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ON THE EXTENDED HARDY'S INEQUALITY

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ABSTRACT. We generalize a strengthened version of Hardy's inequality and give a new simpler proof.

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In the recent paper [4], Hardy's inequality was generalized. In this note, the results given in [4] are further generalized and a new much simpler proof is given. The following Hardy's inequality is well known [1, Theorem 349].

Theorem 1 (Hardy's inequality). *Let* $\lambda_n > 0$, $A_n = \sum_{k=1}^{n} \lambda_k$, $a_n \ge 0$ $(n \in \mathbb{N})$, $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$, *then*

$$\sum_{n=1}^{\infty} \lambda_n \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/A_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \tag{1}$$

Recently, [4] gave an improvement of Theorem 1, and the following result was proved.

Theorem 2. *Let* $0 < \lambda_{n+1} \le \lambda_n$, $A_n = \sum_{k=1}^{n} \lambda_k$, $a_n \ge 0$ $(n \in \mathbb{N})$, $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$, *then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/A_n} < e \sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n}{2(A_n + \lambda_n)}\right) \lambda_n a_n. \tag{2}$$

In this note, we will prove the following theorem.

Theorem 3. *Let* $0 < \lambda_{n+1} \le \lambda_n$, $A_n = \sum_{k=1}^{n} \lambda_k$, $a_n \ge 0$ $(n \in \mathbb{N})$, $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$, *then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/A_n} < e \sum_{n=1}^{\infty} \left(1 + \frac{5\lambda_n}{5A_n + \lambda_n}\right)^{-1/2} \lambda_n a_n. \tag{3}$$

To prove Theorem 3, we introduce some lemmas.

Lemma 4. *For* $x > 0$, *then*

$$e\left(1 - \frac{1}{2x+1}\right) < \left(1 + \frac{1}{x}\right)^x < e\left(1 + \frac{5}{5x+1}\right)^{-1/2}. \tag{4}$$

Proof. (i) Define $f(x)$ as

$$f(x)=x\ln\left(1+\frac{1}{x}\right)+\frac{1}{2}\ln\left(1+\frac{5}{5x+1}\right),\quad x\in(0,\ +\infty). \tag{5}$$

It is obvious that when $x>0$, the inequality

$$\left(1+\frac{1}{x}\right)^{x}<e\left(1+\frac{5}{5x+1}\right)^{-1/2} \tag{6}$$

is equivalent to $f(x)<1$. It is easy to see that

$$f'(x)=-\frac{1}{x+1}+\ln\left(1+\frac{1}{x}\right)-\frac{25}{2(5x+6)(5x+1)} \tag{7}$$

and for $x\in(0,+\infty)$, it can be shown that

$$\begin{aligned} f''(x)&=\frac{1}{(x+1)^2}-\frac{1}{x(x+1)}+\frac{25}{2(5x+1)^2}-\frac{25}{2(5x+6)^2}\\ &=\frac{-125x^3-50x^2+35x-72}{2x(x+1)^2(5x+1)^2(5x+6)^2}<0. \end{aligned} \tag{8}$$

Hence $f'(x)$ is decreasing on $(0,+\infty)$. Then for any $x\in(0,\ +\infty)$, we have $f'(x)>\lim_{x\to+\infty}f'(x)=0$, thus, $f(x)$ is increasing on $(0,+\infty)$, and $f(x)<\lim_{x\to+\infty}f(x)=1$ for $x\in(0,+\infty)$. The inequality (6) is valid.

(ii) Define $g(x)$ as

$$g(x)=x\ln\left(1+\frac{1}{x}\right)-\ln\left(1-\frac{1}{2x+1}\right),\quad x\in(0,+\infty). \tag{9}$$

When $x>0$, the inequality

$$e\left(1-\frac{1}{2x+1}\right)<\left(1+\frac{1}{x}\right)^{x} \tag{10}$$

is equivalent to $g(x)>1$. For $x\in(0,+\infty)$, it can be shown that

$$\begin{aligned} g'(x)&=-\frac{1}{x+1}+\ln\left(1+\frac{1}{x}\right)-\frac{1}{x(2x+1)},\\ g''(x)&=\frac{5x^2+5x+1}{x^2(x+1)^2(2x+1)^2}>0. \end{aligned} \tag{11}$$

Hence, $g'(x)$ is increasing on $(0,+\infty)$. Then for any $x\in(0,\ +\infty)$, we have $g'(x)<\lim_{x\to+\infty}g'(x)=0$, therefore, $g(x)$ is decreasing on $(0,+\infty)$ and $g(x)>\lim_{x\to+\infty}g(x)=1$ for $x\in(0,+\infty)$. Inequality (10) is valid.

By virtue of (6) and (10), inequalities (4) are valid. This proves Lemma 4. □

Remark 5. By a direct calculation, we have

$$\left(1+\frac{5}{5x+1}\right)^{-1/2}<1-\frac{1}{2(x+19/20)}\quad (x>0). \tag{12}$$

Then by (4) and (12), we have

$$e\left(1-\frac{1}{2x+1}\right)<\left(1+\frac{1}{x}\right)^{x}<e\left[1-\frac{1}{2(x+19/20)}\right]\quad(x>0). \tag{13}$$

Inequality (13) is equivalent to

$$\frac{e}{2(x+19/20)}<e-\left(1+\frac{1}{x}\right)^{x}<\frac{e}{2x+1}\quad(x>0). \tag{14}$$

Thus, [1, Lemma 2] is contained in Lemma 4. Inequalities (4) and (14) are the new inequalities on the constant e (cf. [3, Theorem 3.8.26]; and [2, page 358]).

Lemma 6 (see [1, Theorem 9]). *Let* $g_m>0$, $\alpha_m\geq 0$ $(m=1,2,\ldots,n)$, $\sum_{m=1}^{n}g_m=1$, *then*

$$\alpha_1^{g_1}\alpha_2^{g_2}\cdots\alpha_n^{g_n}\leq\sum_{m=1}^{n}g_m\alpha_m. \tag{15}$$

Proof of Theorem 3. Setting $c_m>0$, $g_m=\lambda_m/A_n$, $\alpha_m=c_ma_m$ $(m=1,2,\ldots,n)$, by Lemma 6, we have

$$(c_1a_1)^{\lambda_1/A_1}(c_2a_2)^{\lambda_2/A_2}\cdots(c_na_n)^{\lambda_n/A_n}\leq\frac{1}{A_n}\sum_{m=1}^{n}\lambda_mc_ma_m. \tag{16}$$

Then we find that

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}\right)^{1/A_n}&=\sum_{n=1}^{\infty}\lambda_{n+1}\frac{(c_1a_1)^{\lambda_1/A_1}(c_2a_2)^{\lambda_2/A_2}\cdots(c_na_n)^{\lambda_n/A_n}}{\left(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}\right)^{1/A_n}}\\&\leq\sum_{n=1}^{\infty}\left[\frac{\lambda_{n+1}}{\left(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}\right)^{1/A_n}}\right]\frac{1}{A_n}\sum_{m=1}^{n}c_m\lambda_ma_m\\&=\sum_{m=1}^{\infty}\lambda_ma_mc_m\sum_{n=m}^{\infty}\frac{\lambda_{n+1}}{A_n\left(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}\right)^{1/A_n}}.\end{aligned} \tag{17}$$

Define $c_m=((A_{m+1})/A_m)^{A_m/\lambda_m}A_m$ $(m=1,2,\ldots)$ and $A_0=0$. Because $0<\lambda_{n+1}\leq\lambda_n$ $(n=1,2,\ldots)$, we have

$$c_m^{\lambda_m}=\frac{(A_{m+1})^{A_m}}{A_m^{A_{m-1}}};\qquad\left(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}\right)^{1/A_n}=A_{n+1}\quad(n\in\mathbb{N});$$

$$\begin{aligned}c_m\sum_{n=m}^{\infty}\frac{\lambda_{n+1}}{A_n\left(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}\right)^{1/A_n}}&=\left(\frac{A_{m+1}}{A_m}\right)^{A_m/\lambda_m}A_m\sum_{n=m}^{\infty}\frac{\lambda_{n+1}}{A_nA_{n+1}}\\&=\left(1+\frac{\lambda_{m+1}}{A_m}\right)^{A_m/\lambda_m}A_m\sum_{n=m}^{\infty}\left(\frac{1}{A_n}-\frac{1}{A_{n+1}}\right)\\&\leq\left(1+\frac{\lambda_m}{A_m}\right)^{A_m/\lambda_m}.\end{aligned} \tag{18}$$

Then by (4) and (17), we obtain that

$$\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/A_n} \le \sum_{m=1}^{\infty}\left(1+\frac{\lambda_m}{A_m}\right)^{A_m/\lambda_m}\lambda_m a_m \le e\sum_{m=1}^{\infty}\left(1+\frac{5\lambda_m}{5A_m+\lambda_m}\right)^{-1/2}\lambda_m a_m. \tag{19}$$

Hence inequality (3) is valid, and Theorem 3 is proved. □

Remark 7. With inequality (12), Theorem 3 is obviously an improvement and an extension of [4, Theorem 1].

Setting $\lambda_n \equiv 1$, (3) changes into

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1+\frac{5}{5n+1}\right)^{-1/2}a_n. \tag{20}$$

By inequality (12), we have

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left[1-\frac{1}{2(n+19/20)}\right]a_n. \tag{21}$$

Thus, inequalities (20) and (21) are obviously an improvement and extension of [5, Theorem 3.1].

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