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ON THE RECURSIVE SEQUENCE $x_{n+1} = -1/x_n + A/x_{n-1}$

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ABSTRACT. We investigate the periodic character of solutions of the nonlinear difference equation $x_{n+1} = -1/x_n + A/x_{n-1}$. We give sufficient conditions under which every positive solution of this equation converges to a period two solution. This confirms a conjecture in the work of DeVault et al. (2000).

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1. Introduction. The behaviour of solutions of the difference equation

$$x_{n+1} = \frac{A}{x_n} + \frac{B}{x_{n-1}}, \tag{1.1}$$

where $A, B \in \mathbb{R}$ and the initial conditions are arbitrary real numbers, has not been thoroughly understood yet.

It was shown in [9], see also [3], that in the case $A, B > 0$ every positive solution of this equation converges to the positive equilibrium $\sqrt{A+B}$. Clearly in this case every negative solution of this equation converges to the negative equilibrium $-\sqrt{A+B}$.

In this paper, we consider the equation

$$x_{n+1} = -\frac{1}{x_n} + \frac{A}{x_{n-1}}, \tag{1.2}$$

where $A > 0$.

As it was observed in [2] for any initial conditions x_0 and x_1 with $x_0 x_1 < 0$, (1.2) is well defined for all $n \in \mathbb{N}$ and $x_n x_{n-1} < 0$, $n \in \mathbb{N}$.

Furthermore the sequence

$$\dots, -\sqrt{A+1}, \sqrt{A+1}, -\sqrt{A+1}, \sqrt{A+1}, \dots \tag{1.3}$$

is two periodic solution of (1.2). It is easy to see that this two cycle is locally asymptotically stable. When $A > 1$, (1.2) has the equilibrium $\sqrt{A-1}$, which is unstable.

Recently there has been a lot of interest in studying the global attractivity, the boundedness character and the periodic nature of nonlinear difference equations. For some recent results concerning, among other problems, the periodic nature of scalar nonlinear difference equations see, for example, [1, 3, 4, 5, 6, 9, 10, 11]. For several open problems and conjectures in this area see [2, 7, 8]. In [5, 10] two closely related, global convergence results, were established which can be applied in considerations of nonlinear difference equations for proving that every solution of these difference equations converges to a period two solution.

In [2], the following conjecture was stated.

Conjecture 1.1. Let (x_n) be a nonequilibrium solution of (1.2) which is well defined for all $n \in \mathbb{N}$. Show that (x_n) converges to the two cycle (1.3).

The main purpose of this paper is to confirm this conjecture when $A \in (0,1]$.

2. A global convergence result.

In proving the main result we need a global convergence result, which is contained in the following theorem.

Theorem 2.1. *Let the sequences (y_n) and (z_n) of positive numbers satisfy the following system*

$$\begin{aligned} y_{n+1} &= F_1(y_n, z_n, y_{n-1}, z_{n-1}), \\ z_{n+1} &= F_2(y_n, z_n, y_{n-1}, z_{n-1}). \end{aligned} \tag{2.1}$$

Assume that for $i = \overline{1,2}$,

(a) *$F_i(u_1, u_2, u_3, u_4)$ is continuous on $\mathbb{R}_+^4$;*

(b) *F_i, is nondecreasing in each of its variables and strictly increasing in the second and third variable;*

(c)
$$F_i(u_1, u_2, u_3, u_4) \le \max\{u_1, u_2, u_3, u_4\}, \tag{2.2}$$
for $u_1, u_2, u_3, u_4 \ge 0$.

Then the sequences (y_n) and (z_n) are convergent.

Proof. From (c) we can see that the sequences (y_n) and (z_n) are bounded. It follows that

$$\begin{aligned} \liminf_{n\to\infty} y_n &= l_y, \qquad \limsup_{n\to\infty} y_n = L_y, \\ \liminf_{n\to\infty} z_n &= l_z, \qquad \limsup_{n\to\infty} z_n = L_z, \end{aligned} \tag{2.3}$$

are finite.

Let $L = \max\{L_y, L_z\}$ and $l_z < L_z$. Since F_i is strictly increasing in the second variable, then there exists $\delta \in (0, L)$ such that

$$F_i(L, l_z, L, L) + \delta < F_i(L, L, L, L) - \delta \tag{2.4}$$

for $i = \overline{1,2}$.

Since F_i is continuous on $\mathbb{R}_+^4$, then for every $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and $i = \overline{1,2}$

$$F_i(L+\varepsilon, l_z+\varepsilon, L+\varepsilon, L+\varepsilon) < F_i(L, l_z, L, L) + \delta. \tag{2.5}$$

In view of the definition of inferior limit, for every $\varepsilon > 0$ and every $n_0 \in \mathbb{N}$ there is $n_1 \ge n_0$ such that

$$z_{n_1} < l_z + \varepsilon. \tag{2.6}$$

Similarly, for every $\varepsilon > 0$ there is $n_2 \in \mathbb{N}$ such that

$$y_n < L + \varepsilon, \qquad z_n < L + \varepsilon, \tag{2.7}$$

for all $n \geq n_2$ and $i = \overline{1,2}$.

Choose $\delta \in (0, L)$ such that (2.4) holds. Choose $\varepsilon = \varepsilon_1$ such that $\varepsilon_1 < \min\{\varepsilon_0, (L - l_z)/2\}$ and find $n_2 + 1 = n_1 = n_1(\varepsilon_1)$ so that (2.6) and (2.7) hold. From (2.1), using (2.4), (2.5), (2.6), (2.7) and by condition (b) we obtain

$$\begin{aligned} y_{n_1+1} &< F_1(L+\varepsilon, l_z+\varepsilon, L+\varepsilon, L+\varepsilon) \\ &< F_1(L, l_z, L, L) + \delta \\ &< F_1(L, L, L, L) - \delta \leq L - \delta. \end{aligned} \tag{2.8}$$

In the same manner we obtain

$$z_{n_1+1} < L - \delta. \tag{2.9}$$

Similarly, since F_i is strictly increasing in the second variable, there exists $\delta_1 \in (0, \delta)$ such that

$$F_i(L, L-\delta, L, L) + \delta_1 < F_i(L, L, L, L) - \delta_1 \tag{2.10}$$

for $i = \overline{1,2}$.

Also, since F_i is continuous on $\mathbb{R}_+^4$, for every $\delta_1 > 0$ there is $\varepsilon_2 > 0$ such that for every $\varepsilon \in (0, \varepsilon_2)$ and $i = \overline{1,2}$

$$F_i(L+\varepsilon, L-\delta, L+\varepsilon, L+\varepsilon) < F_i(L, L-\delta, L, L) + \delta_1. \tag{2.11}$$

We may assume that $\varepsilon_0 = \varepsilon_2$.

By (2.8), (2.9), (2.10), (2.11), and condition (b) we obtain

$$\begin{aligned} y_{n_1+2} &< F_1(L+\varepsilon, L-\delta, L+\varepsilon, L+\varepsilon) \\ &< F_1(L, L-\delta, L, L) + \delta_1 \\ &< F_1(L, L, L, L) - \delta_1 \leq L - \delta_1, \\ z_{n_1+2} &< L - \delta_1. \end{aligned} \tag{2.12}$$

From (2.8), (2.9), and (2.12), we obtain

$$\begin{aligned} y_{n_1+3} &< F_1(L-\delta_1, L-\delta_1, L-\delta, L-\delta) \leq L - \delta_1, \\ z_{n_1+3} &< F_1(L-\delta_1, L-\delta_1, L-\delta, L-\delta) \leq L - \delta_1. \end{aligned} \tag{2.13}$$

It is easy to show by induction that

$$\begin{aligned} y_n &\leq F_1(L-\delta_1, L-\delta_1, L-\delta_1, L-\delta_1) \leq L - \delta_1, \\ z_n &\leq F_2(L-\delta_1, L-\delta_1, L-\delta_1, L-\delta_1) \leq L - \delta_1, \end{aligned} \tag{2.14}$$

for all $n \geq n_1$.

From (2.14) we obtain

$$\limsup_{n\to\infty} y_n \leq L - \delta_1, \qquad \limsup_{n\to\infty} z_n \leq L - \delta_1, \tag{2.15}$$

which is a contradiction with our choice of L. Thus $l_z = L_z$ and so the sequence (z_n) converges. The proof that the sequence (y_n) converges is similar and will be omitted. □

3. Main result. We are now in position to confirm Conjecture 1.1.

Theorem 3.1. *Assume $A \in (0,1]$, then every nonequilibrium solution of (1.2), which is well defined for all $n \in \mathbb{N}$, converges to the two periodic solution (1.3).*

Proof. First we prove that the sequence (x_n) eventually satisfies the condition $x_n x_{n-1} < 0$. As it was observed in [2], if $x_{n_0} x_{n_0-1} < 0$ for some $n_0 \in \mathbb{N}$, then $x_n x_{n-1} < 0$, $n \geq n_0$. Suppose the contrary, then we may suppose that $x_n > 0$, $n \in \mathbb{N}$ or $x_n < 0$, $n \in \mathbb{N}$.

In the first case, by (1.2) we obtain

$$x_{n+1} = \frac{Ax_n - x_{n-1}}{x_n x_{n-1}} > 0, \quad n \in \mathbb{N}. \tag{3.1}$$

Thus we have $0 < x_{n-1} < Ax_n \leq x_n$, $n \in \mathbb{N}$. From that it follows that there exists finite or infinite $\lim_{n\to\infty} x_n = a > 0$. Letting $n \to \infty$ in (1.2) we obtain $a = (A-1)/a$, which is impossible. Thus the result follows in that case.

In the second case, we obtain $0 < -x_{n-1} < A(-x_n) \leq -x_n$, $n \in \mathbb{N}$. The rest of the proof is similar to the previous one and so we omit it. □

In the sequel, we may suppose that $x_0 < 0$ and $x_1 > 0$. By induction we obtain $x_{2n} < 0$ and $x_{2n-1} > 0$ for all $n \in \mathbb{N}$.

Set $y_n = x_{2n+1}$ and $z_n = -x_{2n}$, $n \in \mathbb{N}$. Since

$$x_{n+1} = -\frac{1}{-1/x_{n-1} + A/x_{n-2}} + \frac{A}{-1/x_{n-2} + A/x_{n-3}}, \tag{3.2}$$

we obtain

$$y_{n+1} = \frac{1}{1/y_n + A/z_n} + \frac{A}{1/z_n + A/y_{n-1}}, \tag{3.3}$$

$$z_{n+1} = \frac{1}{1/z_n + A/y_{n-1}} + \frac{A}{1/y_{n-1} + A/z_{n-1}}. \tag{3.4}$$

Let

$$F_1(u_1,u_2,u_3,u_4) = \begin{cases} \dfrac{1}{1/u_1 + A/u_2} + \dfrac{A}{1/u_2 + A/u_3}, & \text{for } u_1u_2u_3 > 0; \\ \dfrac{A}{1/u_2 + A/u_3}, & \text{for } u_1 = 0,\ u_2u_3 > 0; \\ \dfrac{1}{1/u_1 + A/u_2}, & \text{for } u_3 = 0,\ u_1u_2 > 0; \\ 0, & \text{for } u_2 = 0;\ u_1 = u_3 = 0,\ u_2 > 0; \\ & \quad u_1 = u_2 = u_3 = 0; \end{cases}$$

$$F_2(u_1,u_2,u_3,u_4) = \begin{cases} \dfrac{1}{1/u_2+A/u_3} + \dfrac{A}{1/u_3+A/u_4}, & \text{for } u_2u_3u_4 > 0; \\ \dfrac{A}{1/u_3+A/u_4}, & \text{for } u_2 = 0,\ u_3u_4 > 0; \\ \dfrac{1}{1/u_2+A/u_3}, & \text{for } u_4 = 0,\ u_2u_3 > 0; \\ 0, & \text{for } u_3 = 0;\ u_2 = u_4 = 0,\ u_3 > 0; \\ & \quad u_2 = u_3 = u_4 = 0. \end{cases} \tag{3.5}$$

Since, for both F_1 and F_2, all the conditions of Theorem 2.1 are satisfied we have that the sequences (y_n) and (z_n) converge. Let $\lim_{n\to\infty} y_n = y \ge 0$ and $\lim_{n\to\infty} z_n = z \ge 0$. Letting $n \to \infty$ in (3.3), (3.4) we obtain

$$y = \frac{1}{1/y+A/z} + \frac{A}{1/z+A/y}, \qquad z = \frac{1}{1/z+A/y} + \frac{A}{1/y+A/z}. \tag{3.6}$$

From (3.6) it follows that $y = z$. If $y = 0$ we see that $x_n \to 0$ as $n \to \infty$. Letting $n \to \infty$ in (1.2) and by applying the condition $x_n x_{n-1} < 0$ we get a contradiction. In the other case we obtain $y = z = \sqrt{A+1}$. Thus the result follows.

References

[1] A. M. Amleh, E. A. Grove, G. Ladas, and D. A. Georgiou, *On the recursive sequence $x_{n+1} = \alpha + x_{n-1}/x_n$*, J. Math. Anal. Appl. **233** (1999), no. 2, 790–798. MR 2000f:39002. Zbl 962.39004.

[2] R. DeVault, L. Galminas, E. J. Janowski, and G. Ladas, *On the recursive sequence $x_{n+1} = A/x_n + B/x_{n-1}$*, J. Differ. Equations Appl. **6** (2000), no. 1, 121–125. CMP 1 752 159. Zbl 956.39002.

[3] R. DeVault, G. Ladas, and S. W. Schultz, *On the recursive sequence $x_{n+1} = A/x_n^p + B/x_{n-1}^q$*, Advances in Difference Equations (Veszprém, 1995), Gordon and Breach, Amsterdam, 1997, The Proceedings of the Second International Conference on Difference Equations, pp. 125–136. MR 99e:39013. Zbl 890.39013.

[4] ______, *On the recursive sequence $x_{n+1} = A/x_n + 1/x_{n-2}$*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3257–3261. MR 99a:39009. Zbl 904.39012.

[5] H. El-Metwally, E. A. Grove, and G. Ladas, *A global convergence result with applications to periodic solutions*, J. Math. Anal. Appl. **245** (2000), no. 1, 161–170. MR 2001g:39014. Zbl 992.20962.

[6] M. R. S. Kulenović, G. Ladas, and W. S. Sizer, *On the recursive sequence $x_{n+1} = (\alpha x_n + \beta x_{n-1})/(\gamma x_n + \delta x_{n-1})$*, Math. Sci. Res. Hot-Line **2** (1998), no. 5, 1–16. MR 99g:39008. Zbl 960.39502.

[7] G. Ladas, *On the recursive sequence $x_{n+1} = \sum_{i=0}^{k} A_i/(x_{n-2i})$*, J. Differ. Equations Appl. **4** (1998), no. 3, 311–313. Zbl 01208767.

[8] ______, *Progress report on $\chi_{n+1} = (\alpha + \beta\chi_n + \gamma\chi_{n-1})/(A + B\chi_n + C\chi_{n-1})$*, J. Differ. Equations Appl. **5** (1999), no. 2, 211–215.

[9] C. G. Philos, I. K. Purnaras, and Y. G. Sficas, *Global attractivity in a nonlinear difference equation*, Appl. Math. Comput. **62** (1994), no. 2-3, 249–258. MR 95h:39008. Zbl 817.39005.

[10] S. Stević, *A generalization of the Copson's theorem concerning sequences which satisfies linear inequality*, to appear in Indian J. Math.

[11] ______, *On the recursive sequence* $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$, to appear in Appl. Math. Lett.

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