

GRADED SUBALGEBRAS OF THE LIE ALGEBRA OF A SMOOTH MANIFOLD

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ABSTRACT. We construct and study finitely generated graded subalgebras of the Lie algebra of a smooth manifold.

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1. Introduction. Let M be a smooth manifold of dimension $n > 1$. Embed M in \mathbb{R}^m for some $m > n$ so that the resulting point set forms an analytic manifold with no boundary. That this is possible follows from Whitney [1]. Let $\mathcal{S}(M)$ be the Lie algebra of smooth vector fields on M that smoothly go to zero at infinity if M is not compact.

We will look at two types of finitely generated graded subalgebras of $\mathcal{S}(M)$. In one type we look at the Lie algebra generated by n analytic vector fields that span the tangent space at some point of M and such that no subset generates a finite dimensional Lie algebra. It is shown that this is a graded Lie algebra and that the graded subalgebras of two manifolds are isomorphic if and only if the manifolds are diffeomorphic. The other type is constructed from an atlas of the manifold satisfying some conditions.

2. Graded subalgebras with $\dim M$ number of generators. Without loss of generality, we set $n = 2$ in this section. Choose two analytic vector fields $X_j \in \mathcal{S}(M)$ that span the tangent space for some point of M and such that \mathfrak{m} is infinite dimensional where $\mathfrak{m} = \mathfrak{m}(X_1, X_2) = \langle X_1, X_2 \rangle$ is the Lie algebra generated by the X_j . We have $\mathfrak{m} = \sum_{k=1}^{\infty} \mathfrak{m}_k(X_1, X_2)$ where $\mathfrak{m}_k = \mathfrak{m}_k(X_1, X_2)$ is the span over \mathbb{R} of the set

$$\left\{ [X_{j_1}, [X_{j_2}, [\dots [X_{j_{k-1}}, X_{j_k}] \dots]] : j_i = 1, 2 \right\}. \quad (2.1)$$

For a smooth atlas of M define a topology on $\mathcal{S}(M)$ by the metric

$$\rho(v, w) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|v - w\|_k}{1 + \|v - w\|_k}, \quad v, w \in \mathcal{S}(M), \quad (2.2)$$

where $\|v - w\|_k$ is the supremum on M of partial derivatives of order less than $k + 1$ of components of $v - w$. Construct \tilde{X}_j , $\tilde{\mathfrak{m}}$, and $\tilde{\rho}(\cdot, \cdot)$ for a smooth manifold \tilde{M} in the same way that X_j , \mathfrak{m} , and $\rho(\cdot, \cdot)$ were instead for M . Assume there is a Lie algebra epimorphism $\Phi : \mathfrak{m} \rightarrow \tilde{\mathfrak{m}}$.

LEMMA 2.1. *The completion $\tilde{\mathfrak{m}}$ of \mathfrak{m} in the $\rho_2(\cdot, \cdot) = \rho(\cdot, \cdot) + \tilde{\rho}(\Phi(\cdot), \Phi(\cdot))$ topology is $\mathcal{S}(M)$.*

PROOF. Define

$$\Delta_{\lambda\xi} = \sum_{k,l=0}^{\infty} \frac{\lambda^k \xi^l}{k!l!} \text{ad}^k(X_1)(\text{ad}^l(X_2)([X_1, X_2])) \quad (2.3)$$

which can be shown to converge for all λ, ξ in the ρ_2 topology. Define

$$V = \left\{ \sum_{k=1}^{\infty} a_k \Delta_{\lambda_k \xi_k} : \sum_{k=1}^{\infty} a_k^2 < \infty \right\} \subset \mathcal{F}(M). \quad (2.4)$$

Let L be a nonzero element of $\mathcal{F}(M)$ and D a countable dense subset of M . Consider

$$v_{2i-1} = \left(L^{(1)}(p_i), \frac{1}{2} \Delta_{\lambda_1 \xi_1}^{(1)}(p_i), \frac{1}{3} \Delta_{\lambda_2 \xi_2}^{(1)}(p_i), \dots \right) \in l_2, \quad (2.5)$$

$$v_{2i} = \left(L^{(2)}(p_i), \frac{1}{2} \Delta_{\lambda_1 \xi_1}^{(2)}(p_i), \frac{1}{3} \Delta_{\lambda_2 \xi_2}^{(2)}(p_i), \dots \right) \in l_2, \quad p_i \in D, \quad (2.6)$$

where for example $L^{(2)}(p_i)$ is the second component of $L(p_i)$ is a coordinate neighbourhood of p_i and l_2 is the Hilbert space of elements (a_1, a_2, \dots) so that $\sum_{k=1}^{\infty} a_k^2 < \infty$ with inner product (\cdot, \cdot) . Let W be the span of the set of v_k . Since $\mathfrak{m}(X_1, X_2)$ is infinite dimensional there are λ_k, ξ_k so that the map that maps (a_1, a_2, a_3, \dots) to (a_2, a_3, \dots) is injective on W . It will be injective on \bar{W} , the completion of W in the (\cdot, \cdot) topology. If, say, $L^{(1)} p_k \neq 0$ then $(0, 1/2 \Delta_{\lambda_1 \xi_1}^{(1)}(p_k), 1/3 \Delta_{\lambda_2 \xi_2}^{(1)}(p_k), \dots)$ is not an element of \bar{W} . There is then a nonzero $(a, a_1, a_2, \dots) \in l_2$ with $a \neq 0$ so that

$$aL(p_i) + a_1 \Delta_{\lambda_1 \xi_1}(p_i) + a_2 \Delta_{\lambda_2 \xi_2}(p_i) + \dots = 0, \quad \forall p_i \in D. \quad (2.7)$$

Since D is dense this equation holds for all $p \in M$ hence $L \in V$. \square

THEOREM 2.2. *If \mathfrak{m} and $\tilde{\mathfrak{m}}$ are isomorphic then M and \tilde{M} are diffeomorphic.*

PROOF. Let Φ be an isomorphism of \mathfrak{m} and $\tilde{\mathfrak{m}}$. Let \mathfrak{m} have the topology ρ_2 and $\tilde{\mathfrak{m}}$ the topology $\tilde{\rho}_2(\cdot, \cdot) = \tilde{\rho}(\cdot, \cdot) + \rho(\Phi^{-1}(\cdot), \Phi^{-1}(\cdot))$. The Lie bracket and Φ are continuous in this topology.

Let A, B be the closure of open sets of M . By [Lemma 2.1](#), there are vector fields $E, F \in \tilde{\mathfrak{m}}$ with supports A, B , respectively. Let \tilde{A}, \tilde{B} be the supports of $\Phi(E), \Phi(F)$, respectively. If $A \cap B \neq \emptyset$ then there are E, F so that $[E, F] \neq 0$ hence $[\Phi(E), \Phi(F)] \neq 0$ consequently $\tilde{A} \cap \tilde{B} \neq \emptyset$ and vice versa.

Let $p_0 \in M$ and let A_i be the closure of open sets of M and let $E_i \in \tilde{\mathfrak{m}}$ have support A_i and $\Phi(E_i)$ support \tilde{A}_i . It follows from the previous paragraph that if $A_{i+1} \subset A_i$ then $\tilde{A}_{i+1} \subset \tilde{A}_i$. The A_i can be chosen so the diameters of A_i, \tilde{A}_i approach zero and $\{p_0\} = \bigcap_{i=1}^{\infty} A_i$. Let $\{\tilde{p}_0\} = \bigcap_{i=1}^{\infty} \tilde{A}_i$. Let $\{B_i, \tilde{B}_i\}$ be another such sequence so that $\{p_0\} = \bigcap_{i=1}^{\infty} B_i$ and suppose $\{\tilde{p}_1\} = \bigcap_{i=1}^{\infty} \tilde{B}_i$. Now the sequence of sets $\{A_i \cap B_i, \tilde{A}_i \cap \tilde{B}_i\}$ satisfy the same conditions so we must have that $\tilde{p}_0 = \tilde{p}_1$. We thus have a well-defined map $\tau : M \rightarrow \tilde{M}$ so that $\tau(p_0) = \tilde{p}_0$ which on using the previous argument but with Φ^{-1} is a bijection.

Choose a coordinate neighborhood (x, y) of p_0 . Using $e^{\lambda \text{ad}(X_1)}(X_2)$, let $\lambda(x, y)$ be the value of λ required to move the integral curve of X_2 passing through p_0 along

X_1 so that it intersects the point (x, y) . If $X_1(p_0) = 0$ there is then a $Z_1 \in \mathfrak{m}$ so that $X_1(p_0) + Z_1(p_0) \neq 0$. Now use [Lemma 2.1](#) with Φ restricted to $\langle X_1 + Z_1, X_2 \rangle$. The map τ will remain unchanged. Similarly, using $e^{\xi \text{ad}(X_2)}(X_1)$ we can construct $\xi(x, y)$.

Let B be the closure of a neighborhood of p_0 then by [Lemma 2.1](#) there is a sequence $Z_{00}^{(i)} \in V$ that converges to a smooth vector field with support B . Consider a coordinate system so that $\Delta_{\lambda\xi}(x, y) = \Delta_{\xi}(x + \lambda, y)$, $p_0 = (0, 0)$ so $Z_{\lambda 0}^{(i)}$ constructed by replacing $\Delta_{\lambda_k \xi_k}$ by $\Delta_{\lambda_k + \lambda, \xi}$ in $Z_{00}^{(i)}$ will converge to a smooth vector field with support $B - (\lambda, 0)$ for sufficiently small B . Let $\Phi(Z_{00}^{(i)})$ have support \tilde{B} . For a coordinate neighborhood (\tilde{x}, \tilde{y}) of $\tilde{p}_0 = \tau(p_0) = (0, 0)$ such that $\tilde{\Delta}_{\lambda\xi}(\tilde{x}, \tilde{y}) = \tilde{\Delta}_{\xi}(\tilde{x} + \lambda, \tilde{y})$ we have $\Phi(Z_{\lambda 0}^{(i)})$ will converge to a smooth vector field with support $\tilde{B} - (\lambda, 0)$. Using also a coordinate system so that $\Theta_{\lambda\xi}(x, y) = \Theta_{\lambda 0}(x, y + \xi)$ where $\Theta_{\lambda\xi}$ is constructed by interchanging X_1 and X_2 in $\Delta_{\lambda\xi}$ and on choosing smaller and smaller B we can conclude that τ maps the point (x, y) to (\tilde{x}, \tilde{y}) so that

$$\lambda(x, y) = \tilde{\lambda}(\tilde{x}, \tilde{y}), \quad \xi(x, y) = \tilde{\xi}(\tilde{x}, \tilde{y}), \tag{2.8}$$

where for example using $e^{\tilde{\lambda} \text{ad}(\Phi(X_1))}(\Phi(X_2))$, $\tilde{\lambda}(\tilde{x}, \tilde{y})$ is the value of $\tilde{\lambda}$ required to move the integral curve of $\Phi(X_2)$ passing through \tilde{p}_0 along $\Phi(X_1)$ until it intersects the point (\tilde{x}, \tilde{y}) . Now $\lambda(x, y)$, $\tilde{\lambda}(\tilde{x}, \tilde{y})$, $\xi(x, y)$, $\tilde{\xi}(\tilde{x}, \tilde{y})$ are smooth functions so by taking derivatives of (2.8) it follows that τ is a diffeomorphism. \square

It follows from [Theorem 2.2](#) that an automorphism of \mathfrak{m} induces a diffeomorphism of M .

Define $V_N(X_1, X_2) = \sum_{k=1}^N \mathfrak{m}_k(X_1, X_2)$. The dimension of $V_N(X_1, X_2)$ is locally maximal if there is $\epsilon > 0$ such that for analytic vector fields Y_j on M so that $\rho(X_j, Y_j) < \epsilon$ we have $\dim V_N(Y_1, Y_2) \leq \dim V_N(X_1, X_2)$.

Let $\{v_q, v_{qk}\}$ be a set of smooth vector fields on M and let $\|\cdot\|$ be a norm on the span of this set. We also require that as $k \rightarrow \infty$, $\|v_{qk} - v_q\| \rightarrow 0$.

LEMMA 2.3. *If v_1, v_2, \dots, v_l are linearly independent then there is an N such that for $k > N$, $v_{1k}, v_{2k}, \dots, v_{lk}$ are linearly independent.*

PROOF. Assume there are $t \rightarrow \infty$ so that for each t we can find a_{kt} so that

$$a_{1t}v_{1t} + a_{2t}v_{2t} + \dots + a_{lt}v_{lt} = 0, \quad \sum_{k=1}^l a_{kt}^2 = 1. \tag{2.9}$$

There is a subsequence $\{a_{ks}\}$ and a_k such that $|a_{ks} - a_k| < 1/s$, $k = 1, 2, \dots, l$. Taking the limit as $s \rightarrow \infty$ we have $a_1v_1 + a_2v_2 + \dots + a_lv_l = 0$ with not all the a_k zero which contradicts the linear independence of the set $\{v_1, v_2, \dots, v_l\}$. \square

Let $\langle F_1, F_2 \rangle$ be a free Lie algebra so $\dim V_N(X_1, X_2) \leq \dim V_N(F_1, F_2) < \infty$. Since the dimension of $V_N(X_1, X_2)$ is bounded for all analytic X_1, X_2 there are by [Lemma 2.3](#) analytic \hat{X}_j so that the dimension of $V_N(\hat{X}_1, \hat{X}_2)$ is locally maximal. In fact it can be shown that there is a Lie algebra $\mathfrak{m}(\hat{X}_1, \hat{X}_2)$ with analytic \hat{X}_j so that the dimension of $V_N(\hat{X}_1, \hat{X}_2)$ is locally maximal for all N .

THEOREM 2.4. *The Lie algebra $\mathfrak{m}(\hat{X}_1, \hat{X}_2)$ is a graded Lie algebra.*

PROOF. Let Y_j be the projection of $e^{-r^2}(\partial/\partial x_p)$, $p = 1, 2, \dots, m$ onto the tangent space at each point of M . Choose two of the projections say Y_1, Y_2 so that they span the tangent space at some point of M . For $\epsilon > 0$ there is an embedding and coordinates on M so that

$$\left\| Y_j(x_1, x_2) - e^{-r^2} \frac{\partial}{\partial x_j} \right\|_0 < \epsilon, \quad \forall (x_1^2 + x_2^2) \leq 1, \quad j = 1, 2. \quad (2.10)$$

Now by argument of [Theorem 3.1](#) we have that $\mathfrak{m}(e^{-r^2}(\partial/\partial x_1), e^{-r^2}(\partial/\partial x_2))$ is a graded Lie algebra. By [Lemma 2.3](#) with a sufficiently small ϵ we have

$$V_N(Y_1, Y_2) = \mathfrak{m}_1(Y_1, Y_2) \oplus \mathfrak{m}_2(Y_1, Y_2) \oplus \dots \oplus \mathfrak{m}_N(Y_1, Y_2). \quad (2.11)$$

Define $E_j(t) = tY_j + (1-t)\hat{X}_j$ and assume

$$\mathfrak{B}_1(E_1(t), E_2(t)) + \mathfrak{B}_2(E_1(t), E_2(t)) + \dots + \mathfrak{B}_N(E_1(t), E_2(t)) = 0 \quad (2.12)$$

with $\mathfrak{B}_p(E_1(1), E_2(1)) \neq 0$ for some $p \leq N$ where $\mathfrak{B}_k(E_1(t), E_2(t))$ is

$$a_{k1}(t)B_{k1}(E_1(t), E_2(t)) + \dots + a_{kl_k}(t)B_{kl_k}(E_1(t), E_2(t)) \quad (2.13)$$

and $B_{kq}(E_1(t), E_2(t))$ is

$$[E_{j_1}(t), [E_{j_2}(t), [\dots, [E_{j_{k-1}}(t), E_{j_k}(t)]] \dots]] \in \mathfrak{m}_k(E_1(t), E_2(t)), \quad (2.14)$$

where the j_i depend on k and q . Since the dimension of $V_N(\hat{X}_1, \hat{X}_2)$ is locally maximal we have that there is a $t_1 < 1$ so that for $t \in (t_1, 1]$, (2.12) holds. The $a_{kq}(t)$ will then be polynomials in t . By (2.11) and [Lemma 2.3](#) there is a $t_0 > 0$ so that for $t \in [0, t_0)$ we must have $\mathfrak{B}_p(E_1(t), E_2(t)) = 0$. Now $\mathfrak{B}_p(E_1(t), E_2(t))$ is analytic in t so $\mathfrak{B}_p(E_1(1), E_2(1)) = 0$ which is a contradiction hence

$$V_N(\hat{X}_1, \hat{X}_2) = \mathfrak{m}_1(\hat{X}_1, \hat{X}_2) \oplus \mathfrak{m}_2(\hat{X}_1, \hat{X}_2) \oplus \dots \oplus \mathfrak{m}_N(\hat{X}_1, \hat{X}_2). \quad (2.15) \quad \square$$

THEOREM 2.5. *The Lie algebras $\mathfrak{m}(X_1, X_2)$ and $\mathfrak{m}(\hat{X}_1, \hat{X}_2)$ are isomorphic.*

PROOF. Let $\epsilon > 0$ be such that for all analytic Y_j so that $\rho(Y_j, \hat{X}_j) < \epsilon$ we have that $\dim V_N(Y_1, Y_2) = \dim V_N(\hat{X}_1, \hat{X}_2)$. Let $Z_j - \hat{X}_j \in \oplus_{k=2}^{\infty} \mathfrak{m}_k(\hat{X}_1, \hat{X}_2)$ so that $\rho(Z_j, \hat{X}_j) < \epsilon$. Assume $\mathfrak{B}(Z_1, Z_2) = 0$ where

$$\mathfrak{B}(Z_1, Z_2) = \mathfrak{B}_1(Z_1, Z_2) + \dots + \mathfrak{B}_N(Z_1, Z_2) \quad (2.16)$$

and $\mathfrak{B}_k(Z_1, Z_2) = a_{k1}B_{k1}(Z_1, Z_2) + \dots + a_{kl_k}B_{kl_k}(Z_1, Z_2)$. The dimension of $V_N(Z_1, Z_2)$ will be locally maximal so by argument of [Theorem 2.4](#) we have that $\mathfrak{B}_p(Z_1, Z_2) = 0$ for all $p \leq N$. Now $\mathfrak{B}_p(Z_1, Z_2) = \mathfrak{B}_p(\hat{X}_1, \hat{X}_2) + W_{p+1} = 0$ where W_{p+1} is an element of $\oplus_{k=p+1}^{\infty} \mathfrak{m}_k(\hat{X}_1, \hat{X}_2)$ hence $\mathfrak{B}_p(\hat{X}_1, \hat{X}_2) = 0$ for all $p \leq N$ consequently $\mathfrak{B}(\hat{X}_1, \hat{X}_2) = 0$. This holds for all such \mathfrak{B} . Since the dimension of $V_N(\hat{X}_1, \hat{X}_2)$ is locally maximal we have $\mathfrak{B}(Z_1, Z_2) = 0$ if and only if $\mathfrak{B}(\hat{X}_1, \hat{X}_2) = 0$.

Define $E_j(t) = tZ_j + (1-t)\hat{X}_j$. Using [Lemma 2.3](#) there is a $t_0 > 0$ such that for $t \in [0, t_0)$, $\rho(E_j(t), \hat{X}_j) < \epsilon$ and $B(E_1(t), E_2(t))$ is a basis element of $V_N(E_1(t), E_2(t))$ if

and only if $B(\hat{X}_1, \hat{X}_2)$ is a basis element of $V_N(\hat{X}_1, \hat{X}_2)$. Using Lemma 2.1 with $\mathfrak{m} = \tilde{\mathfrak{m}}$ and $\Phi = \text{id}$ we can conclude that $\mathcal{B}(E_1(t), E_2(t)) = 0$ for all $t_0 \in [0, t_0)$. Now $\mathcal{B}(E_1(t), E_2(t))$ is analytic in t so when $t = 1$ we have $\mathcal{B}(X_1, X_2) = 0$. This holds for all N and all such \mathcal{B} so we can conclude there is an epimorphism

$$\Phi : \mathfrak{m}(\hat{X}_1, \hat{X}_2) \rightarrow \mathfrak{m}(X_1, X_2). \tag{2.17}$$

It can be shown that there are $W_j \in \tilde{\mathfrak{m}}$ so that $W_j, \Phi(W_j)$ are analytic and the dimension of $V_N(\Phi(W_1), \Phi(W_2))$ is locally maximal for all N and consequently Φ restricted to $\langle W_1, W_2 \rangle$ is an isomorphism. By Lemma 2.1 the completion of $\langle W_1, W_2 \rangle$ in the ρ_2 topology is $\mathcal{S}(M)$ so Φ restricted to $\mathfrak{m}(\hat{X}_1, \hat{X}_2)$ is an isomorphism. \square

It follows from Theorems 2.4 and 2.5 that $\mathfrak{m}(X_1, X_2)$ is a graded Lie algebra.

By Lemma 2.3, Theorem 2.5, and the argument presented in the first paragraph of Theorem 2.4, it can be shown that if $B(X_1, X_2)$ is a basis element of $\mathfrak{m}(X_1, X_2)$ then the bracket $B(e^{-r^2}(\partial/\partial x_1), e^{-r^2}(\partial/\partial x_2))$ is a basis element of $\mathfrak{m}(e^{-r^2}(\partial/\partial x_1), e^{-r^2}(\partial/\partial x_2))$ and vice versa. It can also be shown that the dimension of $\mathfrak{m}_k(e^{-r^2}(\partial/\partial x_1), e^{-r^2}(\partial/\partial x_2))$ grows polynomially with k hence $\mathfrak{m}(X_1, X_2)$ is not a free Lie algebra.

THEOREM 2.6. *The Lie algebras $\mathfrak{m}(X_1, X_2)$ and $\tilde{\mathfrak{m}}(\tilde{X}_1, \tilde{X}_2)$ are isomorphic if and only if M and \tilde{M} are diffeomorphic.*

PROOF. In Theorem 2.2 we showed that if $\mathfrak{m}(X_1, X_2)$ and $\tilde{\mathfrak{m}}(\tilde{X}_1, \tilde{X}_2)$ are isomorphic then M and \tilde{M} are diffeomorphic.

Let σ be a diffeomorphism of M and \tilde{M} . By the paragraph preceding this theorem we have that $B_1(X_1, X_2), \dots, B_l(X_1, X_2)$ are basis elements of $V_N(X_1, X_2)$ if and only if $B_1(\tilde{X}_1, \tilde{X}_2), \dots, B_l(\tilde{X}_1, \tilde{X}_2)$ are basis elements of $V_N(\tilde{X}_1, \tilde{X}_2)$. As in Theorem 2.5 we must have $\mathcal{B}(\sigma_*^{-1}(\tilde{X}_1), \sigma_*^{-1}(\tilde{X}_2)) = 0$ hence $\mathcal{B}(\tilde{X}_1, \tilde{X}_2) = 0$. This holds for all N and \mathcal{B} so we then have an epimorphism $\mathfrak{m}(X_1, X_2) \rightarrow \tilde{\mathfrak{m}}(\tilde{X}_1, \tilde{X}_2)$. Similarly, there is an epimorphism $\tilde{\mathfrak{m}}(\tilde{X}_1, \tilde{X}_2) \rightarrow \mathfrak{m}(X_1, X_2)$ so $\mathfrak{m}(X_1, X_2)$ and $\tilde{\mathfrak{m}}(\tilde{X}_1, \tilde{X}_2)$ are isomorphic. \square

As a possible application we can look at diffeomorphism classes of S^4 . Let $P = \{T_i(\theta_1, \theta_2, \theta_3, \theta_4) : i = 1, \dots, 8\}$ be a set of polynomials in $\sin \theta_j, \cos \theta_j, j = 1, 2, 3, 4$ so that P defines a homeomorphism of S^4 into a subset S_P^4 of \mathbb{R}^8 . Let $\mathfrak{m}(P)$ be the Lie algebra with k th grade $\mathfrak{m}_k(P)$ constructed by projecting $\partial/\partial x_i, i = 1, 2, \dots, 8$ onto the tangent space of S_P^4 and choose four such projections so that they span the tangent space for some point of S_P^4 . We can construct a finite dimensional graded Lie algebra

$$\mathfrak{m}(P, N) = \frac{\mathfrak{m}(P)}{\bigoplus_{k=N}^{\infty} \mathfrak{m}_k(P)} \tag{2.18}$$

for some positive integer N . If we can find two polynomial sets P_1, P_2 , and an N so that $\mathfrak{m}(P_1, N)$ and $\mathfrak{m}(P_2, N)$ are not isomorphic then we have at least two diffeomorphism classes of S^4 .

3. Graded subalgebras constructed from atlases. Let $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ be an atlas of M such that

(1) U_α is a neighborhood of O_α where the O_α are disjoint open connected subsets of M , the union of the closure of all the O_α is M .

- (2) $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ are onto.
 (3) $\phi_\alpha \circ \phi_\beta^{-1}$ are analytic.
 (4) For $\sigma, \beta \in I$, a $y \in I$ can be found such that $U_\sigma \cap U_y = \emptyset$ and $U_\beta \cap U_y \neq \emptyset$.
 (5) For any set $\{\alpha_1, \alpha_2, \dots, \alpha_q\}$ such that if $U = U_{\alpha_1} \cap U_{\alpha_2} \cap \dots \cap U_{\alpha_q} \neq \emptyset$ then U is connected and is a proper subset of $U_{\alpha_1} \cap \dots \cap U_{\alpha_{i-1}} \cap U_{\alpha_{i+1}} \cap \dots \cap U_{\alpha_q}$ for all i .

Without loss of generality, in this section we consider manifolds so that $|I| < \infty$. Define

$$\mathfrak{m} = \langle X_{\alpha p} : \alpha \in I, p \in \mathbb{N} = \{1, 2, \dots, n\} \rangle \subset \mathcal{G}(M) \quad (3.1)$$

to be the Lie algebra generated by $X_{\alpha p}$ where $X_{\alpha p}$ is zero outside U_α and for points of U_α the push forward by ϕ_α of $X_{\alpha p}$ is $e^{-r^2}(\partial/\partial x_p)$ so $\mathfrak{m} = \sum_{k=1}^{\infty} \mathfrak{m}_k$ where \mathfrak{m}_k is the span over \mathbb{R} of the set

$$\{[X_{\beta_1 j_1}, [X_{\beta_2 j_2}, [\dots [X_{\beta_{k-1} j_{k-1}}, X_{\beta_k j_k}]] \dots]] : \beta_i \in I, j_m \in \mathbb{N}\}. \quad (3.2)$$

If all the $\phi_\alpha \circ \phi_\beta^{-1}$ are rational functions it can be shown that the dimension of \mathfrak{m}_k grows polynomially with k .

THEOREM 3.1. *The Lie algebra \mathfrak{m} is a graded Lie algebra.*

PROOF. Consider for example an equation

$$\begin{aligned} a_1 [X_{y_1 j_1}, [X_{y_2 j_2}, [\dots [X_{y_{k-1} j_{k-1}}, X_{y_k j_k}]] \dots]] \\ + \dots + a_l [X_{\sigma_1 i_1}, [X_{\sigma_2 i_2}, [\dots [X_{\sigma_{m-1} i_{m-1}}, X_{\sigma_m i_m}]] \dots]] = 0. \end{aligned} \quad (3.3)$$

We can write a component of (3.3) in the coordinates of U_β as

$$a_1 [P_{11} F_{11} + \dots + P_{1\tau_1} F_{1\tau_1}] e^{-q_1 r^2} + \dots + a_l [P_{l1} F_{l1} + \dots + P_{l\tau_l} F_{l\tau_l}] e^{-q_l r^2} = 0, \quad (3.4)$$

where $P_{ij}(x_1, \dots, x_n)$ is a polynomial and $F_{ij}(x_1, \dots, 0c, x_n)$ is made of factors of partial derivatives of components of $X_{\alpha p}$, $\alpha \neq \beta$. q_i is the number of times the factors of the form $X_{\beta p}$ appear in the i th term of (3.3). Since the $X_{\alpha p}$, $\alpha \neq \beta$ are analytic for some point on the boundary of U_β we must have $q_i = q_j$. Consequently writing equations like (3.3) in the coordinates of U_y for each $y \in I$ and using condition 5 on the atlas allows us to conclude, $\mathfrak{m} = \bigoplus_{k=1}^{\infty} \mathfrak{m}_k$ and $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j}$ follows using properties of the Lie bracket. \square

Construct $\{(\tilde{U}_{\tilde{\alpha}}, \tilde{\phi}_{\tilde{\alpha}}) : \tilde{\alpha} \in \tilde{I}\}$, $\tilde{X}_{\tilde{\alpha} p}$, $\tilde{\mathfrak{m}}$, $\tilde{\rho}(\cdot, \cdot)$ for a smooth manifold \tilde{M} in the same way that $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$, $X_{\alpha p}$, \mathfrak{m} , $\rho(\cdot, \cdot)$ were for M . Assume there is a Lie algebra isomorphism $\Phi : \mathfrak{m} \rightarrow \tilde{\mathfrak{m}}$. We can take Φ so that $\Phi(\mathfrak{m}_1) = \tilde{\mathfrak{m}}_1$. We now show this implies M and \tilde{M} are diffeomorphic.

LEMMA 3.2. *There is a bijection $B : I \rightarrow \tilde{I}$ such that for all $p \in \mathbb{N}$, the support of $\Phi(X_{\alpha p})$ is the closure of $\tilde{U}_{B(\alpha)}$.*

PROOF. Write $\Phi(X_{\alpha p}) = \tilde{Z}_{\tilde{\alpha}_1} + \dots + \tilde{Z}_{\tilde{\alpha}_m}$ where $\tilde{Z}_{\tilde{\alpha}_i} \in \tilde{\mathfrak{m}}_1$ and $\text{supp } \tilde{Z}_{\tilde{\alpha}_i} = \tilde{U}_{\tilde{\alpha}_i}$. Assume $m > 1$. There is a $\tilde{Z}_{\tilde{\alpha}_1}$, say $\tilde{Z}_{\tilde{\alpha}_1}$, so that $\text{supp } \Phi^{-1}(\tilde{Z}_{\tilde{\alpha}_1}) \supset U_\alpha$. By condition 4 on the atlas there is a $\tilde{U}_{\tilde{y}}$ such that $\tilde{U}_{\tilde{\alpha}_1} \cap \tilde{U}_{\tilde{y}} = \emptyset$ and $\tilde{U}_{\tilde{\alpha}_m} \cap \tilde{U}_{\tilde{y}} \neq \emptyset$. We can then find a $\tilde{X}_{\tilde{y} q}$ so that $[\Phi(X_{\alpha p}), [\tilde{X}_{\tilde{y} q}, \tilde{Z}_{\tilde{\alpha}_m}]] \neq 0$, hence $\Phi^{-1}([\tilde{X}_{\tilde{y} q}, \tilde{Z}_{\tilde{\alpha}_m}])$ has support on

U_α . It can be checked that the bracket of an element of \mathfrak{m}_1 with an element of \mathfrak{m}_2 is not zero if they have common support. The intersection of U_α and the supports of $\Phi^{-1}(\tilde{Z}_{\tilde{\alpha}_1})$ and $\Phi^{-1}([\tilde{X}_{\tilde{y}q}, \tilde{Z}_{\tilde{\alpha}_m}])$ is then not empty so $[\Phi^{-1}(\tilde{Z}_{\tilde{\alpha}_1}), \Phi^{-1}([\tilde{X}_{\tilde{y}q}, \tilde{Z}_{\tilde{\alpha}_m}])] \neq 0$ hence $[\tilde{Z}_{\tilde{\alpha}_1}, [\tilde{X}_{\tilde{y}q}, \tilde{Z}_{\tilde{\alpha}_m}]] \neq 0$ which is a contradiction since $\tilde{U}_{\tilde{\alpha}_1} \cap \tilde{U}_{\tilde{y}} = \emptyset$ so $m = 1$. Use the argument as just presented on $\Phi(X_{\alpha p} + X_{\alpha q})$ to conclude that $\text{supp} \Phi(X_{\alpha p}) = \tilde{U}_{\tilde{\alpha}_1}$, for each $p \in \mathbb{N}$. Define $B : I \rightarrow \tilde{I}$ by $B(\alpha) = \tilde{\alpha}_1$. If $\text{supp} \Phi(X_{\beta_1 p_1}) = \text{supp} \Phi(X_{\beta_2 p_2})$ then by letting $\tilde{Z}_{B(\beta_1)} = \Phi(X_{\beta_1 p_1} + X_{\beta_2 p_2})$ and using a similar argument as just presented on $\Phi^{-1}(\tilde{Z}_{B(\beta_1)})$ allows us to conclude that B is bijective. \square

THEOREM 3.3. *If \mathfrak{m} and $\hat{\mathfrak{m}}$ are isomorphic then M and \tilde{M} are diffeomorphic.*

PROOF. We can write $L \in \mathcal{G}(M)$ as $L = \sum_{\alpha \in I} L_\alpha$ where $L_\alpha \in \mathcal{G}(M)$ has support, a compact subset of U_α . By **Lemmas 3.2** and **2.1** we have that L_α is in the completion of $\langle X_{\alpha p} : p \in \mathbb{N} \rangle$ in the ρ_2 topology. Now use the argument of **Theorem 2.2**. \square

We now look at how the graded Lie algebras constructed from different atlases satisfying the five conditions differ. Let $\{(U_\alpha, \phi_\alpha), (\hat{U}_\beta, \hat{\phi}_\beta) : \alpha \in I \setminus \{\beta\}\}$ be an atlas of M satisfying the five conditions on an atlas and require also that $U_\beta \cap U_\gamma \neq \emptyset$ if and only if $\hat{U}_\beta \cap U_\gamma \neq \emptyset$ for all $\gamma \in I$. Define $\hat{X}_{\beta p}$ to be the push forward of $X_{\beta p}$ by $\hat{\phi}_\beta^{-1} \circ \phi_\beta$. Define

$$\hat{\mathfrak{m}} = \langle X_{\alpha p}, \hat{X}_{\beta p} : \alpha \in I \setminus \{\beta\}, p \in \mathbb{N} \rangle \tag{3.5}$$

with k th grade $\hat{\mathfrak{m}}_k$.

We can in fact choose \hat{U}_β and $\hat{\phi}_\beta$ so that the dimension of $\sum_{k=1}^N \hat{\mathfrak{m}}_k$ is locally maximum for all N .

THEOREM 3.4. *There is an epimorphism $\hat{\mathfrak{m}} \rightarrow \mathfrak{m}$ mapping $\hat{X}_{\beta p}$ to $X_{\beta p}$ and $X_{\alpha p}$ to $X_{\alpha p}$ for $\alpha \neq \beta$.*

PROOF. Without loss of generality, we can let $n = 1$ and define $X_\alpha = X_{\alpha 1}$. Define a diffeomorphism

$$\theta^{-1} : (-1, 1) \rightarrow \mathbb{R}, \quad x \mapsto \frac{x}{1-x^2}. \tag{3.6}$$

Write $\hat{\phi}_\beta^{-1} \circ \phi_\beta$ in the $\theta \circ \phi_\beta$ coordinates of U_β as $x \mapsto \hat{x} = x + T(x)$ and $x = \hat{x} + S(\hat{x})$. Let

$$T^{(N)}(x) = e^{-1/(1+\delta)^2 - x^2} \sum_{l=0}^N b_l^{(N)} x^l, \quad \delta > 0 \tag{3.7}$$

be such that as $N \rightarrow \infty$, $T^{(N)}$ converges uniformly to T on $[-1, 1]$. Define $\hat{X}_\beta^{(N)}$ and $\hat{\mathfrak{m}}^{(N)}$ by replacing $T(x)$ by $T^{(N)}(x)$ in the definition of \hat{X}_β and $\hat{\mathfrak{m}}$. Consider for example an equation of the form

$$\begin{aligned} & a_1^{(N)} [\hat{X}_\beta^{(N)}, [X_{y_2}, [\dots [X_{y_{k-1}}, \hat{X}_\beta^{(N)}] \dots]] \\ & + \dots + a_l^{(N)} [X_{v_1}, [\hat{X}_\beta^{(N)}, [\dots [\hat{X}_\beta^{(N)}, X_{v_k}] \dots]]] = 0. \end{aligned} \tag{3.8}$$

By the argument of **Theorem 3.1**, each term will contain the same number of factors of the form $\hat{X}_\beta^{(N)}$. We can write $X_\beta \mapsto \hat{X}_\beta$ in the $\theta \circ \phi_\beta$ coordinates of U_β as

$$e^{-[x/(1-x^2)]^2} \frac{(1-x^2)^2}{1+x^2} \frac{d}{dx} \mapsto \frac{e^{-[(x+S(x))/(1-(x+S(x))^2)]^2} (1-(x+S(x))^2)^2}{1+S'(x)} \frac{d}{dx}. \tag{3.9}$$

With this in mind, we can write (3.8) in the $\theta \circ \phi_\beta$ coordinates of U_β after clearing denominators and exponentials as

$$\begin{aligned}
 & a_1^{(N)} \left[\left(P_{11}^{(N)} + Q_{11}^{(N)} \right) F_{11}^{(N)} + \cdots + \left(P_{1\tau_1}^{(N)} + Q_{1\tau_1}^{(N)} \right) F_{1\tau_1}^{(N)} \right] + \cdots \\
 & + a_l^{(N)} \left[\left(P_{l1}^{(N)} + Q_{l1}^{(N)} \right) F_{l1}^{(N)} + \cdots + \left(P_{l\tau_l}^{(N)} + Q_{l\tau_l}^{(N)} \right) F_{l\tau_l}^{(N)} \right] = 0,
 \end{aligned}
 \tag{3.10}$$

where the $F_{ij}^{(N)}(x)$ are made of factors of derivatives of the component of X_α , $\alpha \neq \beta$. $P_{ij}^{(N)}(x)$ is a polynomial and $Q_{ij}^{(N)}$ is a polynomial in x , $S^{(N)}$, and derivatives of $S^{(N)}$. Each term of $Q_{ij}^{(N)}$ contains a $S^{(N)}$ or a derivative of $S^{(N)}$ as a factor. Since $S^{(N)}$ is analytic on $(-1-\delta, 1+\delta)$, (3.10) holds for all points of $(-1-\delta, 1+\delta)$. On the boundary of $(-1-\delta, 1+\delta)$, $S^{(N)}$ and all its derivatives are zero hence for all points of $(-1, 1)$

$$a_1^{(N)} \left[P_{11}^{(N)} F_{11}^{(N)} + \cdots + P_{1\tau_1}^{(N)} F_{1\tau_1}^{(N)} \right] + \cdots + a_l^{(N)} \left[P_{l1}^{(N)} F_{l1}^{(N)} + \cdots + P_{l\tau_l}^{(N)} F_{l\tau_l}^{(N)} \right] = 0. \tag{3.11}$$

Equation (3.8) then holds when $\hat{X}_\beta^{(N)}$ is replaced by X_β . There is then an epimorphism $\hat{\mathfrak{m}}^{(N)} \rightarrow \mathfrak{m}$ taking $\hat{X}_\beta^{(N)}$ to X_β and X_α to X_α for $\alpha \neq \beta$. Now let $N \rightarrow \infty$ and use the local maximality of the dimension of $\sum_{k=1}^N \hat{\mathfrak{m}}_k$ for all N . □

Let $\hat{\mathfrak{f}} = \langle \hat{L}_{\alpha p} : \alpha \in I, p \in \mathbb{N} \rangle$ be a graded Lie algebra with k th grade $\hat{\mathfrak{f}}_k$ constructed as was \mathfrak{m} and such that the dimension of $\sum_{k=1}^N \hat{\mathfrak{f}}_k$ is locally maximum for all N . We can use Theorem 3.4 to show that there is an $\epsilon > 0$ such that for all such $\hat{\mathfrak{f}}$ with $\rho(\hat{L}_{\alpha p}, X_{\alpha p}) < \epsilon$ for all $\alpha \in I, p \in \mathbb{N}$ we have an epimorphism $\hat{\mathfrak{f}} \rightarrow \mathfrak{m}$.

REFERENCES

[1] H. Whitney, *Differentiable manifolds*, Ann. of Math. II. Ser. **37** (1936), 645–680. [Zbl 015.32001](#).

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