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EXISTENCE OF SOLUTIONS FOR NON-NECESSARILY COOPERATIVE SYSTEMS INVOLVING SCHRÖDINGER OPERATORS

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ABSTRACT. We study the existence of a solution for a non-necessarily cooperative system of n equations involving Schrödinger operators defined on $\mathbb{R}^N$ and we study also a limit case (the Fredholm Alternative (FA)). We derive results for semilinear systems.

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1. Introduction. We consider the following elliptic system defined on $\mathbb{R}^N$, for $1 \le i \le n$,

$$L_{q_i} u_i := (-\Delta + q_i) u_i = \sum_{j=1}^{n} a_{ij} u_j + f_i \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where n and N are two integers not equal to 0 and Δ is the Laplacian operator

(H1) for $1 \le i, j \le n$, $a_{ij} \in L^\infty(\mathbb{R}^N)$,

(H2) for $1 \le i \le n$, q_i is a continuous potential defined on $\mathbb{R}^N$ such that $q_i(x) \ge 1$, for all $x \in \mathbb{R}^N$ and $q_i(x) \to +\infty$ when $|x| \to +\infty$,

(H3) for $1 \le i \le n$, $f_i \in L^2(\mathbb{R}^N)$.

We do not make here any assumptions on the sign of a_{ij}. Recall that (1.1) is called cooperative if $a_{ij} \ge 0$ a.e. for $i \ne j$.

Our paper is organized as follow, in Section 2, we recall some results about M-matrices and about the maximum principle for cooperative systems involving Schrödinger operators $-\Delta + q_i$ in $\mathbb{R}^N$. In Section 3, we show the existence of a solution for a non-necessarily cooperative system of n equations. After that we study a limit case (FA) and finally we study the existence of a solution for a (non-necessarily cooperative) semilinear system.

2. Definitions and notations

2.1. M-matrix. We recall some results about the M-matrix (see [4, Theorem 2.3, page 134]). We say that a matrix is positive if all its coefficients are nonnegative and we say that a symmetric matrix is positive definite if all its principal minors are strictly positive.

DEFINITION 2.1 (see [4]). A matrix $M = sI - B$ is called a nonsingular M-matrix if B is a positive matrix (i.e., with nonnegative coefficients) and $s > \rho(B) > 0$ the spectral radius of B.

Proposition 2.2 (see [4]). *If M is a matrix with nonpositive off-diagonal coefficients, the conditions (P0), (P1), (P2), (P3), and (P4) are equivalent.*

(P0) *M is a nonsingular M-matrix,*

(P1) *all the principal minors of M are strictly positive,*

(P2) *M is semi-positive (i.e., there exists $X \gg 0$ such that $MX \gg 0$), where $X \gg 0$ signify for all i, $X_i > 0$ if $X = (X_1, \dots, X_n)$,*

(P3) *M has a positive inverse,*

(P4) *there exists a diagonal matrix D, $D > 0$, such that $MD + D^t M$ is positive definite.*

Remark 2.3. If M is a nonsingular M-matrix, then ${}^tM$ is also a nonsingular M-matrix.

So condition (P4) holds if and only if condition (P5) holds where (P5): there exists a diagonal matrix D, $D > 0$, such that ${}^tMD + DM$ is positive definite.

2.2. Schrödinger operators. Let $\mathcal{D}(\mathbb{R}^N) = \mathcal{C}_0^\infty(\mathbb{R}^N) = \mathcal{C}_c^\infty(\mathbb{R}^N)$ be the set of functions $\mathcal{C}^\infty$ on $\mathbb{R}^N$ with compact support.

Let q be a continuous potential defined on $\mathbb{R}^N$ such that $q(x) \geq 1$, for all $x \in \mathbb{R}^N$, and $q(x) \to +\infty$ when $|x| \to +\infty$. The variational space is, $V_q(\mathbb{R}^N)$, the completion of $\mathcal{D}(\mathbb{R}^N)$ for the norm $\|\cdot\|_q$ where $\|u\|_q = [\int_{\mathbb{R}^N} |\nabla u|^2 + q|u|^2]^{1/2}$

$$V_q(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N),\ \sqrt{q}u \in L^2(\mathbb{R}^N)\}, \tag{2.1}$$

$(V_q(\mathbb{R}^N), \|\cdot\|_q)$ is a Hilbert space. (See [1, Proposition I.1.1].)

Moreover, we have the following proposition.

Proposition 2.4 (see [1, Proposition I.1.1] and [8, Proposition 1, page 356]). *The embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact with dense range.*

To the form

$$a(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + \int_{\mathbb{R}^N} quv, \quad \forall (u,v) \in \left(V_q(\mathbb{R}^N)\right)^2, \tag{2.2}$$

we associate the operator $L_q := -\Delta + q$ defined on $L^2(\mathbb{R}^N)$ by variational methods.

Here $D(L_q)$ denotes the domain of the operator L_q. $D(L_q) = \{u \in V_q(\mathbb{R}^N), (-\Delta + q),\ u \in L^2(\mathbb{R}^N)\}$ (see [3, Theorem 1.1, page 4]).

We have that, for all $u \in D(L_q)$, for all $v \in V_q(\mathbb{R}^N)$, $a(u,v) = \int_{\mathbb{R}^N} L_q u \cdot v$. The embedding of $D(L_q)$ into $V_q(\mathbb{R}^N)$ is continuous and with dense range. (See [1, page 24] and [3, pages 5–6].)

Proposition 2.5 (see [1, pages 25–27]; [3, Theorem 1.1, pages 4, 6, 8, and 11]; [2, page 3, Theorem 3.2, page 45]; [7, pages 488–489]; [9, pages 346–350], and [10, Theorem XIII.16, page 120 and Theorem XIII.47, page 207]). *L_q, is considered as an operator in $L^2(\mathbb{R}^N)$, positive, selfadjoint, and with compact inverse. Its spectrum is discrete and consists of an infinite sequence of positive eigenvalues tending to $+\infty$. The smallest one, denoted by $\lambda(q)$, is simple and associated with an eigenfunction ϕ_q which does not change sign in $\mathbb{R}^N$. The eigenvalue $\lambda(q)$ is a principal eigenvalue if it is positive and simple.*

Furthermore,

$$L_q\phi_q = \lambda(q)\phi_q \quad \text{in } \mathbb{R}^N, \quad \phi_q(x) \longrightarrow 0 \quad \text{when } x \longrightarrow +\infty; \\ \phi_q > 0 \quad \text{in } \mathbb{R}^N; \quad \lambda(q) > 0, \tag{2.3}$$

$$\forall u \in V_q(\mathbb{R}^N), \quad \lambda(q)\int_{\mathbb{R}^N} |u|^2 \le \int_{\mathbb{R}^N} [|\nabla u|^2 + q|u|^2]. \tag{2.4}$$

Moreover, the equality holds if and only if u is collinear to ϕ_q. If $a \in L^\infty(\mathbb{R}^N)$, let $a^* = \sup_{x\in\mathbb{R}^N} a(x)$, $a_* = \inf_{x\in\mathbb{R}^N} a(x)$ and

$$\lambda(q-a) = \inf\left\{\frac{\int_{\mathbb{R}^N}[|\nabla\phi|^2 + (q-a)\phi^2]}{\int_{\mathbb{R}^N}\phi^2} \phi \in \mathcal{D}(\mathbb{R}^N)\phi \not\equiv 0\right\}. \tag{2.5}$$

The operator $-\Delta + q - a$ in $\mathbb{R}^N$ has a unique selfadjoint realization (see [2, page 3]) in $L^2(\mathbb{R}^N)$ which is denoted L_{q-a}. (Indeed, q is a continuous potential, $a \in L^\infty(\mathbb{R}^N)$, so the condition in [2] $(q-a)_- \in L^p_{\text{loc}}(\mathbb{R}^N)$ for a $p > N/2$ is satisfied.) We also note that $\lambda(q-a) \le \lambda(q) - a_*$ and for all $m \in \mathbb{R}^{*+}$, $\lambda(q-a+m) = \lambda(q-a) + m$.

The following theorem is classical.

Theorem 2.6 (see [1, 6, 10, page 204])**.** *Consider the equation*

$$(-\Delta + q)u = au + f \quad \text{in } \mathbb{R}^N, \quad \text{where } a \in \mathbb{R},\ f \in L^2(\mathbb{R}^N),\ f \ge 0 \tag{2.6}$$

and q is a continuous potential on $\mathbb{R}^N$ such that $q \ge 1$ and $q(x) \to +\infty$ when $|x| \to +\infty$. If $a < \lambda(q)$ then $\exists! u \in V_q(\mathbb{R}^N)$ solution of (2.6). Moreover, $u \ge 0$.

2.3. Cooperative systems. In this section, we consider the system (1.1) and we assume that it is cooperative, that is,

(H1*) $a_{ij} \in L^\infty(\mathbb{R}^N)$; $a_{ij} \ge 0$ a.e. for $i \ne j$.

We recall here a sufficient condition for the maximum principle and existence of solutions for such cooperative systems.

We say that (1.1) satisfies the maximum principle if for all $f_i \ge 0$, $1 \le i \le n$, any solution $u = (u_1,\dots,u_n)$ of (1.1) is nonnegative.

Let $E = (e_{ij})$ be the $n \times n$ matrix such that for all $1 \le i \le n$, $e_{ii} = \lambda(q_i - a_{ii})$, and for all $1 \le i, j \le n$, $i \ne j$ implies $e_{ij} = -a^*_{ij}$.

Theorem 2.7 (see [6])**.** *Assume that (H1*), (H2), and (H3) are satisfied. If E is a nonsingular M-matrix, then (1.1) satisfies the maximum principle.*

Theorem 2.8 (see [6])**.** *Assume that (H1*), (H2), and (H3) are satisfied. If E is a nonsingular M-matrix and if $f_i \ge 0$ for each $1 \le i \le n$, then (1.1) has a unique solution which is nonnegative.*

3. Study of a non-necessarily cooperative system

3.1. Study of a non-necessarily cooperative system of n equations with bounded coefficients. We adapt here an approximation method used in [5] for problems defined on bounded domains.

We consider the following elliptic system defined on $\mathbb{R}^N$; for $1 \le i \le n$,

$$L_{q_i}u_i := (-\Delta + q_i)u_i = \sum_{j=1}^{n} a_{ij}u_j + f_i \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

Let $G = (g_{ij})$ be the $n \times n$ matrix such that for all $1 \le i \le n$, $g_{ii} = \lambda(q_i - a_{ii})$ and for each $1 \le i$, $j \le n$, $i \ne j$ implies that $g_{ij} = -|a_{ij}|^*$, where $|a_{ij}|^* = \sup_{x\in\mathbb{R}^N} |a_{ij}(x)|$.

We make the following hypothesis:

(H) G is a nonsingular M-matrix.

Theorem 3.1. *Assume that (H1), (H2), (H3), and (H) are satisfied. Then system (1.1) has a weak solution $(u_1,\dots,u_n) \in V_{q_1}(\mathbb{R}^N)\times\cdots\times V_{q_n}(\mathbb{R}^N)$.*

First, we prove the following lemma.

Lemma 3.2. *Assume that (H), (H1), (H2), and (H3) are satisfied. Let $(u_1,\dots,u_n) \in V_{q_1}(\mathbb{R}^N)\times\cdots\times V_{q_n}(\mathbb{R}^N)$ be the solution of*

$$L_{q_i}u_i := (-\Delta + q_i)u_i = \sum_{j=1}^{n} a_{ij}u_j \quad \text{in } \mathbb{R}^N. \tag{3.2}$$

Then $(u_1,\dots,u_n) = (0,\dots,0)$.

Proof of Lemma 3.2. Let $m \in \mathbb{R}^{*+}$ be such that for all $1 \le i \le n$, $m - a_{ii} > 0$. Let $q_i' = q_i + m - a_{ii} \ge 1$. For any $1 \le i \le n$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\left[|\nabla u_i|^2 + q_i'|u_i|^2\right] &= \int_{\mathbb{R}^N} m|u_i|^2 + \sum_{j;j\ne i}\int_{\mathbb{R}^N} a_{ij}u_ju_i \\ &\le \int_{\mathbb{R}^N} m|u_i|^2 + \sum_{j;j\ne i}\int_{\mathbb{R}^N} |a_{ij}u_ju_i|,\end{aligned} \tag{3.3}$$

and by the characterization (2.4) of the first eigenvalue $\lambda(q_i')$ we get that $(\lambda(q_i') - m)\int_{\mathbb{R}^N}|u_i|^2 \le \sum_{j;j\ne i}|a_{ij}|^*(\int_{\mathbb{R}^N}|u_j|^2)^{1/2}(\int_{\mathbb{R}^N}|u_i|^2)^{1/2}$. So $(\lambda(q_i') - m)(\int_{\mathbb{R}^N}|u_i|^2)^{1/2} \le \sum_{j;j\ne i}|a_{ij}|^*(\int_{\mathbb{R}^N}|u_j|^2)^{1/2}$.

Let

$$X = \begin{pmatrix} \left(\int_{\mathbb{R}^N} u_1^2\right)^{1/2} \\ \vdots \\ \left(\int_{\mathbb{R}^N} u_n^2\right)^{1/2} \end{pmatrix}. \tag{3.4}$$

We have $X \ge 0$ and $GX \le 0$. Since G is a nonsingular M-matrix, by Proposition 2.2, we deduce that $X \le 0$. So $X = 0$, that is, for all $1 \le i \le n$, $u_i = 0$. □

Proof of Theorem 3.1. Let $m \in \mathbb{R}^{*+}$ such that for all $1 \le i \le n$, $m - a_{ii} > 0$. Let $q_i' = q_i - a_{ii} + m \ge 1$. ($m$ exists because for all $1 \le i \le n$, $a_{ii} \in L^\infty(\mathbb{R}^N)$.)

First, we note that $(u_1,\dots,u_n)\in V_{q_1}(\mathbb{R}^N)\times\cdots\times V_{q_n}(\mathbb{R}^N)$ is a weak solution of (1.1) if and only if $(u_1,\dots,u_n)$ is a weak solution of (3.5) where, for $1\le i\le n$,

$$(-\Delta+q_i')u_i=mu_i+\sum_{j;j\neq i}a_{ij}u_j+f_i \quad \text{in } \mathbb{R}^N. \tag{3.5}$$

Let $\epsilon\in]0,1[$, $B_\epsilon=B(0,1/\epsilon)=\{x\in\mathbb{R}^N,\ |x|<1/\epsilon\}$, and 1_{B_ϵ} be the indicator function of B_ϵ.

Let $T: L^2(\mathbb{R}^N)\times\cdots\times L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)\times\cdots\times L^2(\mathbb{R}^N)$ be defined by $T(\xi_1,\dots,\xi_n)=(\omega_1,\dots,\omega_n)$ where for any $1\le i\le n$,

$$(-\Delta+q_i')\omega_i=m\frac{\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon}+\sum_{j;j\neq i}a_{ij}\frac{\xi_j}{1+\epsilon|\xi_j|}1_{B_\epsilon}+f_i \quad \text{in } \mathbb{R}^N. \tag{3.6}$$

(i) First, we prove that T is well defined. Let for all $(\xi_1,\dots,\xi_n)\in L^2(\mathbb{R}^N)\times\cdots\times L^2(\mathbb{R}^N)$, for all $1\le i\le n$,

$$\psi_i(\xi_1,\dots,\xi_n)=m\frac{\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon}+\sum_{j;j\neq i}a_{ij}\frac{\xi_j}{1+\epsilon|\xi_j|}1_{B_\epsilon}. \tag{3.7}$$

We have

$$\left|\frac{\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon}\right|=\frac{1}{\epsilon}\left|\frac{\epsilon\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon}\right|\le\frac{1}{\epsilon}1_{B_\epsilon}. \tag{3.8}$$

Since $1_{B_\epsilon}\in L^2(\mathbb{R}^N)$ and $a_{ij}\in L^\infty(\mathbb{R}^N)$, we deduce that for any $1\le i\le n$, $\psi_i(\xi_1,\dots,\xi_n)\in L^2(\mathbb{R}^N)$. By (H3), $f_i\in L^2(\mathbb{R}^N)$ and therefore $\psi_i(\xi_1,\dots,\xi_n)+f_i\in L^2(\mathbb{R}^N)$.

By Theorem 2.6, we deduce the existence (and uniqueness) of $(\omega_1,\dots,\omega_n)\in V_{q_1}(\mathbb{R}^N)\times\cdots\times V_{q_n}(\mathbb{R}^N)$. So T is well defined.

(ii) We note that for all $(\xi_1,\dots,\xi_n)$, $|\psi_i(\xi_1,\dots,\xi_n)|\le n\max_{j;j\neq i}(m,|a_{ij}|^*)(1/\epsilon)1_{B_\epsilon}$.

Let $h=(n/\epsilon)\max_{i,j;i\neq j}(m,|a_{ij}|^*)$, $1_{B_\epsilon}\in L^2(\mathbb{R}^N)$, and $h+f_i\in L^2(\mathbb{R}^N)$, so, by the scalar case, we deduce that there exists a unique $\xi_i^0\in V_{q_i}(\mathbb{R}^N)$ such that $(-\Delta+q_i')\xi_i^0=h+f_i$ in $\mathbb{R}^N$, $(\xi_1^0,\dots,\xi_n^0)$ is an upper solution of (3.5), for all $1\le i\le n$,

$$(-\Delta+q_i')\xi_i^0\ge\psi_i(\xi_1,\dots,\xi_n)+f_i. \tag{3.9}$$

In the same way, we construct a lower solution of (3.5), for all $1\le i\le n$, there exists a unique $\xi_{i,0}\in V_{q_i}(\mathbb{R}^N)$ such that $(-\Delta+q_i')\xi_{i,0}=-h+f_i$ in $\mathbb{R}^N$, $(\xi_{1,0},\dots,\xi_{n,0})$ is a lower solution of (3.5), for all $1\le i\le n$,

$$(-\Delta+q_i')\xi_{i,0}\le\psi_i(\xi_1,\dots,\xi_n)+f_i. \tag{3.10}$$

We note that for all i, $\xi_{i,0}\le\xi_i^0$ (because $(-\Delta+q_i')(\xi_i^0-\xi_{i,0})=2h\ge 0$). We consider now the restriction of T, denoted by T^*, at $[\xi_{1,0},\xi_1^0]\times\cdots\times[\xi_{n,0},\xi_n^0]$. We prove that T^* has a fixed point by the Schauder fixed point theorem.

(iii) First, we prove that $[\xi_{1,0},\xi_1^0]\times\cdots\times[\xi_{n,0},\xi_n^0]$ is invariant by T^*. Let $(\xi_1,\dots,\xi_n)\in[\xi_{1,0},\xi_1^0]\times\cdots\times[\xi_{n,0},\xi_n^0]$. We put $T^*(\xi_1,\dots,\xi_n)=(\omega_1,\dots,\omega_n)$. We have $(-\Delta+q_i')(\xi_i^0-\omega_i)=h-\psi_i(\xi_1,\dots,\xi_n)\ge 0$. By the scalar case, we deduce that $\xi_i^0\ge\omega_i$ a.e. By the same way we get $(-\Delta+q_i')(\omega_i-\xi_{i,0})=\psi_i(\xi_1,\dots,\xi_n)+h\ge 0$ and $\omega_i\ge\xi_{i,0}$ a.e. So $[\xi_{1,0},\xi_1^0]\times\cdots\times[\xi_{n,0},\xi_n^0]$ is invariant by T^*.

(iv) We prove that T^* is a compact continuous operator. T^* is continuous if and only if for all i, ψ_i^* is continuous where ψ_i^* is the restriction of ψ_i to $[\xi_{1,0},\xi_1^0]\times\cdots\times[\xi_{n,0},\xi_n^0]$.

Let $(\xi_1,\dots,\xi_n)\in[\xi_{1,0},\xi_1^0]\times\cdots\times[\xi_{n,0},\xi_n^0]$. Let $(\xi_1^p,\dots,\xi_n^p)_p$ be a sequence in $[\xi_{1,0},\xi_1^0]\times\cdots\times[\xi_{n,0},\xi_n^0]$ converging to $(\xi_1,\dots,\xi_n)$ for $\|\cdot\|_{(L^2(\mathbb{R}^N))^n}$. We have for all $1\le i\le n$,

$$\left\|\frac{\xi_i^p}{1+\epsilon|\xi_i^p|}1_{B_\epsilon}-\frac{\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon}\right\|_{L^2(\mathbb{R}^N)}\le\frac{1}{\epsilon}\left\|\frac{\epsilon\xi_i^p}{1+\epsilon|\xi_i^p|}-\frac{\epsilon\xi_i}{1+\epsilon|\xi_i|}\right\|_{L^2(\mathbb{R}^N)}. \tag{3.11}$$

However, the function l defined on $\mathbb{R}$ by for all $x\in\mathbb{R}$, $l(x)=x/(1+|x|)$ is Lipschitz and satisfies for all $x,y\in\mathbb{R}$, $|l(x)-l(y)|\le|x-y|$. So

$$\left\|\frac{\xi_i^p}{1+\epsilon|\xi_i^p|}-\frac{\xi_i}{1+\epsilon|\xi_i|}\right\|_{L^2(\mathbb{R}^N)}\le\frac{1}{\epsilon}\|\epsilon\xi_i^p-\epsilon\xi_i\|_{L^2(\mathbb{R}^N)}=\|\xi_i^p-\xi_i\|_{L^2(\mathbb{R}^N)}. \tag{3.12}$$

Hence,

$$\frac{\xi_i^p}{1+\epsilon|\xi_i^p|}1_{B_\epsilon}-\frac{\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon}\longrightarrow 0\quad\text{in } L^2(\mathbb{R}^N)\text{ when } p\longrightarrow+\infty. \tag{3.13}$$

So ψ_i^* is continuous and therefore T^* is a continuous operator. Moreover, by Proposition 2.5, $(-\Delta+q_i')^{-1}$ is a compact operator. So T^* is compact.

(v) $[\xi_{1,0},\xi_1^0]\times\cdots\times[\xi_{n,0},\xi_n^0]$ is a closed convex subset. Hence, by the Schauder fixed point theorem, we deduce the existence of $(\xi_1,\dots,\xi_n)\in[\xi_{1,0},\xi_1^0]\times\cdots\times[\xi_{n,0},\xi_n^0]$ such that $T^*(\xi_1,\dots,\xi_n)=(\xi_1,\dots,\xi_n)$ for all i, ξ_i depends of ϵ, so we denote $\xi_i=u_{i,\epsilon}$ and $u_{1,\epsilon},\dots,u_{n,\epsilon}$ satisfy for $1\le i\le n$,

$$(-\Delta+q_i')u_{i,\epsilon}=m\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon}+\sum_{j;j\ne i}a_{ij}\frac{u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon}+f_i\quad\text{in } \mathbb{R}^N. \tag{3.14}$$

(vi) Now we prove that for all i, $(\epsilon u_{i,\epsilon})_\epsilon$ is a bounded sequence in $V_{q_i'}(\mathbb{R}^N)$. Let $\|u\|_{q_i'}=[\int_{\mathbb{R}^N}|\nabla u|^2+q_i'|u|^2]^{1/2}$. Multiply (3.14) by $\epsilon^2u_{i,\epsilon}$ and integrate over $\mathbb{R}^N$. So we get

$$\begin{aligned}\|\epsilon u_{i,\epsilon}\|_{q_i'}^2\le{}& m\int_{\mathbb{R}^N}\left|\frac{\epsilon u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon}\epsilon u_{i,\epsilon}\right|\\&+\sum_{j;j\ne i}|a_{ij}|^*\int_{\mathbb{R}^N}\left|\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon}\epsilon u_{i,\epsilon}\right|+\int_{\mathbb{R}^N}|\epsilon f_i\epsilon u_{i,\epsilon}|.\end{aligned} \tag{3.15}$$

But for all j, $|\epsilon u_{j,\epsilon}/(1+\epsilon|u_{j,\epsilon}|)|<1$. So there exists a strictly positive constant K such that $\|\epsilon u_{i,\epsilon}\|_{q_i'}^2\le K\|\epsilon u_{i,\epsilon}\|_{L^2(\mathbb{R}^N)}\le K\|\epsilon u_{i,\epsilon}\|_{q_i'}$ and therefore, $\|\epsilon u_{i,\epsilon}\|_{q_i'}\le K$.

(vii) We prove now that $\epsilon u_{i,\epsilon}\to 0$ when $\epsilon\to 0$ strongly in $L^2(\mathbb{R}^N)$ and weakly in $V_{q_i'}(\mathbb{R}^N)$. We know that the imbedding of $V_{q_i'}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact. The sequence $(\epsilon u_{i,\epsilon})_\epsilon$ is bounded in $V_{q_i'}(\mathbb{R}^N)$ so (for a subsequence), we deduce that there exist u_i^* such that $\epsilon u_{i,\epsilon}\to u_i^*$ when $\epsilon\to 0$ strongly in $L^2(\mathbb{R}^N)$ and weakly in $V_{q_i'}(\mathbb{R}^N)$. Multiplying (3.14) by ϵ, we get

$$(-\Delta+q_i')\epsilon u_{i,\epsilon}=m\frac{\epsilon u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon}+\sum_{j;j\ne i}a_{ij}\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon}+\epsilon f_i\quad\text{in } \mathbb{R}^N. \tag{3.16}$$

But $\epsilon u_{i,\epsilon} \to u_i^*$ weakly in $V_{q_i}(\mathbb{R}^N)$. So for all $\phi \in \mathcal{D}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} [\nabla(\epsilon u_{i,\epsilon}) \cdot \nabla \phi + q_i' \epsilon u_{i,\epsilon} \phi] \longrightarrow \int_{\mathbb{R}^N} [\nabla u_i^* \cdot \nabla \phi + q_i' u_i^* \phi] \quad \text{when } \epsilon \longrightarrow 0. \tag{3.17}$$

Moreover, for all $\phi \in \mathcal{D}(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} \epsilon f_i \phi \to 0$ when $\epsilon \to 0$. Moreover, we have for all j

$$\begin{aligned} &\left\| \frac{\epsilon u_{j,\epsilon}}{1+\epsilon |u_{j,\epsilon}|} 1_{B_\epsilon} - \frac{u_j^*}{1+|u_j^*|} \right\|_{L^2(\mathbb{R}^N)}^2 \\ &\quad = \int_{B_\epsilon} \left[\frac{\epsilon u_{j,\epsilon}}{1+\epsilon |u_{j,\epsilon}|} - \frac{u_j^*}{1+|u_j^*|} \right]^2 + \int_{\mathbb{R}^N - B_\epsilon} \left(\frac{u_j^*}{1+|u_j^*|} \right)^2. \end{aligned} \tag{3.18}$$

Since $|u_j^*/(1+|u_j^*|)| \le |u_j^*|$, $u_j^*/(1+|u_j^*|) \in L^2(\mathbb{R}^N)$, hence $\int_{\mathbb{R}^N - B_\epsilon} (u_j^*/(1+|u_j^*|))^2 \to 0$ when $\epsilon \to 0$. Moreover,

$$\begin{aligned} \int_{B_\epsilon} \left[\frac{\epsilon u_{j,\epsilon}}{1+\epsilon |u_{j,\epsilon}|} - \frac{u_j^*}{1+|u_j^*|} \right]^2 &\le \int_{\mathbb{R}^N} \left[\frac{\epsilon u_{j,\epsilon}}{1+\epsilon |u_{j,\epsilon}|} - \frac{u_j^*}{1+|u_j^*|} \right]^2 \\ &\le \| \epsilon u_{j,\epsilon} - u_j^* \|_{L^2(\mathbb{R}^N)}^2. \end{aligned} \tag{3.19}$$

But $\epsilon u_{j,\epsilon} \to u_j^*$ when $\epsilon \to 0$ strongly in $L^2(\mathbb{R}^N)$. So, $(\epsilon u_{j,\epsilon}/1+\epsilon|u_{j,\epsilon}|)1_{B_\epsilon} \to u_j^*/(1+|u_j^*|)$ when $\epsilon \to 0$ strongly in $L^2(\mathbb{R}^N)$. Therefore, we can pass through the limit and we get for all $1 \le i \le n$,

$$(-\Delta + q_i') u_i^* = m \frac{u_i^*}{1+|u_i^*|} + \sum_{j;j \ne i} a_{ij} \frac{u_j^*}{1+|u_j^*|} \quad \text{in } \mathbb{R}^N. \tag{3.20}$$

We prove now that for any i, $u_i^* = 0$. Multiply (3.20) by u_i^*, integrate over $\mathbb{R}^N$, and obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \left[|\nabla u_i^*|^2 + q_i' |u_i^*|^2 \right] &= \int_{\mathbb{R}^N} m \frac{|u_i^*|^2}{1+|u_i^*|} + \sum_{j;j \ne i} \int_{\mathbb{R}^N} a_{ij} \frac{u_j^* u_i^*}{1+|u_j^*|} \\ &\le \int_{\mathbb{R}^N} m \frac{|u_i^*|^2}{1+|u_i^*|} + \sum_{j;j \ne i} \int_{\mathbb{R}^N} |a_{ij}|^* \frac{|u_j^*| |u_i^*|}{1+|u_j^*|}. \end{aligned} \tag{3.21}$$

But for all j, $1/(1+|u_j^*|) \le 1$. So we get

$$\lambda(q_i') \int_{\mathbb{R}^N} |u_i^*|^2 \le m \int_{\mathbb{R}^N} |u_i^*|^2 + \sum_{j;j \ne i} |a_{ij}|^* \left(\int_{\mathbb{R}^N} |u_j^*|^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} |u_i^*|^2 \right)^{1/2}. \tag{3.22}$$

Replacing u_i by u_i^*, we proceed exactly as in Lemma 3.2 and we get that for all $1 \le i \le n$, $u_i^* = 0$.

(viii) We prove now by contradiction that for all $1 \le i \le n$, $(u_{i,\epsilon})_\epsilon$ is bounded in $V_{q_i}(\mathbb{R}^N)$. We suppose that there exists i_0, $\|u_{i_0,\epsilon}\|_{q_{i_0}} \to +\infty$ when $\epsilon \to 0$. Let for all $1 \le i \le n$,

$$t_\epsilon = \max_i \left(\|u_{i,\epsilon}\|_{q_i} \right), \qquad v_{i,\epsilon} = \frac{1}{t_\epsilon} u_{i,\epsilon}. \tag{3.23}$$

We have $\|v_{i,\epsilon}\|_{q_i} \le 1$ so $(v_{i,\epsilon})_\epsilon$ is a bounded sequence in $V_{q_i}(\mathbb{R}^N)$. Since the imbedding of $V_{q_i}(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ is compact (see Proposition 2.4), there exists v_i such that $v_{i,\epsilon} \to v_i$ when $\epsilon \to 0$ strongly in $L^2(\mathbb{R}^N)$ and weakly in $V_{q_i}(\mathbb{R}^N)$.

In a weak sense, we have for all $1 \le i \le n$,

$$(-\Delta + q_i')v_{i,\epsilon} = m\frac{v_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon} + \sum_{j;j\neq i} a_{ij}\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon} + \frac{1}{t_\epsilon}f_i \quad \text{in } \mathbb{R}^N. \tag{3.24}$$

We have for all $\phi \in \mathcal{D}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}[\nabla v_{i,\epsilon}\cdot\nabla\phi + q_i'v_{i,\epsilon}\phi] \longrightarrow \int_{\mathbb{R}^N}[\nabla v_i\cdot\nabla\phi + q_i'v_i\phi] \quad \text{when } \epsilon \longrightarrow 0. \tag{3.25}$$

Moreover, $t_\epsilon \to +\infty$ when $\epsilon \to 0$ so, for all $\phi \in \mathcal{D}(\mathbb{R}^N)$, $\int_{\mathbb{R}^N}(1/t_\epsilon)f_i\phi \to 0$ when $\epsilon \to 0$. We also have for all $1 \le j \le n$,

$$\left\|\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon} - v_j\right\|^2_{L^2(\mathbb{R}^N)} = \int_{B_\epsilon}\left[\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - v_j\right]^2 + \int_{\mathbb{R}^N - B_\epsilon} v_j^2. \tag{3.26}$$

But $v_j \in L^2(\mathbb{R}^N)$ so, $\int_{\mathbb{R}^N - B_\epsilon} v_j^2 \to 0$ when $\epsilon \to 0$. Moreover,

$$\begin{aligned}\int_{B_\epsilon}\left[\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - v_j\right]^2 &\le \int_{\mathbb{R}^N}\left[\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - v_j\right]^2 \\ &\le 2\left[\int_{\mathbb{R}^N}\frac{(v_{j,\epsilon}-v_j)^2}{(1+\epsilon|u_{j,\epsilon}|)^2} + \int_{\mathbb{R}^N}\frac{(\epsilon v_j|u_{j,\epsilon}|)^2}{(1+\epsilon|u_{j,\epsilon}|)^2}\right].\end{aligned} \tag{3.27}$$

But $1+\epsilon|u_{j,\epsilon}| \ge 1$. So, $\int_{\mathbb{R}^N}(v_{j,\epsilon}-v_j)^2/(1+\epsilon|u_{j,\epsilon}|)^2 \le \int_{\mathbb{R}^N}(v_{j,\epsilon}-v_j)^2$. Since $v_{j,\epsilon} \to v_j$ in $L^2(\mathbb{R}^N)$, we get $\int_{\mathbb{R}^N}(v_{j,\epsilon}-v_j)^2/(1+\epsilon|u_{j,\epsilon}|)^2 \to 0$ when $\epsilon \to 0$. Moreover,

$$\frac{(\epsilon v_j|u_{j,\epsilon}|)^2}{(1+\epsilon|u_{j,\epsilon}|)^2} \longrightarrow 0 \quad \text{a.e. when } \epsilon \longrightarrow 0. \tag{3.28}$$

(At least for a subsequence because $\epsilon u_{j,\epsilon} \to 0$ when $\epsilon \to 0$.) By using the dominated convergence theorem, we deduce that $\int_{\mathbb{R}^N}(\epsilon v_j|u_{j,\epsilon}|)^2/(1+\epsilon|u_{j,\epsilon}|)^2 \to 0$ when $\epsilon \to 0$. So we can pass through the limit and we get for all $1 \le i \le n$,

$$(-\Delta + q_i')v_i = mv_i + \sum_{j;j\neq i} a_{ij}v_j \quad \text{in } \mathbb{R}^N. \tag{3.29}$$

By Lemma 3.2, we deduce that for all $1 \le i \le n$, $v_i = 0$. However, there exists a sequence (ϵ_n) such that there exists i_1, $\|v_{i_1,\epsilon_n}\|_{q_{i_1}} = 1$. But $v_{i_1,\epsilon_n} \to v_{i_1}$ when $n \to +\infty$. So we get a contradiction.

(ix) There exists u_i^0 such that $u_{i,\epsilon} \to u_i^0$ strongly in $L^2(\mathbb{R}^N)$ and weakly in $V_{q_i}(\mathbb{R}^N)$. We have in a weak sense

$$(-\Delta + q_i')u_{i,\epsilon} = m\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon} + \sum_{j;j\neq i} a_{ij}\frac{u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon} + f_i \quad \text{in } \mathbb{R}^N. \tag{3.30}$$

But $u_{i,\epsilon} \rightharpoonup u_i^0$ when $\epsilon \to 0$ weakly in $V_{q_i}(\mathbb{R}^N)$. Hence, for all $\phi \in \mathcal{D}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} [\nabla u_{i,\epsilon}.\nabla\phi + q_i' u_{i,\epsilon}\phi] \longrightarrow \int_{\mathbb{R}^N} [\nabla u_i^0 \cdot \nabla\phi + q_i' u_i^0 \phi] \quad \text{when } \epsilon \longrightarrow 0. \tag{3.31}$$

We also have

$$\left\| \frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} 1_{B_\epsilon} - u_i^0 \right\|^2_{L^2(\mathbb{R}^N)} = \int_{B_\epsilon} \left[\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} - u_i^0 \right]^2 + \int_{\mathbb{R}^N - B_\epsilon} |u_i^0|^2. \tag{3.32}$$

By $u_i^0 \in L^2(\mathbb{R}^N)$ we derive $\int_{\mathbb{R}^N-B_\epsilon} |u_i^0|^2 \to 0$ when $\epsilon \to 0$. Moreover,

$$\begin{aligned} \int_{B_\epsilon} \left[\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} - u_i^0 \right]^2 &\le \int_{\mathbb{R}^N} \left[\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} - u_i^0 \right]^2 \\ &\le 2\left[\int_{\mathbb{R}^N} \frac{(u_{i,\epsilon}-u_i^0)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} + \int_{\mathbb{R}^N} \frac{(\epsilon u_i^0|u_{i,\epsilon}|)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \right]. \end{aligned} \tag{3.33}$$

Since $1+\epsilon|u_{i,\epsilon}| \ge 1$ we get $\int_{\mathbb{R}^N} (u_{i,\epsilon}-u_i^0)^2/(1+\epsilon|u_{i,\epsilon}|)^2 \le \int_{\mathbb{R}^N} (u_{i,\epsilon}-u_i^0)^2$. But $u_{i,\epsilon} \to u_i^0$ in $L^2(\mathbb{R}^N)$. So $\int_{\mathbb{R}^N} (u_{i,\epsilon}-u_i^0)^2/(1+\epsilon|u_{i,\epsilon}|)^2 \to 0$ when $\epsilon \to 0$. Moreover,

$$\frac{(\epsilon u_i^0|u_{i,\epsilon}|)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \longrightarrow 0 \quad \text{a.e. when } \epsilon \longrightarrow 0. \tag{3.34}$$

(At least for a subsequence because $\epsilon u_{i,\epsilon} \to 0$ when $\epsilon \to 0$) and $(\epsilon u_i^0|u_{i,\epsilon}|)^2/(1+\epsilon|u_{i,\epsilon}|)^2 \le |u_i^0|^2$ and $|u_i^0|^2 \in L^1(\mathbb{R}^N)$.

By using the dominated convergence theorem, we deduce that

$$\int_{\mathbb{R}^N} \frac{(\epsilon u_i^0|u_{i,\epsilon}|)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \longrightarrow 0 \quad \text{when } \epsilon \longrightarrow 0. \tag{3.35}$$

So we can pass through the limit and we get for all $1 \le i \le n$,

$$(-\Delta + q_i')u_i^0 = m u_i^0 + \sum_{j;j\ne i} a_{ij} u_j^0 + f_i \quad \text{in } \mathbb{R}^N. \tag{3.36}$$

So we get $(-\Delta+q_i)u_i^0 = a_{ii}u_i^0 + \sum_{j;j\ne i} a_{ij}u_j^0 + f_i$ in $\mathbb{R}^N$, $(u_1^0,\dots,u_n^0)$ is a weak solution of (1.1). □

3.2. Study of a limit case. We use again a method in [5]. We rewrite system (1.1), assuming for all $1 \le i \le n$, $q_i = q$

$$L_q u_i := (-\Delta + q)u_i = \sum_{j=1}^n a_{ij}u_j + f_i(x,u_1,\dots,u_n) \quad \text{in } \mathbb{R}^N. \tag{3.37}$$

Each a_{ij} is a real constant. We denote $A = (a_{ij})$ the $n\times n$ matrix, I the $n \times n$ identity matrix, ${}^tU = (u_1,\dots,u_n)$ and ${}^tF = (f_1,\dots,f_n)$.

Theorem 3.3. *Suppose that (H1), (H2), and (H3) are satisfied. Suppose that A has only real eigenvalues. Suppose also that $\lambda(q)$, the principal eigenvalue of $-\Delta+q$, is the largest eigenvalue of A and that it is simple.*

Let $X \in \mathbb{R}^N$ such that ${}^tX(\lambda(q)I-A)=0$. Then (3.37) has a solution if and only if $\int_{\mathbb{R}^N} {}^tXF\phi_q=0$, where ϕ_q is the eigenfunction associated to $\lambda(q)$.

Proof of Theorem 3.3. Let P be a $n\times n$ nonsingular matrix such that the last line of P is ${}^tX$ and such that $T=PAP^{-1}:=(t_{ij})$ where, $t_{ij}=0$ if $i>j$; $t_{nn}=\lambda(q)$ and for all $1\le i\le n-1$, $t_{ii}<\lambda(q)$.

Let $W=PU$. The system (3.37) is equivalent to the system (3.2) $(-\Delta+q)W=TW+PF$. Let ${}^tW=(w_1,\dots,w_n)$ and $\pi_i=(\delta_{ij})$ where, $\delta_{ij}=0$ if $i\neq j$ and $\delta_{ii}=1$. So (3.2) is

$$L_qw_i:=(-\Delta+q)w_i=t_{ii}w_i+\sum_{j;j>i}t_{ij}w_j+\pi_iPF \quad \text{in } \mathbb{R}^N, \tag{3.38}$$

for $1\le i\le n$. We have

$$(-\Delta+q)w_n=\lambda(q)w_n+{}^tXF \quad \text{in } \mathbb{R}^N. \tag{3.39}$$

Equation (3.39) has a solution if and only if $\int_{\mathbb{R}^N} {}^tXF\phi_q=0$. If $\int_{\mathbb{R}^N} {}^tXF\phi_q=0$ is satisfied, first we solve $(2n)$, then we solve $(2n-1)$ until $n=1$ because for all $1\le i\le n-1$, $t_{ii}<\lambda(q)$. Then we deduce U (because matrix P is a nonsingular matrix). □

3.3. Study of a non-necessarily cooperative semilinear system of n equations. We rewrite system (3.37), for $1\le i\le n$,

$$L_{q_i}u_i:=(-\Delta+q_i)u_i=\sum_{j=1}^{n}a_{ij}u_j+f_i(x,u_1,\dots,u_n) \quad \text{in } \mathbb{R}^N. \tag{3.40}$$

We recall that the $n\times n$ matrix $G=(g_{ij})$ defined by $g_{ii}=\lambda(q_i-a_{ii})$, for all $1\le i\le n$, and

$$\forall 1\le i,\ j\le n,\ i\neq j \Longrightarrow g_{ij}=-|a_{ij}|^*, \quad \text{where } |a_{ij}|^*=\sup_{x\in\mathbb{R}^N}|a_{ij}(x)|. \tag{3.41}$$

Let I be the identity matrix.

Theorem 3.4. *Assume that (H1), (H2), and (H3) are satisfied. Also assume that hypothesis (H4), (H5), and (H6) are satisfied, where*

(H4) $\exists s>0$ *such that $F-sI$ is a nonsingular M-matrix,*

(H5) *for all $1\le i\le n$, $\exists\theta_i\in L^2(\mathbb{R}^N)$, $\theta_i>0$, such that for all $1\le i\le n$, for all $u_1,\dots,u_n$, $0\le f_i(x,u_1,\dots,u_n)\le su_i+\theta_i$,*

(H6) *for all $1\le i\le n$, f_i is Lipschitz for $(u_1,\dots,u_n)$, uniformly in x.*

Then (3.40) has at least a solution.

Proof of Thorem 3.4. (a) Construction of an upper and lower solution. We consider the following system (3.42)

$$\forall 1\le i\le n, \quad L_{q_i}u_i:=(-\Delta+q_i)u_i=a_{ii}u_i+\sum_{j;j\neq i}|a_{ij}|u_j+su_i+\theta_i \quad \text{in } \mathbb{R}^N. \tag{3.42}$$

By hypothesis (H4) and (H5) we can apply Theorem 2.8. We deduce the existence of a positive solution $U^0 = (u_1^0, \dots, u_n^0)$ in $V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$ for the system (3.42). U^0 is an upper solution of (3.40).

Let $U_0 = -U^0 = (-u_1^0, \dots, -u_n^0)$. We have for all $1 \le i \le n$, $(-\Delta + q_i)(-u_i^0) = -(-\Delta + q_i)u_i^0$. Hence, $(-\Delta + q_i)(-u_i^0) = -a_{ii}u_i^0 - \sum_{j;j\neq i}|a_{ij}|u_j^0 - su_i^0 - \theta_i$. So, for all $1 \le i \le n$,

$$(-\Delta + q_i)(-u_i^0) \le a_{ii}(-u_i^0) + \sum_{j;j\neq i} a_{ij}(-u_j^0) + f_i(x, -u_1^0, \dots, -u_n^0). \tag{3.43}$$

Therefore, U_0 is a lower solution of (3.40).

(b) We first recall the definition of a compact operator. Let $m \in \mathbb{R}^{*+}$ be such that for all $1 \le i \le n$, $m - a_{ii} > 0$. Let $q_i' = q_i - a_{ii} + m$. Let $T : (L^2(\mathbb{R}^N))^n \to (L^2(\mathbb{R}^N))^n$ defined by $T(u_1, \dots, u_n) = (w_1, \dots, w_n)$ such that for all $1 \le i \le n$,

$$(-\Delta + q_i')w_i = mu_i + \sum_{j=1;j\neq i}^{n} a_{ij}u_j + f_i(x, u_1, \dots, u_n) \quad \text{in } \mathbb{R}^N. \tag{3.44}$$

We easily prove that T is a well-defined operator by the scalar case, continuous by (H6) and compact (because $(-\Delta + q_i')^{-1}$ is compact). We prove now that $T([U_0, U^0]) \subset [U_0, U^0]$. Let $U = (u_1, \dots, u_n) \in [U_0, U^0]$. We have for all $1 \le i \le n$, $-u_i^0 \le u_i \le u_i^0$. We have

$$\begin{aligned}(-\Delta + q_i')(u_i^0 - w_i) = m(u_i^0 - u_i) &+ \sum_{j;j\neq i}|a_{ij}|u_j^0 \\ &- \sum_{j;j\neq i} a_{ij}u_j + su_i^0 + \theta_i - f_i(x, u_1, \dots, u_n).\end{aligned} \tag{3.45}$$

So $m(u_i^0 - u_i) \ge 0$. By (H5), we have $f_i(x, u_1, \dots, u_n) \le su_i + \theta_i \le su_i^0 + \theta_i$. Moreover, $|a_{ij}u_j| \le |a_{ij}|u_j^0$ so, $a_{ij}u_j \le |a_{ij}|u_j^0$. So, $(-\Delta + q_i')(u_i^0 - w_i) \ge 0$ and by the scalar case $u_i^0 - w_i \ge 0$. In the same way, we have

$$\begin{aligned}(-\Delta + q_i')(w_i - (-u_i^0)) = m(u_i^0 + u_i) &+ \sum_{j;j\neq i}|a_{ij}|u_j^0 \\ &+ \sum_{j;j\neq i} a_{ij}u_j + su_i^0 + \theta_i + f_i(x, u_1, \dots, u_n).\end{aligned} \tag{3.46}$$

But $-u_i^0 \le u_i$. So $m(u_i^0 + u_i) \ge 0$. Moreover, $-a_{ij}u_j \le |a_{ij}|u_j^0$. By using (H5), we conclude that $(-\Delta + q_i')(w_i + u_i^0) \ge 0$ and hence, $w_i \ge -u_i^0$. So $T([U_0, U^0]) \subset [U_0, U^0]$. $[U_0, U^0]$ is a convex, closed, and bounded subset of $(L^2(\mathbb{R}^N))^n$, so by the Schauder fixed point theorem, we deduce that T has a fixed point. Therefore, (3.40) has at least a solution. □

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