

ASYMPTOTIC DECAY OF NONOSCILLATORY SOLUTIONS OF GENERAL NONLINEAR DIFFERENCE EQUATIONS

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ABSTRACT. The authors consider the m th order nonlinear difference equations of the form $D_m y_n + q_n f(y_{\sigma(n)}) = e_n$, where $m \geq 1$, $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, $a_n^i > 0$ for $i = 1, 2, \dots, m-1$, $a_n^m \equiv 1$, $D_0 y_n = y_n$, $D_i y_n = a_n^i \Delta D_{i-1} y_n$, $i = 1, 2, \dots, m$, $\sigma(n) \rightarrow \infty$ as $n \rightarrow \infty$, and $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$. They give sufficient conditions to ensure that all bounded nonoscillatory solutions tend to zero as $n \rightarrow \infty$ without assuming that $\sum_{n=0}^{\infty} 1/a_n^i = \infty$, $i = 1, 2, \dots, m-1$, $\{q_n\}$ is positive, or $e_n \equiv 0$ as is often required. If $\{q_n\}$ is positive, they prove another such result for all nonoscillatory solutions.

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1. Introduction. Consider the m th order nonlinear difference equation

$$D_m y_n + q_n f(y_{\sigma(n)}) = e_n, \tag{1.1}$$

where $m \geq 1$, $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, $\{q_n\}$, $\{e_n\}$, and $\{a_n^1\}$, $\{a_n^2\}$, \dots , $\{a_n^{m-1}\}$ are real sequences, $a_n^i > 0$ for $i = 1, 2, \dots, m-1$ and all $n \in \mathbb{N}$, $a_n^m \equiv 1$, $D_0 y_n = y_n$, $D_i y_n = a_n^i \Delta D_{i-1} y_n$ for $i = 1, 2, \dots, m$, $\{\sigma(n)\}$ is a sequence of positive integers with $\sigma(n) \rightarrow \infty$ as $n \rightarrow \infty$, and $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$. Throughout, we will assume that

$$\rho_i(n) = \sum_{s=n+1}^{\infty} \frac{\rho_{i-1}(s)}{a_s^i}, \quad i = 1, 2, \dots, m-1, \quad \rho_0(n) \equiv 1, \tag{1.2}$$

satisfies

$$\lim_{n \rightarrow \infty} \rho_i(n) = 0 \quad \text{for } i = 1, 2, \dots, m-1. \tag{1.3}$$

Note that condition (1.3) is satisfied if

$$\sum_{n=N}^{\infty} \frac{1}{a_n^i} < \infty \quad \text{for each } i = 1, 2, \dots, m-1. \tag{1.4}$$

By a *solution* of (1.1) we mean a nontrivial real sequence $\{y_n\}$ defined for $n \geq N_0 - \min_{n \in \mathbb{N}} \sigma(n)$, $N_0 \in \mathbb{N}$, and satisfying (1.1) for $n \geq N_0$. Such a solution is said to be *oscillatory* if for every $N \in \mathbb{N}$ there exist $n_1, n_2 \in \mathbb{N}$ with $n_2 > n_1 > N$ and $y_{n_1} y_{n_2} \leq 0$, and it is said to be *nonoscillatory* otherwise.

An important problem in the study of oscillation theory of difference equations is to determine sufficient conditions for all nonoscillatory solutions or all bounded nonoscillatory solutions to converge to zero as $n \rightarrow \infty$. This problem has received a

good deal of attention in the literature, and for recent results of this type, we refer the reader to the monographs of Agarwal [1], Agarwal and Wong [2] as well as the papers of Cheng et al. [3], Graef et al. [4], Graef and Spikes [5, 6], Szmanda [7], Thandapani and Lalli [8], Thandapani and Pandian [9], and Zhang [10]. Most of these results, however, are obtained under the assumptions that $\sum_{n=N}^{\infty} 1/a_n^i = \infty$, $i = 1, 2, \dots, m-1$, and/or $e_n \equiv 0$. It is these last two restrictions that provide the motivation for our work here. That is, we do not require that either of these conditions hold in our results below.

Our results are of two types. First, if the sequence $\{q_n\}$ is allowed to oscillate, we provide sufficient conditions for all bounded nonoscillatory solutions of (1.1) to converge to zero as $n \rightarrow \infty$. Second, in the case where $\{q_n\}$ is a nonnegative sequence, we give sufficient conditions for all nonoscillatory solutions of (1.1) to approach zero as $n \rightarrow \infty$. Examples to illustrate our results are also included.

2. Asymptotic decay of nonoscillatory solutions. We begin with a lemma that will be used in the proofs of our main results.

LEMMA 2.1. *Consider the difference equation*

$$\Delta u_n - \frac{\Delta \rho(n)}{\rho(n)} u_n + \frac{\Delta \rho(n)}{\rho(n)} \phi_n = 0, \quad (2.1)$$

where $\{\phi_n\}$ and $\{\rho(n)\}$ are real sequences defined for $n \geq N$, for some $N \in \mathbb{N}$,

$$\rho(n) > 0, \quad \Delta \rho(n) < 0, \quad \lim_{n \rightarrow \infty} \rho(n) = 0. \quad (2.2)$$

Let $\{u_n\}$ be the solution of (2.1) defined for $n \geq N$ and satisfying $u_N = 0$. Then

$$\lim_{n \rightarrow \infty} \phi_n = \infty(-\infty) \quad \text{implies} \quad \lim_{n \rightarrow \infty} u_n = \infty(-\infty). \quad (2.3)$$

PROOF. The solution $\{u_n\}$ of (2.1) is given by

$$u_n = -\rho(n) \sum_{s=N}^{n-1} \frac{\Delta \rho(s)}{\rho(s)\rho(s+1)} \phi_s, \quad n \geq N. \quad (2.4)$$

If $\lim_{n \rightarrow \infty} \phi_n = \infty(-\infty)$, then clearly

$$\lim_{n \rightarrow \infty} \sum_{s=N}^{n-1} \frac{\Delta \rho(s)}{\rho(s)\rho(s+1)} \phi_s = -\infty(\infty). \quad (2.5)$$

Hence, by Stolz's theorem [1],

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left| \frac{\Delta(-\sum_{s=N}^{n-1} (\Delta \rho(s)/\rho(s)\rho(s+1)) \phi_s)}{\Delta(1/\rho(n))} \right| = \lim_{n \rightarrow \infty} \phi_n = \infty(-\infty), \quad (2.6)$$

and this completes the proof of the lemma. \square

In our results that follow, we will make use of the notation $q_n^+ = \max\{q_n, 0\}$ and $q_n^- = \max\{-q_n, 0\}$.

THEOREM 2.2. *Assume that*

$$\sum_{n=N}^{\infty} \rho_{m-1}(n) a_n^+ = \infty, \quad (2.7)$$

$$\sum_{n=N}^{\infty} \rho_{m-1}(n) a_n^- < \infty, \quad (2.8)$$

$$\sum_{n=N}^{\infty} \rho_{m-1}(n) |e_n| < \infty.$$

Then all bounded nonoscillatory solutions of (1.1) tend to zero as $n \rightarrow \infty$.

PROOF. Let $\{y_n\}$ be a bounded nonoscillatory solution of (1.1). Without loss of generality, we may assume that $y_n > 0$ and $y_{\sigma(n)} > 0$ for $n \geq N_1$ for some $N_1 \in \mathbb{N}$. Define

$$G_0(n) = y_n, \quad G_i(n) = a_n^i \Delta G_{i-1}(n), \quad i = 1, 2, \dots, m-1, \quad (2.9)$$

and observe that

$$G_i(n) = D_i y_n \quad \text{for } i = 1, 2, \dots, m-1, \quad \Delta G_{m-1}(n) = D_m y_n. \quad (2.10)$$

Next, we define the family of sequences

$$u_k(n) = \sum_{s=N_1+1}^n \rho_{m-k-1}(s) \Delta G_{m-k-1}(s), \quad k = 0, 1, \dots, m-1, \quad (2.11)$$

for $n \geq N_1 + 1$.

A summation by parts yields

$$\begin{aligned} u_{k-1}(n) &= \sum_{s=N_1+1}^n \rho_{m-k}(s) \Delta G_{m-k}(s) = \rho_{m-k}(n+1) G_{m-k}(n+1) \\ &\quad - \rho_{m-k}(N_1+1) G_{m-k}(N_1+1) + \sum_{s=N_1+1}^n \frac{\rho_{m-k-1}(s)}{a_s^{m-k}} G_{m-k}(s) \\ &= -\frac{\rho_{m-k}(n+1)}{\Delta \rho_{m-k}(n)} \Delta u_k(n) + \Delta u_k(n) + u_k(n) - 2\rho_{m-k}(N_1+1) G_{m-k}(N_1+1) \\ &= -\frac{\rho_{m-k}(n)}{\Delta \rho_{m-k}(n)} \Delta u_k(n) + u_k(n) - 2\rho_{m-k}(N_1+1) G_{m-k}(N_1+1). \end{aligned} \quad (2.12)$$

This shows that each sequence $\{u_k(n)\}$, $k = 0, 1, \dots, m-1$, satisfies the difference equation

$$\frac{\rho_{m-k}(n)}{\Delta \rho_{m-k}(n)} \Delta u_k(n) - u_k(n) + \phi_k(n) = 0, \quad (2.13)$$

which can be written in the form

$$\Delta u_k(n) - \frac{\Delta \rho_{m-k}(n)}{\rho_{m-k}(n)} u_k(n) + \frac{\Delta \rho_{m-k}(n)}{\rho_{m-k}(n)} \phi_k(n) = 0, \quad (2.14)$$

where $\phi_k(n) = u_{k-1}(n) + 2\rho_{m-k}(N_1 + 1)G_{m-k}(N_1 + 1)$. Since $u_k(N_1) = 0$ by (2.11) and since $\rho_{m-k}(n) > 0$, $\Delta\rho_{m-k}(n) < 0$, and $\lim_{n \rightarrow \infty} \rho_{m-k}(n) = 0$ by (1.3), we can apply Lemma 2.1 to (2.14) to conclude that $\lim_{n \rightarrow \infty} u_{k-1}(n) = \infty$ (or $-\infty$) which in turn implies that $\lim_{n \rightarrow \infty} u_k(n) = \infty$ (or $-\infty$).

Multiplying (1.1) by $\rho_{m-1}(n)$ and summing from $N_1 + 1$ to n , we have

$$\begin{aligned} & \sum_{s=N_1+1}^n \rho_{m-1}(s)\Delta G_{m-1}(s) + \sum_{s=N_1+1}^n \rho_{m-1}(s)q_s^+ f(\gamma_{\sigma(s)}) \\ &= \sum_{s=N_1+1}^n \rho_{m-1}(s)e_s + \sum_{s=N_1+1}^n \rho_{m-1}(s)q_s^- f(\gamma_{\sigma(s)}). \end{aligned} \quad (2.15)$$

We consider the following two cases:

$$\sum_{n=N_1+1}^{\infty} \rho_{m-1}(n)q_n^+ f(\gamma_{\sigma(n)}) = \infty \quad (2.16)$$

or

$$\sum_{n=N_1+1}^{\infty} \rho_{m-1}(n)q_n^+ f(\gamma_{\sigma(n)}) < \infty. \quad (2.17)$$

Suppose (2.16) holds. In view of (2.8) and the boundedness of $\{\gamma_n\}$, the right-hand side of (2.15) tends to a finite limit as $n \rightarrow \infty$. From (2.15), we see that $\lim_{n \rightarrow \infty} u_0(n) = -\infty$. Hence, applying Lemma 2.1 to (2.14) with $k = 1$, we have $\lim_{n \rightarrow \infty} u_1(n) = -\infty$. Again applying Lemma 2.1 to (2.14), this time with $k = 2$, we see that $\lim_{n \rightarrow \infty} u_2(n) = -\infty$. Repeating this procedure, we can conclude that $\lim_{n \rightarrow \infty} u_{m-1}(n) = -\infty$, which implies that $\lim_{n \rightarrow \infty} \gamma_n = -\infty$. This, however, contradicts the assumption that $\{\gamma_n\}$ is positive, and thus (2.16) cannot hold.

Next, letting $n \rightarrow \infty$ in (2.15) and using (2.17), we see that $\lim_{n \rightarrow \infty} u_0(n)$ is finite. From (2.13), with $k = 1$, we have

$$\frac{\rho_{m-1}(n)}{\Delta\rho_{m-1}(n)} \Delta u_1(n) = u_1(n) - \phi_1(n) \quad (2.18)$$

or

$$u_1(n) = \frac{\rho_{m-1}(n)}{\rho_{m-1}(N_1)} \left(u_1(N_1) - \rho_{m-1}(N_1) \sum_{s=N_1}^{n-1} \frac{\Delta\rho_{m-1}(s)}{\rho_{m-1}(s)\rho_{m-1}(s+1)} \phi_1(s) \right). \quad (2.19)$$

Taking the limit as $n \rightarrow \infty$ and using (1.3), we obtain

$$\lim_{n \rightarrow \infty} u_1(n) = - \lim_{n \rightarrow \infty} \rho_{m-1}(n) \sum_{s=N_1}^{n-1} \frac{\Delta\rho_{m-1}(s)}{\rho_{m-1}(s)\rho_{m-1}(s+1)} \phi_1(s). \quad (2.20)$$

This limit must be finite since $\lim_{n \rightarrow \infty} u_1(n) = -\infty$ implies $\lim_{n \rightarrow \infty} \gamma_n = -\infty$, which contradicts the positivity of $\{\gamma_n\}$, and $\lim_{n \rightarrow \infty} u_1(n) = \infty$ implies $\lim_{n \rightarrow \infty} \gamma_n = \infty$, which contradicts the boundedness of $\{\gamma_n\}$. Continuing in this way, $\lim_{n \rightarrow \infty} u_{m-1}(n)$ is finite. Therefore, $\lim_{n \rightarrow \infty} \gamma_n$ exists as a finite number. On the other hand, in view of (2.7) and (2.17), it is easy to verify that

$$\liminf_{n \rightarrow \infty} \gamma_{\sigma(n)} = \liminf_{n \rightarrow \infty} \gamma_n = 0. \quad (2.21)$$

Thus, it follows that $\lim_{n \rightarrow \infty} \gamma_n = 0$, and this completes the proof of the theorem. \square

EXAMPLE 2.3. Consider the difference equation

$$\Delta(n\Delta(n\Delta(n(n+1)\Delta y_n))) + y_{yn} = \frac{1}{yn}, \quad n \geq 1, \quad (2.22)$$

where y is a positive integer. We have $\rho_1(n) = \rho_2(n) = \rho_3(n) = 1/(n+1)$ and we see that all conditions of [Theorem 2.2](#) are satisfied. Hence, all bounded nonoscillatory solutions of (2.22) tend to zero as $n \rightarrow \infty$. In fact, $\{y_n\} = \{1/n\}$ is a solution of (2.22) having this property.

In the following theorem, we show that the conclusion of [Theorem 2.2](#) still holds if the roles of the sequences $\{q_n^+\}$ and $\{q_n^-\}$ are interchanged.

THEOREM 2.4. *All bounded nonoscillatory solutions of (1.1) tend to zero as $n \rightarrow \infty$ if the following conditions are satisfied:*

$$\sum_{n=N}^{\infty} \rho_{m-1}(n) q_n^+ < \infty, \quad (2.23)$$

$$\sum_{n=N}^{\infty} \rho_{m-1}(n) q_n^- = \infty, \quad (2.24)$$

$$\sum_{n=N}^{\infty} \rho_{m-1}(n) |e_n| < \infty. \quad (2.25)$$

PROOF. Let $\{y_n\}$ be a bounded nonoscillatory solution of (1.1), say, $y_n > 0$ and $y_{\sigma(n)} > 0$ for $n \geq N_1 \geq N_0$. Define $G_i(n)$ and $u_k(n)$ as in (2.9) and (2.11). Assume that

$$\sum_{n=N_1+1}^{\infty} \rho_{m-1}(n) q_n^- f(y_{\sigma(n)}) = \infty. \quad (2.26)$$

Letting $n \rightarrow \infty$ in (2.15) and using (2.23), (2.25), and the boundedness of $\{y_n\}$, we obtain $\lim_{n \rightarrow \infty} u_0(n) = \infty$. Applying [Lemma 2.1](#) to (2.14) with $k = 1$, we see that $\lim_{n \rightarrow \infty} u_1(n) = \infty$. Repeated applications of this argument yield $\lim_{n \rightarrow \infty} u_{m-1}(n) = \infty$, which implies that $\lim_{n \rightarrow \infty} y_n = \infty$. This contradicts the boundedness of $\{y_n\}$, and so we must have

$$\sum_{n=N_1+1}^{\infty} \rho_{m-1}(n) q_n^- f(y_{\sigma(n)}) < \infty. \quad (2.27)$$

The remainder of the proof is similar to the proof of [Theorem 2.2](#) and will be omitted. \square

EXAMPLE 2.5. Consider the equation

$$\Delta^4(2^{n+1}\Delta y_n) - 2^n y_{n-2}^3 = -\frac{1}{4^{n-3}}, \quad n \geq 0. \quad (2.28)$$

It is easy to verify that the hypotheses of [Theorem 2.4](#) are satisfied with $\rho_1(n) = \rho_2(n) = \rho_3(n) = 1/2^{n+1}$. It follows that all bounded nonoscillatory solutions of (2.28) approach zero as $n \rightarrow \infty$. One such solution is $\{y_n\} = \{1/2^n\}$.

As an example where $\{q_n\}$ is oscillatory, we have the following example.

EXAMPLE 2.6. Consider the equation

$$\begin{aligned} \Delta^3(2^{n+1}\Delta y_n) + \left\{ 2^{n-1}[1+(-1)^n] - \frac{[1+(-1)^{n+1}]}{2n^2} \right\} y_n \\ = \frac{[1+(-1)^n]}{2} - \frac{[1+(-1)^{n+1}]}{2^{n+1}n^2}, \quad n \geq 1. \end{aligned} \quad (2.29)$$

Observe that $q_n^+ = 2^n$, $q_n^- = -1/n^2$, and $\rho_1(n) = \rho_2(n) = \rho_3(n) = 1/2^{n+1}$. All the hypotheses of [Theorem 2.2](#) are satisfied so all bounded nonoscillatory solutions of (2.29) approach zero as $n \rightarrow \infty$. Here, $\{y_n\} = \{1/2^n\}$ is such a solution. Clearly, a simple modification of this equation will yield an example of [Theorem 2.4](#).

In our final result, we examine (1.1) in the case where $\{q_n\}$ is positive and establish conditions under which all nonoscillatory solutions are bounded and tend to zero as $n \rightarrow \infty$.

THEOREM 2.7. *Assume that condition (1.4) holds, $\{q_n\}$ is positive, $\liminf_{u \rightarrow \infty} f(u) > 0$, and $\limsup_{u \rightarrow \infty} f(u) < 0$. If*

$$\sum_{n=N}^{\infty} \rho_{m-1}(n)q_n = \infty, \quad (2.30)$$

$$\sum_{n=N}^{\infty} |e_n| < \infty, \quad (2.31)$$

then all nonoscillatory solutions of (1.1) tend to zero as $n \rightarrow \infty$.

PROOF. Let $\{y_n\}$ be a nonoscillatory solution of (1.1), say, $y_n > 0$ and $y_{\sigma(n)} > 0$ for $n \geq N_1 \geq N_0$. Define $G_i(n)$ and $u_k(n)$ as in (2.9) and (2.11). We will first show that $\{y_n\}$ is bounded above. From (1.1), we obtain

$$G_{m-1}(n) - G_{m-1}(N_1) + \sum_{s=N_1}^{n-1} q_s f(y_{\sigma(s)}) = \sum_{s=N_1}^{n-1} e_s. \quad (2.32)$$

Since the first sum in (2.32) is positive, and by (2.31), the second sum is bounded, there exists a constant K_{m-1} such that

$$G_{m-1}(n) = a_n^{m-1} \Delta G_{m-2}(n) \leq K_{m-1} \quad \text{for } n \geq N_1. \quad (2.33)$$

Dividing the last inequality by a_n^{m-1} and summing from N_1 to $n-1$, we obtain

$$G_{m-2}(n) - G_{m-2}(N_1) \leq K_{m-1} \sum_{s=N_1}^{n-1} \frac{1}{a_s^{m-1}} \quad \text{for } n \geq N_1, \quad (2.34)$$

which, in view of (1.4), implies there exists a constant K_{m-2} such that

$$G_{m-2}(n) = a_n^{m-2} \Delta G_{m-3}(n) \leq K_{m-2} \quad \text{for } n \geq N_1. \quad (2.35)$$

Repeatedly applying the above argument, we obtain constants K_{m-3}, \dots, K_1, K_0 such that

$$G_{m-3}(n) \leq K_{m-3}, \dots, G_1(n) \leq K_1, G_0(n) \leq K_0 \quad \text{for } n \geq N_1. \quad (2.36)$$

It follows that $\{y_n\}$ is bounded from above.

Now, we argue as in the proof of [Theorem 2.2](#) using

$$\sum_{s=N_1+1}^n \rho_{m-1}(s)\Delta G_{m-1}(s) + \sum_{s=N_1+1}^n \rho_{m-1}(s)q_s f(y_{\sigma(s)}) = \sum_{s=N_1+1}^n \rho_{m-1}(s)e_s \quad (2.37)$$

in place of (2.15). Noting that (2.31) implies the right-hand side of (2.37) tends to a finite limit as $n \rightarrow \infty$, we claim that

$$\sum_{s=N_1+1}^n \rho_{m-1}(s)q_s f(y_{\sigma(s)}) < \infty. \quad (2.38)$$

If this was not the case, we could use [Lemma 2.1](#) to obtain $\lim_{n \rightarrow \infty} u_k(n) = -\infty$ for $k = 0, 1, \dots, m-1$, and contradict the boundedness of $\{y_n\}$. Next, using (2.37) and (2.38) we can show that $\lim_{n \rightarrow \infty} u_k(n)$ is finite for each $k = 0, 1, \dots, m-1$. Thus, $\lim_{n \rightarrow \infty} y_n$ exists and is finite. On the other hand, from (2.30) and (2.38), we see that $\liminf_{n \rightarrow \infty} y_n = 0$. Hence, $\{y_n\}$ tends to zero as $n \rightarrow \infty$, and this completes the proof of the theorem. \square

We conclude this paper with some examples of [Theorem 2.7](#).

EXAMPLE 2.8. Consider the equation

$$\Delta(2^n \Delta(2^n \Delta(2^n \Delta y_n))) + 8^n y_{n+k}^3 = \frac{1}{8k}, \quad n \geq 0, \quad (2.39)$$

where k is a positive integer. In this case, $\rho_1(n) = 1/2^n$, $\rho_2(n) = (1/3)(1/4^n)$, and $\rho_3(n) = (1/21)(1/8^n)$. Since all conditions of [Theorem 2.7](#) are satisfied, every nonoscillatory solution of (2.39) tends to zero as $n \rightarrow \infty$, and $\{y_n\} = \{1/2^n\}$ is such a solution.

EXAMPLE 2.9. Consider the equation

$$\Delta(n(n+1)\Delta((n+2)(n+3)\Delta(n(n+1)\Delta y_n))) + n^4 y_{kn}^3 = \frac{n}{k^3(kn+1)^3}, \quad n \geq 1, \quad (2.40)$$

where k is a positive integer. All the hypotheses of [Theorem 2.7](#) are satisfied with $\rho_1(n) = 1/(n+1)$, $\rho_2(n) = 1/2(n+1)(n+2)$, and $\rho_3(n) = 1/6(n+1)(n+2)(n+3)$, so every nonoscillatory solution of (2.40) tends to zero as $n \rightarrow \infty$. Here, $\{y_n\} = \{1/n(n+1)\}$ is a solution of (2.40).

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