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DEVELOPMENT OF SINGULARITIES IN SOLUTIONS OF A HYPERBOLIC SYSTEM

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ABSTRACT. We consider a special type of a hyperbolic system and show that classical solutions blow up in finite time even for small initial data.

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1. Introduction. For the system of nonlinear elasticity

$$u_t(x,t) = \varphi(v(x,t))v_x(x,t), \qquad v_t(x,t) = u_x(x,t), \tag{1.1}$$

it is well known that C^1-solutions break down in finite time however smooth and small the initial data are. This was shown by Lax [4] in 1964. In his work, the author studied (1.1), for $\varphi > 0$ and $\varphi' > 0$, and established a blowup result. MacCamy and Mizel [7] in 1967 considered the same system and proved a similar result, allowing φ' to change sign. They also showed, under appropriate conditions on φ, that there are x-intervals, for which the solution must exist for all time even though it blows up for values of x outside these intervals.

Messaoudi [9] discussed the following system:

$$u_t(x,t) = \alpha(x)\varphi(v(x,t))v_x(x,t), \qquad v_t(x,t) = u_x(x,t), \tag{1.2}$$

which models a transverse motion of a string with variable density. He showed that C^1-solutions develop singularities in finite time if the initial data are taken with large enough gradients. He also discussed, in [8], a system with dissipation of the form

$$\theta_t + c(\theta)q_x = 0, \qquad q_t + \sigma(\theta)\theta_x = -\lambda(\theta)q, \tag{1.3}$$

which describes heat propagation in materials that predict finite propagation speed. This phenomenon is called second sound. Here θ is the difference temperature and q is the heat flux. He studied the Cauchy problem and proved a blowup result of the classical solutions. We should note that, for λ constant and $c(\theta) = -1$, (1.3) reduces to a system describing steady shearing flows in nonlinear viscoelastic fluids. This problem was studied by Slemrod [11] and a blowup result for classical solutions has been established. A similar problem was also discussed by Nishibata [10], Kosiński [3], and Zheng [12] and results concerning global existence and nonexistence have been accomplished.

For more general systems, it is worth mentioning the work of Li et al. [6], in which they discussed

$$u_t(x,t) = A(u(x,t))u_x(x,t), \tag{1.4}$$

associated with decaying initial data. Here $u : I \times (0,T) \to \mathbb{R}^n$ is a vector-valued function, A is an $(n\times n)$-matrix, and I is an interval (bounded or unbounded). They proved a global C^1-solution for the Cauchy problem if, in addition to the local strict hyperbolicity condition, (1.4) is weakly linearly degenerate and the initial data satisfy, for $\mu > 0$, $\sup_x\{(1+|x|)^{1+\mu}|u_0'(x)|+|u_0(x)|\}$ is small enough. They also established a blowup result to C^1-solutions for nonweakly linearly degenerate systems. As they pointed out, their work generalizes their result of [5] to the case of initial data with no compact support but they possess certain decay properties.

In this work, we are concerned with a quasilinear hyperbolic system of the form

$$u_t(x,t) = \varphi\left(\frac{v(x,t)}{1+au(x,t)}\right)v_x(x,t), \qquad v_t(x,t) = u_x(x,t), \tag{1.5}$$

where the constant $a \neq 0$. In addition to its importance from the mathematical technique point of view, this system can be regarded as a relative generalization of the one-dimensional wave equation in the sense if $a = 0$, (1.5) reduces to (1.1). We will consider (1.5) together with initial conditions and show that C^1-solutions blowup even for small initial data. Our result cannot be directly deduced from the results of [6] since we do not impose the same conditions regarding the size and the regularity of the initial data (cf. [6, Theorem 1.2] and Theorem 3.1 below). This work is divided into two parts. In part one we state, without proof, a local existence theorem. In part two our main result is stated and proved.

2. Local existence.

We consider the following Cauchy problem

$$u_t(x,t) = \varphi\left(\frac{v(x,t)}{1+au(x,t)}\right)v_x(x,t), \tag{2.1}$$

$$v_t(x,t) = u_x(x,t), \quad \forall x \in \mathbb{R},\ t > 0, \tag{2.2}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \forall x \in \mathbb{R}, \tag{2.3}$$

where $a \neq 0$ and φ is a function satisfying

$$\varphi(\xi) \geq \beta > 0, \quad \forall \xi \in \mathbb{R}. \tag{2.4}$$

Proposition 2.1. *Assume that φ is a C^1 function satisfying (2.4) and let u_0 and v_0 in $H^2(\mathbb{R})$ be given such that*

$$|1+au_0(x)| \geq \lambda > 0, \quad \forall x \in \mathbb{R}. \tag{2.5}$$

Then the problem (2.1), (2.2), and (2.3) has a unique local solution (u,v), on a maximal time interval $[0,T)$, satisfying

$$u,v \in C([0,T),H^2(\mathbb{R})) \cap C^1([0,T),H^1(\mathbb{R})). \tag{2.6}$$

This result can be proved by applying a classical energy argument [1] or the nonlinear semigroup theory [2].

Remark 2.2. The functions u, v are C^1 functions by the standard Sobolev embedding theory.

3. Formation of singularities. We introduce the quantities and the differential operators

$$\begin{aligned} r &:= \frac{1}{a}\ln|1+au| + \int_0^{v/(1+au)} \alpha(\xi)\,d\xi, \\ s &:= \frac{1}{a}\ln|1+au| - \int_0^{v/(1+au)} \beta(\xi)\,d\xi, \\ \partial_t &:= \frac{\partial}{\partial t} - \rho\left(\frac{v}{1+au}\right)\frac{\partial}{\partial x}, \\ D_t &:= \frac{\partial}{\partial t} + \rho\left(\frac{v}{1+au}\right)\frac{\partial}{\partial x}, \end{aligned} \tag{3.1}$$

where

$$\rho(\xi) = \sqrt{\varphi(\xi)}, \qquad \alpha(\xi) = \frac{\sqrt{\varphi(\xi)}}{1+a\xi\sqrt{\varphi(\xi)}}, \qquad \beta(\xi) = \frac{\sqrt{\varphi(\xi)}}{1-a\xi\sqrt{\varphi(\xi)}}. \tag{3.2}$$

The following lemma shows, for initial data appropriately chosen, that r, s, and ρ are well defined and $|v(x,t)/(1+au(x,t))|$ is uniformly bounded.

Theorem 3.1. *Let a and φ be as in Proposition 2.1. Then there exist initial data in $H^2(\mathbb{R})$ satisfying (2.5), for which*

$$\left|\frac{av(x,t)}{1+au(x,t)}\sqrt{\varphi\left(\frac{v(x,t)}{1+au(x,t)}\right)}\right| < 1, \quad |1+au(x,t)| > 0, \tag{3.3}$$

and $|v(x,t)/(1+au(x,t))|$ is uniformly bounded on $\mathbb{R}\times[0,T)$.

Proof. We first choose $\delta > 0$ such that if

$$|u_0(x)| < \delta, \quad |v_0(x)| < \delta, \quad \forall x \in \mathbb{R}, \tag{3.4}$$

then

$$\left|\frac{av_0(x)}{1+au_0(x)}\sqrt{\varphi\left(\frac{v_0(x)}{1+au_0(x)}\right)}\right| < 1, \quad |1+au_0(x)| > 0,\ \forall x \in \mathbb{R}. \tag{3.5}$$

Of course, this is possible by taking δ small enough. Then the continuity of u, v, and φ implies that there exists $T' \le T$, such that (3.3) holds on $\mathbb{R}\times[0,T')$. Let $T_0 := \sup\{T' : (3.3)$ holds for all $x \in \mathbb{R},\ t \in [0,T')\}$. We have two cases, either $T_0 = T$, this completes the proof. Or $T_0 < T$; in this case we estimate

$$\begin{aligned}\partial_t r &= \frac{u_t}{1+au} + \alpha\left[\frac{v_t}{1+au} - \frac{v}{(1+au)^2}au_t\right] \\ &\quad -\rho\left[\frac{u_x}{1+au} + \alpha\frac{v_x}{1+au} - \alpha\frac{v}{(1+au)^2}au_x\right] \\ &= \frac{1}{1+au}\left[\left(1-a\alpha\frac{v}{1+au}\right)u_t - \alpha\rho v_x\right] \\ &\quad + \frac{1}{1+au}\left[\alpha v_t - \rho\left(1-a\alpha\frac{v}{1+au}\right)u_x\right], \quad \forall x\in\mathbb{R},\ t\in[0,T_0).\end{aligned} \tag{3.6}$$

We recall that, unless otherwise stated, α, β, ρ, and φ are functions of $v/(1+au)$. By noting that $\alpha\rho = (1-a\alpha v/(1+au))\varphi$, $(1-a\alpha 1-a\alpha v/(1+au))\rho = \alpha$, and using (2.1) and (2.2), we obtain

$$\partial_t r = 0, \quad \forall x\in\mathbb{R},\ t\in[0,T_0). \tag{3.7}$$

Similar calculations also yield

$$D_t s = 0, \quad \forall x\in\mathbb{R},\ t\in[0,T_0). \tag{3.8}$$

Therefore, on $\mathbb{R}\times[0,T_0)$, r and s remain constant along backward and forward characteristics, respectively; hence $\|r\|_\infty = \|r_0\|_\infty$ and $\|s\|_\infty = \|s_0\|_\infty$. It is easy to see that

$$r(x,t) - s(x,t) = \phi\left(\frac{v(x,t)}{1+au(x,t)}\right), \quad \forall x\in\mathbb{R},\ t\in[0,T_0), \tag{3.9}$$

where $\phi(\tau) = 2\int_0^\tau \sqrt{\varphi(\xi)}/(1-a^2\xi^2\varphi(\xi))\,d\xi$ is strictly monotone and continuous at least in a neighborhood of zero, so it admits a continuous inverse ψ near zero. Since the function $g(\xi) = 1-a^2\xi^2\varphi(\xi)$ is continuous and $g(0) = 1$, one can choose γ so that $g(\xi) \ge \varepsilon > 0$, for all $|\xi| < \gamma$ and choose $\delta_1 > 0$ so that $|\psi(\tau)| < \gamma$, for all $|\tau| < \delta_1$. Therefore, by choosing δ small enough so that (3.4) holds and $\|r_0\|_\infty + \|s_0\|_\infty < \delta_1$, we get

$$|r(x,t) - s(x,t)| \le ||r_0||_\infty + ||s_0||_\infty < \delta_1, \tag{3.10}$$

consequently

$$\left|\frac{v(x,t)}{1+au(x,t)}\right| = |\psi(r-s)| < \gamma, \tag{3.11}$$

which yields

$$\left|\frac{av(x,t)}{1+au(x,t)}\sqrt{\varphi\left(\frac{v(x,t)}{1+au(x,t)}\right)}\right| \le 1-\varepsilon < 1, \quad \forall x\in\mathbb{R},\ t\in[0,T_0]. \tag{3.12}$$

We then use (3.1), the boundedness of r, and the fact that $1+a\xi\sqrt{\varphi(\xi)} \ge \varepsilon$ to conclude that $\ln|1+au|$ is bounded on $\mathbb{R}\times[0,T_0]$; hence $|1+au| > 0$. Again by continuity, there exists $T_1 > T_0$ such that (3.3) holds on $\mathbb{R}\times[0,T_1)$. This contradicts the maximality of T_0; hence T_0 must be equal to T. Therefore (3.3) and (3.11) hold. This completes the proof. □

Theorem 3.2. *Assume that, in addition to (2.4), φ satisfies $\varphi'(0) > 0$. Then there exist initial data u_0, v_0 in $H^2(\mathbb{R})$ satisfying (3.4), for which the solution of the problem (2.1), (2.2), and (2.3) blows up in finite time.*

Proof. We take an x-partial derivative of (3.7) to get

$$(\partial_t r)_x = r_{xt} - \rho r_{xx} - r_x\rho_x = 0 \tag{3.13}$$

which, in turn, implies

$$\partial_t r_x = r_x\rho_x = \frac{\varphi'}{2\sqrt{\varphi}}r_x\frac{\partial}{\partial x}\left(\frac{v}{1+au}\right). \tag{3.14}$$

We then use

$$r_x = \frac{u_x}{1+au} + \alpha\cdot\frac{\partial}{\partial x}\left(\frac{v}{1+au}\right), \qquad s_x = \frac{u_x}{1+au} - \beta\cdot\frac{\partial}{\partial x}\left(\frac{v}{1+au}\right), \tag{3.15}$$

and substitute in (3.14) to arrive at

$$\begin{aligned}\partial_t r_x &= \frac{\varphi'}{2\sqrt{\varphi}(\alpha+\beta)}r_x(r_x - s_x)\\ &= \frac{\varphi'}{4\varphi}\left(1 - a^2\left(\frac{v}{1+au}\right)^2\varphi\right)r_x^2 - \frac{\varphi'}{4\varphi}\left(1 - a^2\left(\frac{v}{1+au}\right)^2\varphi\right)r_x s_x.\end{aligned} \tag{3.16}$$

To handle the last term in (3.16), we set $W := \varphi^{1/4}r_x$ and substitute in (3.16), to get

$$\begin{aligned}\partial_t W = \varphi^{1/4}\frac{\varphi'}{4a\varphi}\left(1-a^2\left(\frac{v}{1+au}\right)^2\varphi\right)r_x^2 - \varphi^{1/4}\frac{\varphi'}{4a\varphi}\left(1-a^2\left(\frac{v}{1+au}\right)^2\varphi\right)r_x s_x\\ + \frac{1}{4}\varphi^{-3/4}\varphi' r_x\partial_t\left(\frac{v}{1+au}\right).\end{aligned} \tag{3.17}$$

By using (2.1) and (2.2), we see that

$$\begin{aligned}\partial_t\left(\frac{v}{1+au}\right) &= \frac{(1+au)(v_t - \sqrt{\varphi}v_x) - av(u_t - \sqrt{\varphi}u_x)}{(1+au)^2}\\ &= \frac{(1+au)(u_x - \sqrt{\varphi}v_x) - av(\varphi v_x - \sqrt{\varphi}u_x)}{(1+au)^2}\\ &= \frac{(u_x - \sqrt{\varphi}v_x)(1+au+a\sqrt{\varphi}v)}{(1+au)^2}.\end{aligned} \tag{3.18}$$

Also straightforward computations lead to

$$s_x = \frac{1}{\sqrt{\varphi}}\frac{\beta}{1+au}(u_x - \sqrt{\varphi}v_x) = \frac{(u_x - \sqrt{\varphi}v_x)}{1+au-av\sqrt{\varphi}}. \tag{3.19}$$

By combining (3.17), (3.18), and (3.19), we arrive at

$$\partial_t W = \varphi^{-5/4}\frac{\varphi'}{4}\left(1-a^2\left(\frac{v}{1+au}\right)^2\varphi\right)W^2. \tag{3.20}$$

If we choose δ sufficiently small, the coefficient of the quadratic term in (3.20) remains bounded away from zero; that is, $\varphi^{-5/4}\varphi'(1-a^2(v/(1+au))^2\varphi)/4 \ge k > 0$. Consequently, (3.20) gives

$$\partial_t W \ge kW^2. \tag{3.21}$$

Therefore, by choosing initial data small enough and satisfying (3.4) with derivatives such that $W_0 > 0$, (3.21) shows that W (hence r_x) blows up in finite time. This completes the proof. □

Remark 3.3. Similar result can be obtained for $\varphi'(0) < 0$. In this case consider the evolution of s_x on the forward characteristics.

Remark 3.4. A simple integration of (3.21) shows that the larger W_0 is, the quicker the blowup takes place.

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