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ON CHARACTERIZATIONS OF FIXED POINTS

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ABSTRACT. We give some necessary and sufficient conditions for the existence of fixed points of a family of self mappings of a metric space and we establish an equivalent condition for the existence of fixed points of a continuous compact mapping of a metric space.

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1. Introduction. Jungck [2] first gave a necessary and sufficient condition for the existence of fixed points of a continuous self mapping of a complete metric space. Afterwards, Park [8], Leader [4], and Khan and Fisher [3] established a few theorems similar to that of Jungck. Janos [1] and Park [9] proved fixed point theorems for compact self mappings of a metric space. Recently, Liu [5] established criteria for the existence of fixed points of a family of self mappings of a metric space. The aim of this paper is to offer some characterizations for the existence of fixed points of a family of self mappings and a continuous compact mappings of metric spaces, respectively. We also establish a fixed point theorem for two compact mappings, which extends properly the results of Janos [1] and Park [9].

Let ω and N denote the sets of nonnegative and positive integers, respectively. Suppose that (X,d) is a metric space. For $x,y \in X$, define

$$\begin{aligned} C_f &= \{g \mid g : X \longrightarrow X \text{ and } fg = gf\}, \\ H_f &= \{g \mid g : X \longrightarrow X \text{ and } g\cap_{n\in\omega} f^n X \subseteq \cap_{n\in\omega} f^n X\}, \\ H_f(x) &= \{hx \mid h \in H_f\}, \quad H_f(x,y) = H_f(x) \cup H_f(y), \\ O(x,f) &= \{f^n x \mid n \in \omega\}, \quad O(x,y,f) = O(x,f) \cup O(y,f). \end{aligned} \tag{1.1}$$

Obviously, $C_f \subseteq H_f$. Let Φ be a family of self mappings of X. A point $x \in X$ is said to be a *fixed point* of Φ if $fx = x$ for all $f \in \Phi$. Let $F : X \times X \to [0,+\infty)$ be continuous and $F(x,y) = 0$ if and only if $x = y$. For $A,B \subset X$, define

$$\delta(A,B) = \sup\{F(x,y) \mid x \in A,\ y \in B\} \tag{1.2}$$

and $\delta(A) = \delta(A,A)$. Particularly, $d(A) = \sup\{d(x,y) \mid x,y \in A\}$. Let $M(X)$ denote the set of all metrics on X that are topologically equivalent to d for a given metric space (X,d). A self mapping f of a metric space (X,d) is said to be *compact* if there exists a compact set Y satisfying $fX \subseteq Y \subseteq X$.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 (see [6]). *Let f be a continuous compact self mapping of a metric space (X,d). If $A = \cap_{n\in\omega} f^n X$, then*
(a1) *A is compact,*
(a2) *$fA = A \neq \emptyset$,*
(a3) *$d(f^n X) \to d(A)$ as $n \to \infty$,*
(a4) *$\{f^n \mid n \in \omega\} \subseteq H_f$.*

Lemma 1.2 (see [7]). *Let f be a continuous self mappings of a metric space (X,d) with the following properties:*
(a5) *f has a unique fixed point w in X,*
(a6) *for every $x \in X$, the sequence of iterations $\{f^n x\}_{n=0}^{\infty}$ converges to w,*
(a7) *there exists an open neighborhood U of w with the property that given any open set V containing w, there exists $k \in N$ such that $n \geq k$ implies $f^n U \subset V$.*
Then for each $\alpha \in (0,1)$, there exists a metric $d' \in M(X)$ relative to which f is a contraction with Lipschitz constant α.

2. Main results

Theorem 2.1. *Let Φ be a family of self mappings of a metric space (X,d). Then the following statements are equivalent:*
(b1) *Φ has a fixed point;*
(b2) *there exist $m,n \in N$ and continuous compact self mappings f, g of (X,d) such that either $\Phi \subseteq C_f$ or $\Phi \subseteq C_g$ and*

$$F(f^m x, g^n y) < \delta(H_f(x), H_g(y)), \tag{2.1}$$

for all $x,y \in X$ with $f^m x \neq g^n y$;
(b3) *there exist $m,n \in N$ and continuous self mappings f, g of (X,d) such that fg is compact, $f \in C_g$, $\Phi \in C_{fg}$, and*

$$F(f^m x, g^n y) < \delta(H_{fg}(x,y)), \tag{2.2}$$

for all $x,y \in X$ with $f^m x \neq g^n y$,
(b4) *there exists a continuous compact self mapping of (X,d) with $\Phi \subseteq C_f$ such that*

$$\begin{aligned} F(fx,fy) < \max\Big\{ & F(x,y), F(x,fx), F(y,fy), \frac{F(x,fx)F(y,fy)}{F(x,y)}, \\ & \frac{F(fx,fy)F(x,fx)}{F(x,y)}, \frac{F(x,fy)F(fx,y)}{F(x,y)} \Big\}, \end{aligned} \tag{2.3}$$

for all $x,y \in X$ with $x \neq y$.

Moreover, if (b2) holds, then f, g, and Φ have a unique common fixed point; if (b3) holds, then fg and Φ have a unique common fixed point; if (b4) holds, then f and Φ have a unique common fixed point.

Proof. Let (b1) hold and w be a fixed point of Φ. Define $f,g : X \to X$ by $fx = gx = w$ for all $x \in X$. It is easy to show that (b2), (b3), and (b4) hold.

Assume that (b2) holds. Let $A = \cap_{n\in\omega} f^n X$, $B = \cap_{n\in\omega} g^n X$. Since f and g are continuous compact self mappings of (X,d), it follows from (a1) and (a2) that A and B are compact and $fA = A$, $gB = B$. Consequently, $f^m A = A$, $g^n B = B$. Suppose that $\delta(A,B) > 0$. Then there exist $a \in A$, $b \in B$ with $\delta(A,B) = F(a,b)$ because F is continuous and $A\times B$ is compact. Since $f^m A = A$, $g^n B = B$, there exist $x \in A$, $y \in B$ such that $f^m x = a$, $g^n y = b$. In view of (2.1), we have

$$\delta(A,B) = F(a,b) = F(f^m x, g^n y) < \delta(H_f(x), H_g(y)) < \delta(A,B), \tag{2.4}$$

which is impossible, and hence $\delta(A,B) = 0$. That is, $A = B = \{w\}$ for some $w \in X$ so $fw = gw = w$. If v is another fixed point of f, then $v \in \cap_{n\in\omega} f^n X = \{w\}$, that is, $v = w$. Hence w is the only fixed point of f. Similarly, w is also the only fixed point of g.

Without loss of generality, we assume that $\Phi \subseteq C_f$. It follows from $C_f \subseteq H_f$ that $hA \subseteq A$ for all $h \in \Phi$. That is, $hw = w$ for all $h \in \Phi$. Thus w is the only common fixed point of f, g, and Φ. Therefore (b1) holds.

Assume that (b3) holds. Put $A = \cap_{n\in\omega}(fg)^n X$. Then A is compact and $fgA = A$. Since f is continuous and $f \in C_g$, we infer that

$$fA = f\cap_{n\in\omega}(fg)^n X \subseteq \cap_{n\in\omega}(fg)^n fX \subseteq \cap_{n\in\omega}(fg)^n X = A. \tag{2.5}$$

Similarly, we have

$$gA \subseteq A. \tag{2.6}$$

It follows from $fgA = A$, (2.5), and (2.6) that

$$fA \subseteq A = fgA \subseteq fA. \tag{2.7}$$

That is, $fA = A$. Similarly, we have $gA = A$. Suppose that $\delta(A) > 0$. Because F is continuous and A is compact, then there exist $a,b \in A$ such that $\delta(A) = F(a,b)$. Since $f^m A = g^n A = A$, there exist $x,y \in A$ with $f^m x = a$, $g^n y = b$. Using (2.2), we have

$$\delta(A) = F(a,b) = (f^m x, g^n y) < \delta(H_{FG}(x,y)) \le \delta(A), \tag{2.8}$$

which is a contradiction. Hence $\delta(A) = 0$. That is, $A = \{w\}$ for some $w \in X$. This implies that $fw = gw = fgw = w$. As in the proof of above, we can prove that w is the only fixed point of fg, and w is the unique common fixed point of fg and Φ. So (b1) holds.

Assume that (b4) holds. As above we infer that $A = \cap_{n\in\omega} f^n X$ is compact and $fA = A$. Since F is continuous, the function $\phi(x)$ defined by $\phi(x) = F(x,fx)$ for $x \in A$ is continuous and attains its minimum value at some $w \in A$. Suppose that $w \ne fw$. By virtue of (2.3), we get

$$\begin{aligned}\phi(fw) &= F(fw,ffw)\\ &< \max\Big\{F(w,fw),F(w,fw),F(fw,ffw),\frac{F(w,fw)F(fw,ffw)}{F(w,fw)},\\ &\qquad \frac{F(fw,ffw)F(w,fw)}{F(w,fw)},\frac{F(w,ffw)F(fw,fw)}{F(w,fw)}\Big\}\\ &= F(w,fw)\\ &= \phi(w).\end{aligned} \tag{2.9}$$

This is a contradiction to the definition of w. So w is a fixed point of f. If f has a second distinct fixed point v, by (2.3), we obtain that

$$\begin{aligned}F(w,v) &= F(fw,fv)\\ &< \max\Big\{F(w,v),F(w,w),F(v,v),\frac{F(w,w)F(v,v)}{F(w,v)},\\ &\qquad \frac{F(w,v)F(w,w)}{F(w,v)},\frac{F(w,v)F(w,v)}{F(w,v)}\Big\}\\ &= F(w,v),\end{aligned} \tag{2.10}$$

which is a contradiction. Therefore, w is the only fixed point of f. It is a simple matter to show that w is the unique common fixed point of f and Φ. Thus (b1) holds. This completes the proof. □

Next, we give a theorem about the equivalent condition for the existence of fixed points of a continuous compact self mapping on a metric space.

Theorem 2.2. *Let s be a continuous compact self mapping of a metric space (X,d). Then s has a fixed point if and only if there exists a continuous self mapping f of X such that $f \in C_s$ and*

$$\begin{aligned}F(fx,fy) < \max\Big\{&F(sx,sy),F(sx,fx),F(sy,fy),\frac{F(sx,fx)F(sy,fy)}{F(sx,sy)},\\ &\frac{F(fx,fy)F(sx,fx)}{F(sx,sy)},\frac{F(sx,fy)F(fx,sy)}{F(sx,sy)}\Big\},\end{aligned} \tag{2.11}$$

for all $x,y \in X$ with $sx \neq sy$. Indeed, f and s have a unique common fixed point.

Proof. To see that the stated conditions is necessary, suppose that s has a fixed point $w \in X$. Define $f : X \to X$ by $fx = w$ for all $X \in X$. Then $fsx = w = sw = sfx$ for all $x \in X$, that is, $f \in C_s$. Clearly, (2.11) holds.

On the other hand, suppose that there exists a continuous self mapping f of X such that $f \in C_s$ and (2.11) holds. Let $A = \cap_{n\in\omega} S^n X$. From Lemma 1.1, we infer that A is compact and $sA = A$. Since f is continuous and $s \in C_f$, we have

$$fA = f \cap_{n\in\omega} s^n X \subseteq \cap_{n\in\omega} s^n fX \subseteq \cap_{n\in\omega} s^n X = A = sA. \tag{2.12}$$

Define the function $\phi(x)$ by $\phi(x) = F(sx, fx)$ for all $x \in A$. It is clear that $\phi(x)$ is continuous on A and attains its minimum value at some $w \in A$. We claim that $sw = fw$. If not, from (2.12), there exists $p \in A$ satisfying $fw = sp$. Using (2.11), we conclude that

$$\begin{aligned} \phi(p) &= F(sp,fp) = F(fw,fp) \\ &< \max\left\{F(sw,sp), F(sw,fw), F(sp,fp), \frac{F(sw,fw)F(sp,fp)}{F(sw,sp)}, \right. \\ &\qquad \left. \frac{F(fw,fp)F(sw,fw)}{F(sw,sp)}, \frac{F(sw,fp)F(fw,sp)}{F(sw,sp)}\right\} \\ &= \max\{F(sw,fw), F(sp,fp)\} \\ &= F(sw,fw) \\ &= \phi(w), \end{aligned} \tag{2.13}$$

which is a contradiction to the choice of w. So $sw = fw$. By virtue of $f \in C_s$, we have

$$fsw = sfw = ssw. \tag{2.14}$$

Now suppose that $ssw \neq sw$. By (2.11) and (2.14), we get

$$\begin{aligned} F(ssw,sw) &= F(fsw,fw) \\ &< \max\left\{F(ssw,sw), F(ssw,fsw), F(sw,fw), \frac{F(ssw,fsw)F(sw,fw)}{F(ssw,sw)}, \right. \\ &\qquad \left. \frac{F(fsw,fw)F(ssw,fsw)}{F(ssw,sw)}, \frac{F(ssw,fw)F(fsw,sw)}{F(ssw,sw)}\right\} \\ &= F(ssw,sw), \end{aligned} \tag{2.15}$$

which is a contradiction. Thus $ssw = sw$, that is, sw is a fixed point of s. Therefore, the set M of fixed points of s is not empty. Now s is continuous, so M is closed. Since $M \subseteq A$ and A is compact, M is compact. Moreover, since f and s commute, $f(M) \subseteq M$. Note also that (2.11) restricted to M reduces to (2.3). We can therefore apply Theorem 2.1(b4) to $f_M : M \to M$ to obtain a unique common fixed point u of f and s in M. Since M contains all the fixed points of s, u is a unique common fixed point f and s. This completes the proof. □

Theorem 2.3. *Let f, g be continuous compact self mappings of a metric space (X,d) satisfying (2.1). Then f and g have a unique fixed point, respectively, and furthermore, for any $\alpha \in (0,1)$, there exist metrics d' and $d'' \in M(X)$ relative to which f and g satisfy, respectively,*

$$d'(fx,fy) \le \alpha d'(x,y), \qquad d''(gx,gy) \le \alpha d''(x,y), \tag{2.16}$$

for all $x, y \in X$.

Proof. Let $A = \cap_{n\in\omega} f^n X$, $B = \cap_{n\in\omega} g^n X$, and $U = X$. As in the proof of Theorem 2.1, we have $A = B = \{w\}$. Lemma 1.1 ensures that (a5) and (a6) hold. Note that

$d(f^nX), d(g^nX) \to 0$ as $n \to \infty$. Thus f^nX and g^nX squeeze into any neighborhood of w. That is (a7) is fulfilled. Thus Theorem 2.3 follows from Lemma 1.2. This completes the proof. □

Corollary 2.4. *Let f be a continuous compact self mapping of a metric space (X,d) satisfying*

$$F(fx, fy) < \delta(H_f(x), H_f(y)), \tag{2.17}$$

for all $x, y \in X$ with $x \neq y$. Then f has a unique fixed point.

Furthermore, for any $\alpha \in (0,1)$, there exists a metric $d' \in M(X)$ relative to which f satisfies

$$d'(fx, fy) \le \alpha d'(x,y), \tag{2.18}$$

for all $x, y \in X$.

The following simple example reveals that Corollary 2.4 extends properly Theorem 1.1 of Janos [1] and Theorem 1 of Park [9].

Example 2.5. Let $X = \{0,2,4,6,9\}$ with the usual metric. Define a mapping $f : X \to X$ by $f0 = f4 = f6 = 6$, $f2 = 0$, and $f9 = 2$. Then f is a continuous compact self mapping of X. It is easy to check that

$$d(fx, fy) \le 6 < 9 = \delta(H_f(x), H_f(y)), \tag{2.19}$$

for all $x, y \in X$ with $x \neq y$. So the conditions of Corollary 2.4 are satisfied. But Theorem 1.1 of Janos [1] and Theorem 1 of Park [9] are not applicable since

$$\begin{aligned} d(f2, f4) &= 6 > 2 = \frac{1}{2}[d(2,f2) + d(4,f4)], \\ d(f2, f4) &= 6 = \delta(O(2,4,f)). \end{aligned} \tag{2.20}$$

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