{n_i+1}}(t_0)) + g(F_{y_{m_i},y_{n_i+1},a}(t_0)) + g(F_{y_{n_i+1},y_{n_i},a}(t_0)). \end{aligned} \tag{1.28}$$

Now, consider $g(F_{y_{m_i},y_{n_i+1},a}(t_0))$ in (1.28), assume that both n_i and m_i are even. Then by (iv), we have

$$
\begin{aligned}
g(F_{y_{m_i},y_{n_i+1},a}(t_0)) &= g(F_{Ax_{m_i},Bx_{n_i+1},a}(t_0)) \\
&\le \phi(\max\{g(F_{Sx_{m_i},Tx_{n_i+1},a}(t_0)), \\
&\qquad g(F_{Sx_{m_i},Ax_{m_i},a}(t_0)), g(F_{Tx_{n_i+1},Bx_{n_i+1},a}(t_0)), \\
&\qquad \frac{1}{2}(g(F_{Sx_{m_i},Bx_{n_i+1},a}(t_0)) + g(F_{Tx_{n_i+1},Bx_{n_i+1},a}(t_0)))\}) \\
&= \phi(\max\{g(F_{y_{m_i-1},y_{n_i},a}(t_0)), \\
&\qquad g(F_{y_{m_i-1},y_{m_i},a}(t_0)), g(F_{y_{n_i},y_{n_i+1},a}(t_0)), \\
&\qquad \frac{1}{2}(g(F_{y_{m_i-1},y_{n_i+1},a}(t_0)) + g(F_{y_{n_i},y_{m_i},a}(t_0)))\}).
\end{aligned}
\tag{1.29}
$$

By (1.27), (1.28), and (1.29), letting $i \to \infty$ in (1.29), we have

$$
g(1-\epsilon_0) \le \phi(\max\{g(1-\epsilon_0), 0, 0, g(1-\in_0)\}) = \phi(g(1-\epsilon_0)) < g(1-\epsilon_0), \tag{1.30}
$$

which is a contradiction. Therefore, $\{y_n\}$ is a Cauchy sequence in X. □

Now, we prove our main theorem.

If we prove $\lim_{n\to\infty} g(F_{y_n,y_{n+1},a}(t)) = 0$ for all $t > 0$, then by Lemma 1.19, the sequence $\{y_n\}$ defined by (1.24) is a Cauchy sequence in X.

First we prove $\lim_{n\to\infty} g(F_{y_n,y_{n+1},a}(t)) = 0$ for all $t > 0$. In fact, by Theorem 1.18(iv) and (1.24), we have

$$
\begin{aligned}
g(F_{y_{2n},y_{2n+1},a}(t)) &= g(F_{Ax_{2n},Bx_{2n+1},a}(t)) \\
&\le \phi(\max\{g(F_{Sx_{2n},Tx_{2n+1},a}(t)), \\
&\qquad g(F_{Sx_{2n},Ax_{2n},a}(t)), g(F_{Tx_{2n},Bx_{2n+1},a}(t)), \\
&\qquad \frac{1}{2}(g(F_{Sx_{2n},Bx_{2n+1},a}(t)) + g(F_{Tx_{2n+1},Ax_{2n},a}(t)))\}) \\
&= \phi(\max\{g(F_{y_{2n-1},y_{2n},a}(t)), \\
&\qquad g(F_{y_{2n-1},y_{2n},a}(t)), g(F_{y_{2n},y_{2n+1},a}(t)), \\
&\qquad \frac{1}{2}(g(F_{y_{2n-1},y_{2n+1},a}(t)) + g(1))\}) \\
&\le \phi(\max\{g(F_{y_{2n-1},y_{2n},a}(t)), g(F_{y_{2n},y_{2n+1},a}(t)), \\
&\qquad g(F_{y_{2n-1},y_{2n},a}(t)) + g(F_{y_{2n},y_{2n+1},a}(t))\})
\end{aligned}
\tag{1.31}
$$

if $g(F_{y_{2n-1},y_{2n},a}(t)) \le g(F_{y_{2n},y_{2n+1},a}(t))$ for all $t > 0$, then by Theorem 1.18(iv), $g(F_{y_{2n},y_{2n+1},a}(t)) \le \phi(g(F_{y_{2n},y_{2n+1},a}(t)))$ and thus, by Lemma 1.8, $g(F_{y_{2n},y_{2n+1},a}(t)) = 0$ for all $t > 0$. Similarly, we have $g(F_{y_{2n+1},y_{2n+2},a}(t)) = 0$, thus we have $\lim_{n\to\infty} g(F_{y_n,y_{n+1},a}(t)) = 0$ for all $t > 0$. On the other hand, if $g(F_{y_{2n-1},y_{2n},a}(t)) \ge g(F_{y_{2n},y_{2n+1},a}(t))$, then by Theorem 1.18(iv), we have

$$
g(F_{y_{2n},y_{2n+1},a}(t)) \le \phi(g(F_{y_{2n-1},y_{2n},a}(t))) \quad \forall t > 0. \tag{1.32}
$$

Similarly,

$$g(F_{y_{2n+1},y_{2n+2},a}(t)) \le \phi(g(F_{y_{2n},y_{2n+1},a}(t))) \quad \forall t > 0, \tag{1.33}$$

hence

$$g(F_{y_n,y_{n+1},a}(t)) \le \phi(g(F_{y_{n-1},y_n,a}(t))) \quad \forall t > 0,\ n = 1,2,3,\dots, \tag{1.34}$$

therefore by Lemma 1.8,

$$\lim_{n\to\infty} g(F_{y_n,y_{n+1},a}(t)) = 0 \quad \forall t > 0, \tag{1.35}$$

which implies that $\{y_n\}$ is a Cauchy sequence in X by Lemma 1.19. Since (X,F,Δ) is complete, the sequence $\{y_n\}$ converges to a point $z \in X$ and so the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$, and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the limit z.

Now, suppose that T is continuous. Since B and T are weak compatible of type (A), by Proposition 1.17, BTx_{2n+1}, TTx_{2n+1} tend to Tz as n tends to ∞. Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in Theorem 1.18(iv), we have

$$\begin{aligned} &g(F_{Ax_{2n},BTx_{2n+1},a}(t)) \\ &\quad \le \phi(\max\{g(F_{Sx_{2n},TTx_{2n+1},a}(t)), g(F_{Sx_{2n},Ax_{2n},a}(t)), g(F_{TTx_{2n+1},BTx_{2n+1},a}(t)), \\ &\qquad\quad \frac{1}{2}(g(F_{Sx_{2n},BTx_{2n+1},a}(t)) + g(F_{TTx_{2n+1},Ax_{2n},a}(T)))\}) \quad \forall t > 0. \end{aligned} \tag{1.36}$$

Letting $n \to \infty$ in (1.36), we get

$$\begin{aligned} g(F_{z,Tz,a}(t)) &\le \phi(\max\{g(F_{z,Tz,a}(t)), g(F_{z,z,a}(t)), g(F_{Tz,Tz,a}(t)), \\ &\qquad \frac{1}{2}(g(F_{z,Tz,a}(t)) + g(F_{Tz,z,a}(t)))\}) \\ &= \phi(g(F_{z,Tz,a}(t))) \quad \forall t > 0, \end{aligned} \tag{1.37}$$

which means that $g(F_{z,Tz,a}(t)) = 0$ for all $t > 0$ by Lemma 1.8 and so we have $Tz = z$. Again replacing x by x_{2n} and y by z in Theorem 1.18(iv), we have

$$\begin{aligned} g(F_{Ax_{2n},Bz,a}(t)) &\le \phi(\max\{g(F_{Sx_{2n},Tz,a}(t)), g(F_{Sx_{2n},Ax_{2n},a}(t)), g(F_{Tz,Bz,a}(t)), \\ &\qquad \frac{1}{2}(g(F_{Sx_{2n},Bz,a}(t)) + g(F_{Tz,Ax_{2n},a}(t)))\}) \quad \forall t > 0. \end{aligned} \tag{1.38}$$

Letting $n \to \infty$ in (1.38), we get

$$\begin{aligned} g(F_{z,Bz,a}(t)) &\le \phi(\max\{g(F_{z,z,a}(t)), g(F_{z,Bz,a}(t)), g(F_{z,Bz,a}(t)), \\ &\qquad \frac{1}{2}(g(F_{z,Bz,a}(t)) + g(F_{z,z,a}(t)))\}) \quad \forall t > 0, \end{aligned} \tag{1.39}$$

which implies that $g(F_{z,Bz,a}(t)) \le \phi(g(F_{z,Bz,a}(t)))$ for all $t > 0$ and so we have $Bz = z$. Since $B(x) \subset S(X)$, there exists a point $w \in X$ such that $Bz = Sw = z$. By using condition Theorem 1.18(iv) again, we have

$$\begin{aligned} g(F_{Aw,z,a}(t)) &= g(F_{Aw,Bz,a}(t)) \\ &\le \phi(\max\{(F_{Sw,Tz,a}(t)), g(F_{Sw,Aw,a}(t)), g(F_{Tz,Bz,a}(t)), \\ &\qquad \frac{1}{2}(g(F_{Sw,Bz,a}(t)) + g(F_{Tz,Aw,a}(t)))\}) \\ &\le \phi(g(F_{Aw,z,a}(t))) \quad \forall t > 0, \end{aligned} \tag{1.40}$$

which means that $Aw = z$. Since A and S are weak compatible mappings of type (A) and $Aw = Sw = z$, by Proposition 1.16, $Az = ASw = SSw = Sz$. Again by using Theorem 1.18(iv), we have $Az = z$.

Therefore, $Az = Bz = Sz = Tz = z$, that is, z is a common fixed point of the given mappings A, B, S, T. The uniqueness of the common fixed point z follows easily from Theorem 1.18(iv).

Remark 1.20. In Theorem 1.18, if S and T are continuous, then by Proposition 1.15, the theorem is true even though the pairs A, S and B, T are compatible of type (A) instead of the condition (ii).

Application

Theorem 1.21. *Let (X, F, t) be a complete 2 NA Menger PM-space and A, B, S, and T be the mappings from the product $X \times X$ to X such that*

$$\begin{gathered} A(X \times \{y\}) \subseteq T(X \times \{y\}), \qquad B(X \times \{y\}) \subseteq S(X \times \{y\}), \\ g(F_{A(T(x,y),y),T(A(x,y),y),a}(t)) \le g(F_{A(x,y),T(x,y),a}(t)), \\ g(F_{B(S(x,y),y),S(B(x,y),y),a}(t)) \le g(F_{B(x,y),S(x,y),a}(t)), \end{gathered} \tag{1.41}$$

for all $t > 0$. If S and T are continuous with respect to their direct argument and

$$\begin{aligned} & g(F_{A(x,y),B(x',y'),a}(t)) \\ & \quad \le \phi\big(\max\{g(F_{S(x,y),T(x',y'),a}(t)), \\ & \qquad\qquad g(F_{S(x,y),A(x,y),a}(t)), g(F_{T(x',y'),B(x',y'),a}(t)), \\ & \qquad\qquad \frac{1}{2}(g(F_{S(x,y),B(x',y'),a}(t)) + g(F_{T(x',y'),A(x,y),a}(t)))\}\big) \end{aligned} \tag{1.42}$$

for all $t > 0$ and x, y, x', y' in X, then there exists only one point b in X such that

$$A(b,y) = S(b,y) = B(b,y) = T(b,y) \quad \forall y \text{ in } X. \tag{1.43}$$

Proof. By (1.42),

$$\begin{aligned} & g(F_{A(x,y),B(x',y'),a}(t)) \\ & \quad \le \phi\big(\max\{g(F_{S(x,y),T(x',y'),a}(t)), \\ & \qquad\qquad g(F_{S(x,y),A(x,y),a}(t)), g(F_{T(x',y'),B(x',y'),a}(t)), \\ & \qquad\qquad \frac{1}{2}(g(F_{S(x,y),B(x',y'),a}(t)) + g(F_{T(x',y'),A(x,y),a}(t)))\}\big) \end{aligned} \tag{1.44}$$

for all $t > 0$, therefore by Theorem 1.18, for each y in X, there exists only one $x(y)$ in X such that

$$A(x(y),y) = S(x(y),y) = B(x(y),y) = T(x(y),y) = x(y), \tag{1.45}$$

for every y, y' in X

$$
\begin{aligned}
g(F_{x(y),x(y'),a}(t)) &= g(F_{A(x(y),y),A(x(y'),y'),a}(t)) \\
&\le \phi(\max\{g(F_{A(x,y),A(x',y'),a}(t)), \\
&\qquad g(F_{A(x,y),A(x,y),a}(t)), g(F_{T(x',y'),A(x',y'),a}(t)), \\
&\qquad \frac{1}{2}(g(F_{A(x,y),A(x',y'),a}(t)) + g(F_{A(x',y'),A(x,y),a}(t)))\}) \\
&= g(F_{x(y),x(y'),a}(t)).
\end{aligned}
\tag{1.46}
$$

This implies that $x(y) = x(y')$ and hence $x(\cdot)$ is some constant $b \in X$ so that

$$
A(b,y) = b = T(b,y) = S(b,y) = B(b,y) \quad \forall y \text{ in } X. \tag{1.47}
$$

□

References

[1] Y. J. Cho, K. S. Ha, and S.-S. Chang, *Common fixed point theorems for compatible mappings of type (A) in non-Archimedean Menger PM-spaces*, Math. Japon. **46** (1997), no. 1, 169–179. CMP 1 466 131. Zbl 888.47038.

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