IJMMS 27:9 (2001) 541–546
PII. S0161171201006597
http://ijmms.hindawi.com

ON SOLUTIONS OF THE GOŁĄB-SCHINZEL EQUATION

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(Received 29 January 2001)

ABSTRACT. We determine the solutions $f:(0,\infty)\to[0,\infty)$ of the functional equation $f(x+f(x)y)=f(x)f(y)$ that are continuous at a point $a>0$ such that $f(a)>0$. This is a partial solution of a problem raised by Brzdęk.

2000 Mathematics Subject Classification. 39B22.

The well-known Gołąb-Schinzel functional equation

$$f(x+f(x)y)=f(x)f(y) \tag{1}$$

has been studied by many authors (cf. [1, 3, 5, 7, 10]) in many classes of functions. Recently Aczél and Schwaiger [2], motivated by a problem of Kahlig, solved the following conditional version of (1)

$$f(x+f(x)y)=f(x)f(y) \quad \text{for } x\ge 0, y\ge 0, \tag{2}$$

in the class of continuous functions $f:\mathbb{R}\to\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Some further conditional generalizations of (1) have been considered by Reich [9] (see also [8] and Brzdęk [4]).

At the 38th International Symposium on Functional Equations (Noszvaj, Hungary, June 11–17, 2000) Brzdęk raised, among others, the problem (see [6]) of solving the equation

$$f(x+f(x)y)=f(x)f(y), \quad \text{whenever } x,y,x+f(x)y\in\mathbb{R}_+, \tag{3}$$

in the class of functions $f:\mathbb{R}_+\to\mathbb{R}$ that are continuous at a point, where $\mathbb{R}_+=(0,\infty)$. We give a partial solution to the problem, namely we determine the solutions $f:\mathbb{R}_+\to[0,\infty)$ of (3) that are continuous at a point $a\in\mathbb{R}_+$ such that $f(a)>0$. Note that actually equations (1) and (3) have the same solutions in the class of functions $f:\mathbb{R}_+\to[0,\infty)$.

From now on we assume that $f:\mathbb{R}_+\to[0,\infty)$ is a solution of (3), continuous at a point $a\in\mathbb{R}_+$ such that $f(a)>0$.

We start with some lemmas.

Lemma 1. *Suppose that* $y_2>y_1>0$ *and* $f(y_1)=f(y_2)>0$. *Then*

(a) $f(t+(y_2-y_1))=f(t)$ *for* $t\ge y_1$;

(b) *for every* $z>0$ *such that* $f(z)>0$,

$$f(t+f(z)(y_2-y_1))=f(t) \quad \text{for } t\ge z+y_1f(z); \tag{4}$$

(c) *if* $z_1, z_2 > 0$ *and* $f(z_2) > f(z_1) > 0$, *then*

$$f(t + (f(z_2) - f(z_1))(y_2 - y_1)) = f(t) \quad \textit{for } t \geq \max\{z_1 + y_1 f(z_1), z_2 + y_1 f(z_2)\}. \tag{5}$$

Proof. (a) We argue in the same way as in [2, 7]. Namely, for $t \geq y_1$, by (3) we have

$$\begin{aligned} f(t + (y_2 - y_1)) &= f\Big(y_2 + \frac{t - y_1}{f(y_1)} f(y_1)\Big) = f\Big(y_2 + \frac{t - y_1}{f(y_1)} f(y_2)\Big) \\ &= f(y_2) f\Big(\frac{t - y_1}{f(y_1)}\Big) = f(y_1) f\Big(\frac{t - y_1}{f(y_1)}\Big) \\ &= f\Big(y_1 + \frac{t - y_1}{f(y_1)} f(y_1)\Big) = f(t). \end{aligned} \tag{6}$$

(b) For every $z > 0$ such that $f(z) > 0$ we have

$$f(z + y_1 f(z)) = f(z) f(y_1) = f(z) f(y_2) = f(z + y_2 f(z)) \tag{7}$$

and consequently by (a) (with y_1 and y_2 replaced by $z + y_1 f(z)$ and $z + y_2 f(z)$)

$$f(t) = f[t + (z + y_2 f(z) - z - y_1 f(z))] = f(t + f(z)(y_2 - y_1)) \tag{8}$$

for $t \geq z + y_1 f(z)$.

(c) Since $(f(z_2) - f(z_1))(y_2 - y_1) > 0$, $t + (f(z_2) - f(z_1))(y_2 - y_1) \geq \max\{z_1 + y_1 f(z_1), z_2 + y_1 f(z_2)\}$ for $t \geq \max\{z_1 + y_1 f(z_1), z_2 + y_1 f(z_2)\}$. Thus using (b) twice, for $z = z_1$ and $z = z_2$ (the first time with t replaced by $t + (f(z_2) - f(z_1))(y_2 - y_1)$), we have

$$\begin{aligned} &f(t + (f(z_2) - f(z_1))(y_2 - y_1)) \\ &\quad = f[t + (f(z_2) - f(z_1))(y_2 - y_1) + f(z_1)(y_2 - y_1)] \\ &\quad = f(t + f(z_2)(y_2 - y_1)) = f(t) \end{aligned} \tag{9}$$

for $t \geq \max\{z_1 + y_1 f(z_1), z_2 + y_1 f(z_2)\}$. □

Lemma 2. *Let* $y_2 > y_1 > 0$ *and* $f(y_1) = f(y_2) > 0$. *Then there exists* $x_0 > 0$ *such that for every* $d > 0$ *there is* $c \in (0, d)$ *with* $f(t + c) = f(t)$ *for* $t \geq x_0$.

Proof. First suppose that there is a neighbourhood $U = (a - \delta, a + \delta)$ of a on which f is constant. Then for every $x \in U$ such that $a < x$, from Lemma 1(a), we get

$$f(t + (x - a)) = f(t) \quad \text{for } t \geq a. \tag{10}$$

Thus it is enough to take $x_0 = a$.

Now assume that there does not exist any neighbourhood of a on which f is constant. Take $\varepsilon \in (0, f(a))$. The continuity of f at a implies that there exists $\delta \in (0, 1)$ such that for every $x \in U_1 = (a - \delta, a + \delta)$ we have $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$. Take $x_1, x_2 \in U_1$ such that $f(x_1) < f(x_2)$. Then $f(x_2) - f(x_1) < 2\varepsilon$. From $\varepsilon < f(a)$ we infer $f(x_1) > 0$ and by Lemma 1(c) we get

$$f(t + (f(x_2) - f(x_1))(y_2 - y_1)) = f(t) \quad \text{for } t \geq \max\{x_1 + y_1 f(x_1), x_2 + y_1 f(x_2)\}. \tag{11}$$

Next by a suitable choice of ε the value $c := (f(x_2) - f(x_1))(y_2 - y_1)$ can be made arbitrarily small. Moreover, $x_1, x_2 < a + 1$ and $f(x_1), f(x_2) < f(a) + \varepsilon < 2f(a)$, which means that $f(t + c) = f(t)$ for $t \geq x_0 := a + 1 + y_1 2f(a)$. This completes the proof. □

Lemma 3. *If for some $y_2 > y_1 > 0$, $f(y_1) = f(y_2) > 0$, then for every $\varepsilon > 0$ and $e > 0$ there is $c \in (0,e)$ with $f(t+c) = f(t)$ for $t \geq \varepsilon$.*

Proof. By Lemma 2 there exists $x_0 > 0$ such that for arbitrarily small $c > 0$

$$f(t+c) = f(t) \quad \text{for } t \geq x_0. \tag{12}$$

By induction, from Lemma 1(a), we get $f(y_1) = f(y_1 + n(y_2 - y_1))$ for any positive integer n. Consequently there exists $x_1 \in [x_0,\infty)$ with $f(x_1) = f(y_1)$.

Put $B = \{x > x_0 : f(x) > 0\}$. Clearly $x_1 \in B$. Thus (12) implies that $B \cap A \neq \varnothing$ for every nontrivial interval $A \subset [x_0,\infty)$. Define a function $f_1 : [0,\infty) \to [x_0,\infty)$ by

$$f_1(x) = x_1 + x f(x_1). \tag{13}$$

Note that f_1 is increasing. Let $\varepsilon > 0$ and $y_0 \in B \cap (f_1(0), f_1(\varepsilon)) \neq \varnothing$. By the continuity of f_1 there exists $z_0 \in (0,\varepsilon)$ such that $f_1(z_0) = y_0$. Take $d > 0$ with $f(t+d) = f(t)$ for $t \geq x_0$. Then

$$f(y_0) = f(y_0 + d) \neq 0. \tag{14}$$

The form of the function f_1 implies that there exists $z_1 > z_0$ such that $f_1(z_1) = y_0 + d$. Note that (14) yields

$$\begin{aligned} f(x_1 + z_0 f(x_1)) &= f(f_1(z_0)) \\ &= f(y_0) = f(y_0 + d) = f(f_1(z_1)) \\ &= f(x_1 + z_1 f(x_1)) \neq 0. \end{aligned} \tag{15}$$

Further by (3)

$$f(x_1) f(z_0) = f(x_1) f(z_1) \neq 0, \tag{16}$$

and consequently $f(z_0) = f(z_1) > 0$. Hence, in view of Lemma 1(a), we infer that

$$f(t + (z_1 - z_0)) = f(t) \quad \text{for } t \geq z_0. \tag{17}$$

This completes the proof, because $\varepsilon > z_0$ and, choosing sufficiently small d, we can make $c := (z_1 - z_0)$ arbitrarily small. □

Lemma 4. *If there exist $y_2 > y_1 > 0$ such that $f(y_1) = f(y_2) > 0$, then $f \equiv 1$.*

Proof. First we show that $f(x) = f(a) =: b$ for $x \in \mathbb{R}_+$. For the proof by contradiction suppose that there exists $t_0 > 0$ with $f(t_0) \neq f(a)$. Put

$$\varepsilon_0 = |f(t_0) - f(a)|. \tag{18}$$

The continuity of f at a implies that there exists $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon_0$. By Lemma 3 there exists $y_0 > 0$ such that $|y_0 - a| < \delta$ and $f(y_0) = f(t_0)$, which means that $|f(t_0) - f(a)| < \varepsilon_0$, contrary to (18). Thus we have proved that $f \equiv b$. Clearly from (3) we get $b = f(a) = f(a + af(a)) = f(a)^2 = b^2$ and consequently $b = 1$. This completes the proof. □

Lemma 5. *If f is nonconstant then $(f(x) - 1)/x$ is constant for all $x > 0$ with $f(x) > 0$.*

PROOF. Suppose that $x > 0$, $y > 0$, $x \neq y$, $f(x)f(y) > 0$, and

$$\frac{f(x)-1}{x} \neq \frac{f(y)-1}{y}. \tag{19}$$

Then $x + yf(x) \neq y + xf(y)$ and

$$f(x+yf(x)) = f(x)f(y) = f(y+xf(y)) > 0. \tag{20}$$

Thus, by Lemma 4, $f \equiv 1$, a contradiction. □

REMARK 6. If we denote the constant in Lemma 5 by c, then from Lemma 5 we get $f(x) \in \{cx+1, 0\}$ for every $x > 0$. In the case $c < 0$ we have $f(x) = 0$ for every $x \geq -1/c$ (because $f \geq 0$).

LEMMA 7. *Suppose that f is nonconstant. Then,*
(a) *in the case $c := (f(a)-1)/a < 0$, $f(x) = cx+1$ for $x \in (0, -1/c)$;*
(b) *in the case $c := (f(a)-1)/a > 0$, $f(x) = cx+1$ for $x > 0$.*

PROOF. The continuity of f at a implies that there exists $\delta \in (0, a)$ such that $f(x) > 0$ for every $x \in U = [a-\delta, a+\delta]$. Thus, by Remark 6, $f(x) = cx+1$ for $x \in U$.

Let $I = (a, -1/c)$ if $c < 0$ and $I = (a, \infty)$ if $c > 0$. Put $B_1 := \{x \in (0,a) : f(x) = 0\}$, $B_2 := \{x \in I : f(x) = 0\}$, $B = B_1 \cup B_2$,

$$d_1 := \begin{cases} \sup B_1 & \text{if } B_1 \neq \emptyset, \\ a-\delta & \text{if } B_1 = \emptyset, \end{cases} \qquad d_2 := \begin{cases} \inf B_2 & \text{if } B_2 \neq \emptyset, \\ a+\delta & \text{if } B_2 = \emptyset. \end{cases} \tag{21}$$

Clearly $f(x) > 0$ on the interval $A = (d_1, d_2) \supset (a-\delta, a+\delta)$.

(a) For the proof by contradiction suppose that there exists $b_1 \in (0, -1/c)$ with $f(b_1) = 0$. Notice that $d_2 < -1/c$. Indeed, if $B_2 \neq \emptyset$ then, since $B_2 \subset (a, -1/c)$, so $\inf B_2 < -1/c$. If not, then from Remark 6 we have that $a + \delta < -1/c$. Consequently $d_2 < -1/c$. Thus $cd_2 > -1$ and consequently $\delta + \delta c d_2 > 0$. Take $b \in B$ and $z \in A$ such that $|z - b| < \delta + \delta c d_2$. Define functions $h, g : U \to \mathbb{R}$ by

$$\begin{aligned} h(x) &= x + zf(x) \quad \text{for } x \in U, \\ g(x) &= x + bf(x) \quad \text{for } x \in U. \end{aligned} \tag{22}$$

By the continuity of f on U, h is continuous. Next, since $z < d_2$, so $cz > cd_2$ and $\delta + \delta cz > \delta + \delta c d_2 > 0$. Hence

$$\begin{aligned} h(a) - h(a-\delta) &= a + z(ca+1) - a + \delta - z[c(a-\delta)+1] = \delta + \delta cz > 0, \\ h(a+\delta) - h(a) &= a + \delta + z[c(a+\delta)+1] - a - z(ca+1) = \delta + \delta cz > 0. \end{aligned} \tag{23}$$

Moreover $1 > ca+1 = f(a) > 0$, whence

$$\begin{aligned} |h(a) - g(a)| &= |a + z(ca+1) - a - b(ca+1)| \\ &= |z-b||ca+1| < |z-b| < \delta + \delta c d_2 < \delta + \delta cz. \end{aligned} \tag{24}$$

From (23) and (24) we obtain

$$h(a-\delta) < g(a) < h(a+\delta). \tag{25}$$

The continuity of h implies that there exists $x_0 \in (a-\delta, a+\delta)$ such that $h(x_0) = g(a)$. Since $a, x_0, z \in A$ and $b \in B$, so we have

$$\begin{aligned} 0 \neq f(x_0)f(z) &= f(x_0 + zf(x_0)) = f(h(x_0)) \\ &= f(g(a)) = f(a + bf(a)) = f(a)f(b) = 0. \end{aligned} \tag{26}$$

This contradiction ends the proof of (a).

(b) For the proof by contradiction suppose that $f(b_1) = 0$ for some $b_1 > 0$. Since $ca+1 = f(a) > 0$, there are $b \in B$ and $z \in A$ such that $|z-b| < \delta/(ca+1)$. Define functions $h, g : U \to \mathbb{R}$ in the same way as in the proof of (a). Then (23) holds and

$$\left| h(a) - g(a) \right| = |z-b||ca+1| < \frac{\delta}{ca+1}(ca+1) = \delta < \delta + \delta cz. \tag{27}$$

Hence

$$h(a-\delta) < g(a) < h(a+\delta). \tag{28}$$

We obtain a contradiction in a similar way as in the proof of (a). □

Lemma 8. *If $c := (f(a)-1)/a = 0$, then $f(x) = 1$ for $x > 0$.*

Proof. The continuity of f at a implies that there exists $\delta > 0$ such that $f(x) > 0$ for every $x \in [a-\delta, a+\delta]$. Thus, by Lemma 5 and Remark 6, $f(x) = cx+1 = 1$ for every $x \in [a-\delta, a+\delta]$. Hence Lemma 4 implies that $f(x) = 1$ for every $x > 0$. □

Finally from Lemmas 7 and 8 and Remark 6 we get the following theorem.

Theorem 9. *If a function $f : \mathbb{R}_+ \to [0,\infty)$ is continuous at a point a such that $f(a) \neq 0$ and satisfies (3), then*

$$f(x) = \max\{cx+1, 0\} \quad \forall x \in \mathbb{R}_+. \tag{29}$$

Acknowledgement. I wish to thank Professor Janusz Brzdęk for paying my attention to the problem and for his most valuable suggestions during the preparation of this paper.

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