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SOLVABILITY OF KOLMOGOROV-FOKKER-PLANCK EQUATIONS FOR VECTOR JUMP PROCESSES AND OCCUPATION TIME ON HYPERSURFACES

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Abstract. We study occupation time on hypersurface for Markov n-dimensional jump processes. Solvability and uniqueness of integro-differential Kolmogorov-Fokker-Planck with generalized functions in coefficients are investigated. Then these results are used to show that the occupation time on hypersurfaces does exist for the jump processes as a limit in variance for a wide class of piecewise smooth hypersurfaces, including some fractal type and moving surfaces. An analog of the Meyer-Tanaka formula is presented.

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1. Introduction. The local time or the occupation time of stochastic processes have been studied by many authors (cf. bibliography in survey papers [2, 5]). For example, the local time has been studied for the scalar Brownian motion and scalar semimartingales (cf. [9, 11]), for general one-dimensional diffusions (see [13]), for stable processes (see [10]).

For vector continuous processes, the distribution of the occupation time is also well studied. McGill [8] derived an analog of Tanaka formula for a solution of a scalar homogeneous nonlinear diffusion equation. Bass [1] investigated occupation time of multidimensional non-Markovian continuous semimartingales of general type and proved the existence of local time. Rosen and Yor [12] considered the occupation time for processes in the plain at points of intersections. Dokuchaev [3] studied occupation time for degenerating diffusion vector processes.

It appears that many important properties of Brownian local time do not hold for a case the occupation time of n-dimensional diffusion processes on $(n-1)$-dimensional hypersurfaces. For example, this occupation time cannot be presented as occupational measure in a case $n > 1$. Hence it is not a perfect analog of Brownian local time.

The paper studies occupation time on hypersurface for Markov n-dimensional jump processes. As is known, analogs of Kolmogorov-Fokker-Planck equations for the probability distribution for jump processes are second-order integro-differential equations (cf. [4]). We extend solvability and uniqueness results for these equations for a case when there are generalized functions in coefficients (Sections 3 and 4). Then we show that the occupation time exists as a limit in variance, and an analog of the Meyer-Tanaka formula is derived, that is, the occupation time is presented as a stochastic integral (Section 6). Equations for the characteristic function of the occupation time are derived in Section 7.

2. Definitions. Consider a standard probability space $(\Omega,\mathcal{F},\mathbf{P})$, a standard n-dimensional Wiener process with independent components such that $w(0)=0$, and a Poisson measure $\nu(\cdot,t)$ in $\mathbb{R}^m$ such that the process $w(\cdot)$ and the measure $\nu(\cdot,t)$ are independent of each other. We assume that there exists a measure $\Pi(\cdot)$ in $\mathbb{R}^m$ such that $\mathbf{E}\nu(A,t)=t\Pi(A)$ for each measurable set $A\subset\mathbb{R}^m$.

Set $\tilde{\nu}(A,t)\triangleq\nu(A,t)-t\Pi(A)$.

Let a be a random real n-dimensional vector such that a does not depend on $w(\cdot)$ and $\nu(\cdot)$. We assume also that $\mathbf{E}|a|^2<+\infty$.

We consider an n-dimensional stochastic differential equation

$$dy(t)=f(y(t),t)dt+\beta(y(t),t)dw(t)+\int_{\mathbb{R}^m}\theta(y(t),u,t)\tilde{\nu}(du,dt),\qquad y(0)=a. \tag{2.1}$$

The functions $f(x,t):\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$, $\beta(x,t):\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^{n\times n}$, and $\theta(x,u,t):\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^{n\times n}$ are bounded and Borel measurable. We assume that the function $\theta(x,u,t)$ is continuous in u, the derivatives $\partial f(x,t)/\partial x$, $\partial\beta(x,t)/\partial x$, $\theta(x,u,t)/\partial x$ are bounded, and there exists a number $\delta>0$ such that $b(x,t)=(1/2)\beta(x,t)\beta(x,t)^\top\geq\delta I_n>0$ (for all x,t), where I_n is the unit matrix in $\mathbb{R}^{n\times n}$. Also, we assume that $\Pi(\mathbb{R}^m)<+\infty$.

Under these assumptions, (2.1) has the unique strong solution (cf. [6, Theorem 2, page 242]).

Let a bounded hypersurface $\Gamma(t)$ of dimension $n-1$ be given for a.a. $t\in[0,T]$, and let $\partial\Gamma(t)$ be its edge (it can happen that $\partial\Gamma(t)=\emptyset$). Let some number $T>0$ be given. Let Ind denote the indicator function, and let $|\cdot|$ denote the Euclidean norm.

We will study the occupation time of $y(t)$ in $\Gamma(t)$. More precisely, we will study the limit of the random variables

$$l_\varepsilon(T)\triangleq\frac{1}{\varepsilon}\int_0^T\operatorname{Ind}\{y(t)\in\Gamma(\varepsilon,t)\}\,dt \tag{2.2}$$

as $\varepsilon\to0+$, where

$$\Gamma(\varepsilon,t)\triangleq\left\{x\in\mathbb{R}^n:\inf_{y\in\Gamma(t)}|x-y|<\frac{\varepsilon}{2}\right\}. \tag{2.3}$$

Spaces and classes of functions. Below $\|\cdot\|_X$ denotes a norm in a space X, and $(\cdot,\cdot)_X$ denotes the scalar product in a Hilbert space X.

Introduce some spaces of (complex-valued) functions. Let $H^0\triangleq L_2(\mathbb{R}^n)$, $H^1\triangleq W_2^1(\mathbb{R}^n)$, where $W_q^m(\mathbb{R}^n)$ is the Sobolev space of functions which belong to $L_q(\mathbb{R}^n)$ together with first m derivatives, $q\geq1$.

Let H^{-1} be the dual space to H^1, with the norm $\|\cdot\|_{H^{-1}}$ such that $\|u\|_{H^{-1}}$, for $u\in H^0$, is the supremum of $(u,v)_{H^0}$ over all $v\in H^0$ such that $\|v\|_{H^1}\leq1$. Let ℓ_m denote the Lebesgue measure in $\mathbb{R}^m$, and let $\bar{\mathcal{B}}_m$ be the σ-algebra of the Lebesgue sets in $\mathbb{R}^m$. We introduce the following spaces:

$$C^0\triangleq C([0,T];H^0),\quad X^k\triangleq L^2([0,T],\bar{\mathcal{B}}_1,\ell_1;H^k),\quad k=0,\pm1, \tag{2.4}$$

and $Y^1\triangleq X^1\cap C^0$, with the norm $\|u\|_{Y^1}\triangleq\|u\|_{X^1}+\|u\|_{C^0}$.

The scalar product $(u,v)_{H^0}$ is assumed to be well defined for $u \in H^{-1}$ and $v \in H^1$ as well (extending it in a natural manner from $u \in H^0$ and $v \in H^1$).

Let μ be a real number such that

$$\mu \in \begin{cases} (1,2) & \text{if } n = 1, \\ (1, n(n-1)^{-1}) & \text{if } n > 1. \end{cases} \tag{2.5}$$

We introduce the space $\mathcal{W} = W_\mu^1(\mathbb{R}^n)$ with the norm

$$\|u\|_{\mathcal{W}} \triangleq \|u\|_{L_\mu(\mathbb{R}^n)} + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L_\mu(\mathbb{R}^n)} \tag{2.6}$$

and its conjugate space $\mathcal{W}^*$, as well as the space $\mathcal{X} \triangleq L^\infty([0,T], \bar{\mathcal{B}}_1, \ell_1; \mathcal{W}^*)$.

3. On solvability of integro-differential equations. In this section, an integro-differential analog of the Kolmogorov-Fokker-Planck equation is studied. Existence results theorems for these equations can be found in Carrany and Menaldy [4]. However, we will need to extend the existence results for a case when there are generalized functions in coefficients.

Let $\mathcal{A} = \mathcal{A}(t)$ be the parabolic operator generated by the process $y(t)$,

$$\mathcal{A} \triangleq \mathcal{A}_c + \mathcal{J}, \tag{3.1}$$

where

$$\mathcal{A}_c v = \mathcal{A}_c(t) v \triangleq \sum_{i,j=1}^{n} b_{ij}(x,t) \frac{\partial^2 v}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} \hat{f}_i(x,t) \frac{\partial v}{\partial x_i}(x), \tag{3.2}$$

$$\mathcal{J} v \triangleq \mathcal{J}'(t) v - \Pi(\mathbb{R}^m) v, \qquad \mathcal{J}'(t) v \triangleq \int_{\mathbb{R}^m} v(x + \theta(x,u,t)) \Pi(du), \tag{3.3}$$

and where

$$\hat{f}(x,t) \triangleq f(x,t) - \int_{\mathbb{R}^m} \theta(x,u,t) \Pi(du). \tag{3.4}$$

Here b_{ij}, $\hat{f}_i$, x_i are the components of the matrix b and the vectors $\hat{f}$, x.

Let $Q \triangleq \mathbb{R}^n \times [0,T]$. Consider a boundary value problem in Q

$$\frac{\partial V}{\partial t} + \mathcal{A} V + g V = -\varphi, \qquad V(x,T) = R(x). \tag{3.5}$$

As is known, problem (3.5) is uniquely solvable in the class Y^1 for $\varphi \in X^{-1}$, $R \in H^0$ and $g \in L_\infty(Q)$ (cf. [7]).

Condition 3.1. There exists a constant $c_\pi > 0$ such that

$$\mathbf{P}(\eta \in B) \le c_\pi \ell_n(B) \ell_n(D)^{-1} \tag{3.6}$$

for any $t \in [0,T]$, any bounded measurable set $D \subset \mathbb{R}^n$, and any measurable set $B \subseteq D$, where $\eta \triangleq \eta_1 + \theta(\eta_1, \eta_2, t)$, and where $\eta_1 : \Omega \to \mathbb{R}^n$ and $\eta_2 : \Omega \to \mathbb{R}^m$ are independent random vectors such that η_2 has the distribution described by the measure $\Pi(\cdot)/\Pi(\mathbb{R}^m)$, and $\mathbf{P}(\eta_1 \in B) = \ell_n(B)\ell_n(D)^{-1}$.

Notice that Condition 3.1 is satisfied if, for example, $\theta(x,u,t) \equiv \theta(u,t)$, that is, does not depend on x. Another example is described in the following proposition.

Proposition 3.2. *Let there exist an integer $K > 0$, $p_1,\dots,p_K \in \mathbb{R}$ and $u_1,\dots,u_K \in \mathbb{R}^m$ such that*

$$\int_{\mathbb{R}^m} \xi(u)\Pi(du) = \sum_{i=1}^{K} \xi(u_i)p_i \tag{3.7}$$

for any $\xi(\cdot) \in C(\mathbb{R}^m)$. Set $F_i(x,t) \triangleq x + \theta(x,u_i,t)$, $D_i \triangleq F_i(\mathbb{R}^n,t)$. Let there exists the inverse function $F^{-1}(x,t) : D_i \to \mathbb{R}^n$ for any given t (i.e., $F^{-1}(F(x,t),t) \equiv x$). Moreover, let there exists a constant $c > 0$ such that $\ell_n(F_i^{-1}(B,t)) \le c\ell_n(B)$ for any measurable set $B \subset D_i$, $i = 1,2,\dots,n$. Then Condition 3.1 is satisfied.

Proof. Let $D \subset \mathbb{R}^n$, η and η_1 be such as in Condition 3.1. For any measurable set $B \subset D$, we have

$$\begin{aligned} \mathbf{P}(\eta \in B) &= \sum_{i=1}^{n} p_i \mathbf{P}(F_i(\eta_1) \in B) = \sum_{i=1}^{n} p_i \mathbf{P}(\eta_1 \in F_i^{-1}(B)) \\ &= \frac{\ell_n(F_i^{-1}(B))}{\ell_n(D)} \le c\frac{\ell_n(B)}{\ell_n(D)}. \end{aligned} \tag{3.8}$$

This completes the proof. □

The following proposition will be useful.

Proposition 3.3 (see [3]). (i) *If $\xi \in H^1$ and $\eta \in H^0$, then $\xi\eta \in \mathcal{W}$ and $\|\xi\eta\|_{\mathcal{W}} \le c\|\xi\|_{H^1}\|\eta\|_{H^0}$, where $c = c(n,\mu)$ is a constant.*

(ii) *If $\xi \in H^1$ and $g \in \mathcal{W}^* \cap H^0$, then $\xi g \in H^{-1}$ and $\|\xi g\|_{H^{-1}} \le c\|\xi\|_{H^1}\|g\|_{\mathcal{W}^*}$ for a constant $c = c(n,\mu)$.*

Introduce the following parameter:

$$\mathcal{P} \triangleq \left\{ n, T, \delta, \sup_{x,t} |f(x,t)|, \sup_{x,t} |\beta(x,t)|, \sup_{x,t,i} \left| \frac{\partial \beta(x,t)}{\partial x_i} \right|, \Pi(\mathbb{R}^m), c_\pi \right\}. \tag{3.9}$$

Theorem 3.4. *Let Condition 3.1 be satisfied. Let $g \in \mathcal{X}$, $\varphi \in X^{-1}$, and $R \in H^0$ be given. Let $g_\varepsilon \in L_\infty(Q) \cap \mathcal{X}$ be such that $\|g_\varepsilon - g\|_{\mathcal{X}} \to 0$ as $\varepsilon \to 0+$. Then,*

(i) *for any $\varepsilon > 0$, there exists the unique solution $V = V_\varepsilon \in Y_1$ of the problem (3.5) with $g = g_\varepsilon$;*

(ii) *the sequence V_ε has a limit V in Y^1 as $\varepsilon \to 0+$. This limit is uniquely defined by φ, g, and V does not depend on the sequence $\{g_\varepsilon\}$. Moreover, $gV \in X^{-1}$ and*

$$\|V\|_{Y^1} \le c\left(\|\varphi\|_{X^{-1}} + \|R\|_{H^0}\right), \tag{3.10}$$

where $c > 0$ is a constant which depends only on the parameters $\mathcal{P}$, μ, and $\|g\|_{\mathcal{X}}$.

Remark 3.5. It can be seen from the proof of Theorem 3.4 that this theorem holds even if the derivatives $\partial f(x,t)/\partial x$ and $\partial\theta(x,t)/\partial x$ do not exist.

Definition 3.6. The limit V, defined in Theorem 3.4, is said to be the solution in Y^1 of the problem (3.5) with $g \in \mathcal{X}$, $\varphi \in X^{-1}$ and $R \in H^0$.

Note that V depends linearly on (φ, R) for any given g. Moreover, by (3.10), it follows that $V = 0$ if $\varphi = 0$ and $R = 0$. Hence it follows that the operator assigning the solution V to the pair $(\varphi, R) \in X^{-1} \times H^0$ is also linear and homogeneous.

Definition 3.7. For every $g \in \mathscr{X}$, define the linear continuous operators $L(g) : Y^{-1} \to Y^1$ and $\mathscr{L}(g) : H^0 \to Y^1$ such that $V = L(g)R$ for V which is the solution in Y^1 of the problem (3.5) with $g \in \mathscr{X}$, $\varphi = 0$, and $R \in H^0$, and $V = \mathscr{L}(g)R$ for V which is the solution in Y^1 of the problem (3.5) with $g \in \mathscr{X}$, $\varphi \in X^{-1}$ and $R = 0$.

The fact that these operators are continuous follows immediately from Theorem 3.4. Clearly, $V = L(g)\varphi + \mathscr{L}(g)R$ for V which is the solution in Y^1 of the problem (3.5) with $g \in \mathscr{X}$, $\varphi \in X^{-1}$, and $R \in H^0$.

To prove Theorem 3.4, we need first a preliminary lemma.

Lemma 3.8. *Let $\varepsilon > 0$ be such that there exists a solution $V = V_\varepsilon \in Y_1$ of the problem (3.5) with $g = g_\varepsilon$. Then*

$$||V_\varepsilon||_{Y^1} \le c\left(\|\varphi\|_{X^{-1}} + \|R\|_{H^0}\right), \tag{3.11}$$

where $c > 0$ is a constant which depends only on the parameters $\mathscr{P}$, μ, and $\|g\|_{\mathscr{X}}$.

Proof of Lemma 3.8. We use below the elementary estimate $uv \le u^2/(2\gamma) + v^2\gamma 2$ (for all $u, v, \gamma \in \mathbb{R}$, $\gamma > 0$).

Let $v \in H^1 \cap C^2(\mathbb{R}^n)$. For $t \in [0, T]$, we have

$$\begin{aligned}
(v, \mathscr{A}_c(t)v)_{H^0} &= \left(v, \sum_{i,j=1}^n b_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j}\right)_{H^0} + \left(v, \sum_{i=1}^n \hat{f}_i \frac{\partial v}{\partial x_i}\right)_{H^0} \\
&= -\sum_{i,j=1}^n \left(\frac{\partial v}{\partial x_i}, b_{ij} \frac{\partial v}{\partial x_j}\right)_{H^0} - \sum_{i,j=1}^n \left(v, \frac{\partial b_{ij}}{\partial x_i} \frac{\partial v}{\partial x_j}\right)_{H^0} + \sum_{i=1}^n \left(v, \hat{f}_i \frac{\partial v}{\partial x_i}\right)_{H^0} \\
&\le -\delta \sum_{i=1}^n \left\|\frac{\partial v}{\partial x_i}\right\|_{H^0}^2 + \sum_{i,j=1}^n \|v\|_{H^0} \left\|\frac{\partial b_{ij}}{\partial x_i}\right\|_{L_\infty(Q)} \left\|\frac{\partial v}{\partial x_j}\right\|_{H^0} \\
&\quad + \sum_{i=1}^n \|v\|_{H^0} ||\hat{f}_i||_{L_\infty(Q)} \left\|\frac{\partial v}{\partial x_i}\right\|_{H^0} \\
&\le -\delta \sum_{i=1}^n \left\|\frac{\partial v}{\partial x_i}\right\|_{H^0}^2 + \frac{\delta}{4} \sum_{i=1}^n \left\|\frac{\partial v}{\partial x_i}\right\|_{H^0}^2 \\
&\quad + \frac{C}{\delta} \|v\|_{H^0}^2 \sum_{i,j=1}^n \left(\left\|\frac{\partial b_{ij}}{\partial x_i}\right\|_{L_\infty(Q)}^2 + ||f_i||_{L_\infty(Q)}^2\right),
\end{aligned} \tag{3.12}$$

where $C = C(n)$ is a constant. Hence we obtain the inequality

$$(v, \mathscr{A}_c(t)v)_{H^0} \le -\frac{3\delta}{4} \sum_{i=1}^n \left\|\frac{\partial v}{\partial x_i}\right\|_{H^0}^2 + C_1' \|v\|_{H^0}^2 \tag{3.13}$$

for all $v \in H^1$, $t \in [0, T]$, where a constant C_1' depends only on $\mathscr{P}$.

Let $D_K \triangleq \{x \in \mathbb{R}^n : |x| \le K\}$. By Condition 3.1,

$$\begin{aligned}
||\mathcal{J}'(t)v||_{H^0}^2 &= \int_{\mathbb{R}^n} dx \left| \int_{\mathbb{R}^m} v(x+\theta(t,x,u))\Pi(du) \right|^2 \\
&\le \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^m} |v(x+\theta(t,x,u))|^2 \Pi(du) \\
&= \lim_{K\to+\infty} \int_{D_K} dx \int_{\mathbb{R}^m} |v(x+\theta(t,x,u))|^2 \Pi(du) \\
&= \lim_{K\to+\infty} \ell_n(D_K) \int_{\mathbb{R}^n} \mu_K(dy)|v(y)|^2 \\
&\le c_\pi \int_{\mathbb{R}^n} |v(y)|^2 dy = c_\pi \|v\|_{H^0}^2,
\end{aligned} \tag{3.14}$$

where $\mu_K(\cdot)$ is the probability measure which describes the distribution of a random vector $\eta = \eta_K$ such as in Condition 3.1, where $D = D_K$. Then

$$||\mathcal{J}(t)v||_{H^0}^2 \le (c_\pi + \Pi(\mathbb{R}^m))\|v\|_{H^0}^2. \tag{3.15}$$

Thus,

$$(v, \mathcal{A}(t)v)_{H^0} \le -\frac{3\delta}{4} \sum_{i=1}^n \left\| \frac{\partial v}{\partial x_i} \right\|_{H^0}^2 + C_1 \|v\|_{H^0}^2 \tag{3.16}$$

for all $v \in H^1$ and $t \in [0,T]$, where a constant C_1' depends only on $\mathcal{P}$.

Furthermore, we have

$$\begin{aligned}
(v, \varphi_\varepsilon(\cdot,t))_{H^0} &\le \|v\|_{H^1} ||\varphi_\varepsilon(\cdot,t)||_{H^{-1}} \\
&\le \frac{\delta}{4}\left(\sum_{i=1}^n \left\| \frac{\partial v}{\partial x_i} \right\|_{H^0}^2 + \|v\|_{H^0}^2 \right) + C_2 ||\varphi_\varepsilon(\cdot,t)||_{H^{-1}}^2
\end{aligned} \tag{3.17}$$

for all $v \in H^1$ and $t \in [0,T]$, where a constant C_2 also depends only on $\mathcal{P}$.

Proposition 3.3(i) yields

$$\begin{aligned}
(v, g_\varepsilon v)_{H^0} &\le ||v^2||_{\mathcal{W}} ||g_\varepsilon||_{\mathcal{W}*} \le C_3 \|v\|_{H^1} \|v\|_{H^0} ||g_\varepsilon||_{\mathcal{W}*} \\
&\le \frac{\delta}{4} \sum_{i=1}^n \|v\|_{H^1}^2 + \hat{C}_3 \|v\|_{H_0}^2 \\
&\le \frac{\delta}{4} \sum_{i=1}^n \left\| \frac{\partial v}{\partial x_i} \right\|_{H^0}^2 + C_3 \|v\|_{H_0}^2 \quad \forall v \in H^1,
\end{aligned} \tag{3.18}$$

where constants $\hat{C}_3$ and C_3 depend on $\|g_\varepsilon\|_{\mathcal{W}*}$, δ, and n.

For the solution $V = V_\varepsilon$ of the problem (3.5) with $g = g_\varepsilon$, $\varepsilon \in (0, \varepsilon_1]$, we have from (3.12), (3.15), (3.16), (3.17), and (3.18) that

$$\begin{aligned}
&||V_\varepsilon(\cdot,t)||_{H^0}^2 - ||V_\varepsilon(\cdot,T)||_{H^0}^2 \\
&\quad = 2\int_t^T (V_\varepsilon(\cdot,s), \mathcal{A}V_\varepsilon(\cdot,s) + g_\varepsilon V_\varepsilon(\cdot,s) + \varphi(\cdot,s))_{H^0} ds \\
&\quad \le \int_t^T \left\{ -\delta \sum_{i=1}^n \left\| \frac{\partial V_\varepsilon}{\partial x_i}(\cdot,s) \right\|_{H^0}^2 + C_4\left(||V_\varepsilon(\cdot,s)||_{H^0}^2 + ||\varphi(\cdot,s)||_{H^{-1}}^2 \right) \right\} ds,
\end{aligned} \tag{3.19}$$

where a constant C_4 depends on $\mathcal{P}$, μ, and $\|g\|_{\mathcal{X}}$. Thus,

$$||V_\varepsilon||_{Y^1} \le C_*(\|\varphi\|_{X^{-1}} + \|R\|_{H^0}) \quad \forall \varepsilon \in (0,\varepsilon_1], \tag{3.20}$$

where a constant C_* also depends on $\mathcal{P}$, μ, and $\|g\|_{\mathcal{X}}$. This completes the proof of Lemma 3.8. □

Proof of Theorem 3.4. We prove (i). Consider a boundary value problem

$$\frac{\partial V}{\partial t} + \mathcal{A}_c V + g_\varepsilon V = -\varphi, \qquad V(x,T) = R(x). \tag{3.21}$$

The solution $V \in Y^1$ of this problem is well defined. For every $g \in L_\infty(Q)$, introduce linear continuous operators $L_c(g) : X^{-1} \to Y^1$ and $\mathcal{L}_c(g) : H^0 \to Y^1$ such that $V = L_c(g)\varphi + \mathcal{L}_c(g)R$ for V which is the solution in Y^1 of the problem (3.21) with given g, φ, and R.

The solution V of (3.5) (if exists) has the form $V = L_c(g)\varphi + \mathcal{L}_c(g)R + L_c(g)\mathcal{J}V$. Let

$$\begin{gathered} V_0 \triangleq 0 \in Y^1, \\ V_k \triangleq L_c(g)\varphi + \mathcal{L}_c(g)R + L_c(g)\mathcal{J}V_{k-1}, \quad U_k \triangleq V_k - V_{k-1}, \quad k = 1,2,\ldots. \end{gathered} \tag{3.22}$$

It suffices to prove that $U_k \to 0$ in Y^1 as $k \to +\infty$. Set

$$y_k(t) \triangleq ||U_k(\cdot,t)||_{H^0} + \delta \sum_{i=1}^{n} \int_t^T \left\| \frac{\partial U_k}{\partial x_i}(\cdot,s) \right\|_{H^0}^2 ds. \tag{3.23}$$

Similar to (3.19), we have

$$y_k(t) \le c_1 + C_4 \int_t^T (y_k(s) + y_{k-1}(s))ds, \tag{3.24}$$

where $c_1 \triangleq \|R(\cdot)\|_{H^0}$. By the Bellman inequality,

$$y_k(t) \le c_1 e^{C_4(T-t)} \int_t^T y_{k-1}(s)ds. \tag{3.25}$$

It is easy to see that $y_k(t) \le C^k$, where $C > 0$ is a constant independent of k and t. After standard iterations, we have that $\sup_{t\in[0,T]} y_k(t) \to 0$ as $k \to +\infty$. Thus, $\{V_k\}$ is a Cauchy sequence in Y^1. Then (i) follows.

We show that the sequence $\{V_\varepsilon\}$, $\varepsilon \to 0$, is a Cauchy sequence in the space Y^1. Let $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ and let $W = V_{\varepsilon_1} - V_{\varepsilon_2}$, then

$$\frac{\partial W}{\partial t} + \mathcal{A} W + g_{\varepsilon_1} W = -\xi, \qquad W(x,T) = 0, \tag{3.26}$$

where $\xi \triangleq (g_{\varepsilon_1} - g_{\varepsilon_2})V_{\varepsilon_2}$. Furthermore,

$$||g_{\varepsilon_1} - g_{\varepsilon_2}||_{\mathcal{W}^*} \longrightarrow 0, \tag{3.27}$$

because $\{g_\varepsilon\}$ is a Cauchy sequence. By Proposition 3.3(ii) and (3.20), (3.27),

$$\begin{aligned}||(g_{\varepsilon_1}-g_{\varepsilon_2})V_{\varepsilon_2}||_{X^{-1}} &= \int_0^T ||(g_{\varepsilon_1}-g_{\varepsilon_2})V_{\varepsilon_2}(\cdot,t)||_{H^{-1}}\,dt \\ &\le \int_0^T ||g_{\varepsilon_1}-g_{\varepsilon_2}||_{\mathcal{W}^*}||V_{\varepsilon_2}(\cdot,t)||_{H^1}\,dt \\ &\le ||g_{\varepsilon_1}-g_{\varepsilon_2}||_{\mathcal{X}}\int_0^T ||V_{\varepsilon_2}(\cdot,t)||_{H^1}\,dt \\ &= ||g_{\varepsilon_1}-g_{\varepsilon_2}||_{\mathcal{W}^*}||V_{\varepsilon_2}||_{X^1}\longrightarrow 0\end{aligned}\tag{3.28}$$

as $\varepsilon_1\to 0$, $\varepsilon_2\to 0$. Hence $\|\xi\|_{X^{-1}}\to 0$. The estimate (3.20) applied to the solution W of the boundary value problem (3.26) yields

$$\|W\|_{Y^1}\le C_*\|\xi\|_{X^{-1}}\longrightarrow 0.\tag{3.29}$$

Hence the sequence $\{V_\varepsilon\}$, $\varepsilon\to 0$, is a Cauchy sequence (and has a limit) in the Banach space Y^1. The estimate (3.10) and the uniqueness of V follows from (3.20) and (3.26). This completes the proof of Theorem 3.4. □

Corollary 3.9. *Let $V_\varepsilon\triangleq L(g_\varepsilon)\varphi_\varepsilon+\mathcal{L}(g_\varepsilon)R_\varepsilon$, and $V\triangleq L(g)\varphi+\mathcal{L}(g)R$, where $g,g_\varepsilon\in\mathcal{X}$, $\varphi,\varphi_\varepsilon\in X^{-1}$ are such that $\|g_\varepsilon-g\|_{\mathcal{X}}\to 0$, $\|\varphi_\varepsilon-\varphi\|_{X^{-1}}\to 0$ and $\|R_\varepsilon-R\|_{H^0}\to 0$ as $\varepsilon\to 0+$. Then $\|V_\varepsilon-V\|_{Y^1}\to 0$.*

Proof. For the sake of simplicity, assume that $\varphi_\varepsilon\equiv\varphi$. Let $\|\mathcal{L}(g_\varepsilon)\|$ denote the norm of the operator $\mathcal{L}(g_\varepsilon):H^0\to Y^1$. By Theorem 3.4, $\sup_\varepsilon\|\mathcal{L}(g_\varepsilon)\|\le$ const. Then

$$\begin{aligned}||V_\varepsilon-V||_{Y^1}&\le||\mathcal{L}(g_\varepsilon)R_\varepsilon-\mathcal{L}(g_\varepsilon)R||_{Y^1}+||\mathcal{L}(g_\varepsilon)R-\mathcal{L}(g)R||_{Y^1}\\&\le\sup_\varepsilon||\mathcal{L}(g_\varepsilon)||\,||R_\varepsilon-R||_{H^0}+||\mathcal{L}(g_\varepsilon)R-\mathcal{L}(g)R||_{Y^1}.\end{aligned}\tag{3.30}$$

By Theorem 3.4, it also follows that $\|\mathcal{L}(g_\varepsilon)R-\mathcal{L}(g)R\|_{Y^1}\to 0$ as $\varepsilon\to 0$. Then the proof follows. □

4. Adjoint equations. Let $\mathcal{A}_c^*=\mathcal{A}_c^*(t)$ be the operator which is formally adjoint to the operator $\mathcal{A}_c(t)$ defined by (3.2),

$$A_c^*(t)p=\sum_{i,j=1}^n\frac{\partial^2}{\partial x_i\partial x_j}\big(b_{ij}(x,t)p(x)\big)-\sum_{i=1}^n\frac{\partial}{\partial x_i}\big(\hat f_i(x,t)p(x)\big).\tag{4.1}$$

Let $\mathcal{J}^*=\mathcal{J}^*(t):H^0\to H^0$ be the operator which is adjoint to the operator $\mathcal{J}=\mathcal{J}(t):H^0\to H^0$ defined by (3.3). Let $\mathcal{A}^*\triangleq\mathcal{A}_c^*+\mathcal{J}^*$. Consider the following boundary value problem in Q:

$$\frac{\partial p}{\partial t}=\mathcal{A}^*p+gp+\varphi,\qquad p(x,0)=\rho(x).\tag{4.2}$$

Theorem 4.1. *Assume that Condition 3.1 is satisfied. Let $g\in\mathcal{X}$, $\varphi\in X^{-1}$, and $\rho\in H^0$ be given. Let $g_\varepsilon\in L_\infty(Q)\cap\mathcal{X}$ be such that*

$$||g_\varepsilon-g||_{\mathcal{X}}\longrightarrow 0\quad\text{as }\varepsilon\longrightarrow 0+.\tag{4.3}$$

Then,

(i) *for any $\varepsilon > 0$, there exist the unique solution p_ε of (4.2) with $g = g_\varepsilon$;*

(ii) *the sequence p_ε has a unique limit p in Y^1 as $\varepsilon \to 0+$, and*

$$\|p\|_{Y^1} \le c\left(\|\varphi\|_{X^{-1}} + \|\rho\|_{H^0}\right), \tag{4.4}$$

where a constant $c > 0$ depends only on the parameters $\mathcal{P}$, μ, and $\|g\|_{\mathcal{X}}$.

The proof of Theorem 4.1 is similar to the proof of Theorem 4.4. Note only that, by Remark 3.5, it follows that the coefficients of the operator $\mathcal{A}_c^*$ are smooth enough, and, by (3.15), $\|\mathcal{J}^* v\|_{H^0}^2 \le (c_\pi + \Pi(\mathbb{R}^m))\|v\|_{H^0}^2$ for all $v \in H^0$.

For $g \in \mathcal{X}$, introduce a linear continuous operators $\hat{\mathbf{L}}(g) : X^{-1} \to H^0$ and $\mathbf{L}_0(g) : H^0 \to H^0$ such that $V(\cdot,0) = \hat{\mathbf{L}}(g)\varphi + \mathbf{L}_T(g)R$, where $V = L(g)\varphi + \mathcal{L}R$ is the solution of the problem (3.5).

Proposition 4.2. *For p, g, φ, and ρ from Theorem 4.1, $p = L(g)^*\varphi + \hat{\mathbf{L}}(g)^*\rho$ and $p(\cdot,T) = L(g)^*\varphi + \mathbf{L}_T(g)^*\rho$, where $L(g)^* : X^{-1} \to X^1$, $\hat{\mathbf{L}}(g)^* : H^0 \to X^1$ and $\mathbf{L}_T(g)^* : H^0 \to H^0$ are linear continuous operators which are adjoint to the operators $L(g) : X^{-1} \to X^1$ and $\hat{\mathbf{L}}(g) : X^{-1} \to H^0$ and $\mathbf{L}_T(g) : H^0 \to H^0$ correspondingly.*

Proof. Let $\phi \in X^0$, $R \in H^0$ be arbitrary, $V = L(g)\phi + \mathcal{L}(g)R$. Then

$$\begin{aligned}
&(p(\cdot,T),R)_{H^0} - (\rho,\hat{\mathbf{L}}(g)\phi + \mathbf{L}_T R)_{H^0} \\
&\qquad = (p(\cdot,T),V(\cdot,T))_{H^0} - (p(\cdot,0),V(\cdot,0))_{H^0} \\
&\qquad = \left(\frac{\partial p}{\partial t},V\right)_{X^0} + \left(p,\frac{\partial V}{\partial t}\right)_{X^0} \\
&\qquad = (\mathcal{A}^* p + gp + \varphi, V)_{X^0} + (p, -\mathcal{A}V - gV - \phi)_{X^0} \\
&\qquad = (\varphi,V)_{X^0} - (p,\phi)_{X^0} \\
&\qquad = (\varphi, L(g)\phi + \mathcal{L}(g)R)_{X^0} - (p,\phi)_{X^0}.
\end{aligned} \tag{4.5}$$

Then

$$(p(\cdot,T),R)_{H^0} + (p,\phi)_{X^0} = (\rho,\hat{\mathbf{L}}(g)\phi + \mathbf{L}_T R)_{H^0} + (\varphi, L(g)\phi + \mathcal{L}(g)R)_{X^0}. \tag{4.6}$$

Then the proof follows. □

Condition 4.3. There exists uniformly bounded derivatives $\partial^k \beta(x,u,t)/\partial x^k$, $\partial^k f(x,t)/\partial x^k$, and $\partial^k \theta(x,u,t)/\partial x^k$ for $k = 1,2$.

Theorem 4.4. *Let Conditions 3.1 and 4.3 be satisfied, let $g(x,t) : Q \to \mathbb{R}$ be a Borel measurable function which belongs to $\mathcal{X}$ and is bounded together with the derivatives $\partial^k g(x,t)/dx^k$ for $k = 1,2$. Let the vector a in (2.1) have the probability density function $\rho \in H^0$, and let $p \triangleq \hat{\mathbf{L}}(g)^*\rho$. Then*

$$\mathbf{E}R(y(T))\exp\left(\int_0^T g(y(t),t)\,dt\right) = \int_{\mathbb{R}^n} p(x,T)R(x)\,dx \tag{4.7}$$

for all Borel measurable $R(\cdot) \in H^0$. In particular, if $g = 0$ then $p(x,t)$ is the probability density function of the solution $y(t)$ of (2.1).

Proof. It suffices to prove (4.7) with $R(\cdot) \in H^0 \cap C^2(\mathbb{R}^n)$. For $(x,s) \in Q$, set $V(x,s) \triangleq \mathbf{E}\{R(y(T)) \mid y(s) = x\}$. By [6, Theorem 4, page 296], it follows that $V = \mathbf{L}(0)R$. By Proposition 4.2, it follows that

$$\mathbf{E}R(y(T)) = (V(\cdot,0),\rho)_{H^0} = (\mathbf{L}_T R,\rho)_{H^0} = (R,\mathbf{L}_T^*\rho)_{H^0} = (R,p(\cdot,T))_{H^0} \tag{4.8}$$

for all $R(\cdot) \in H^0 \cap C^2(\mathbb{R}^n)$. This completes the proof. □

Corollary 4.5. *Let $V = \mathcal{L}(g)R$, where $R \in H^0$. Then there exist a version of V such that* $\operatorname{esssup}_{x,t} V(x,t) \le \max_x R(x)$ *and* $\operatorname{essinf}_{x,t} V(x,t) \ge \min_x R(x)$.

Proof. If $R(\cdot) \in H^0 \cap C^2(\mathbb{R}^n)$ then $V(x,s) = \mathbf{E}\{R(y(T)) \mid y(s) = x\}$ and the proof follows. For the general case $R \in H^0$, the proof can be obtained by a standard approximation. □

Condition 4.6. (i) There exist uniformly bounded derivatives $\partial^m \beta(x,u,t)/\partial x^m$ for $m \le 4$, $\partial^l f(x,t)/\partial x^l$ for $l = 1,2,3$, and $\partial^k \theta(x,u,t)/\partial x^k$ for $k = 1,2$.

(ii) There exist $c_* \in \mathbb{R}$, a measure $\Pi_*(\cdot)$ in $\mathbb{R}^m$, and a bounded and Borel measurable function $\theta_*(x,u,t) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{n\times n}$ which is continuous in u, such that $\Pi_*(\mathbb{R}^m) < +\infty$ and $\mathcal{J}^*(t)v = \int_{\mathbb{R}^m} v(x + \theta_*(x,u,t))\Pi_*(du) + c_* v$ for any $v \in H^0$.

(iii) The derivatives $\partial^k \theta_*(x,u,t)/\partial x^k$ are bounded for $k = 1,2$ and Condition 3.1 is satisfied with substituting $(\Pi(\cdot),\theta(\cdot)) = (\Pi_*(\cdot),\theta_*(\cdot))$.

Note that if the mapping $z = x + \theta(x,u,t)$ maps $\mathbb{R}^n$ one-to-one onto itself for any (u,t), then $\theta_*(\cdot)$ can be found such that $x = z - \theta_*(z,u,t)$ is the inverse mapping. If the last one is differentiable, then $\Pi_*(\cdot)$ can be found as $\Pi_*(dx) = J(x)\Pi(dx)$, where $J(x)$ is the Jacobian of the transformation $y = x - \theta_*(x,u,t)$ (see [6, page 299]).

Corollary 4.7. *Let Conditions 3.1, 4.3, and 4.6 be satisfied. Let $\rho \in L_\infty(\mathbb{R}^n) \cap H^0$, and let $p \triangleq \hat{\mathbf{L}}(0)^*\rho$. Then $p \in L_\infty(Q)$.*

Proof. It can be seen that equation (4.7) after a change of time variable can be rewritten in the form (3.2), and then Corollary 4.5 is satisfied. This completes the proof. □

5. On a class of acceptable hypersurfaces. We will use the equations from Sections 3 and 4 for the distributions of the occupation time on hypersurfaces. In this section we describe a class of acceptable hypersurfaces.

Let $\Gamma \subset D$ be some $(n-1)$-dimensional hypersurface. By $e^{(i)}$ we denote the ith unit vectors in $\mathbb{R}^n$, $i = 1,\dots,n$. Let $\mathbf{n}(x)$ be the normal to Γ in x, and let $\alpha_i(x)$ be the angle between $e^{(i)}$ and $\mathbf{n}(x)$.

Introduce the functions $\gamma_i : \Gamma \to \mathbb{R}$, $i = 1,\dots,n$, such that

$$\gamma_i(x) = \begin{cases} |\cos\alpha_i(x)| & \text{if the normal } \mathbf{n}(x) \text{ at } x \text{ is uniquely defined,} \\ 0 & \text{if the normal at } x \text{ is not defined.} \end{cases} \tag{5.1}$$

(In fact, $\mathbf{n}(x)$ is not defined at points of violation of smoothness of Γ.)

Denote by $N(x,j,\Gamma)$ the number of intersections of the hypersurface Γ with the ray from $x = (x_1,x_2,\dots,x_n)$ to $(x_1,\dots,x_{j-1},-\infty,x_{j+1},\dots,x_n)$. Let $\hat{x}_k(x,j)$ be the corresponding intersection points.

We assume that $N(x,j,\Gamma) = +\infty$, if the ray is tangential to Γ.

Set

$$
\begin{aligned}
G_j(x) &\triangleq \sum_{k=1}^{N(x,j,\Gamma)} \gamma_j(\hat{x}_k(x,j)), \qquad g = \sum_{j=1}^{n} \frac{\partial G_j}{\partial x_j}, \\
\Gamma(\varepsilon) &\triangleq \left\{ x \in \mathbb{R}^n : \inf_{y\in\Gamma} |x-y| \le \frac{\varepsilon}{2} \right\}, \\
g_\varepsilon(x) &\triangleq \frac{1}{\varepsilon} \operatorname{Ind}\{x \in \Gamma(\varepsilon)\}.
\end{aligned}
\tag{5.2}
$$

Definition 5.1. A set $\hat{\Gamma} \in \mathbb{R}^n$ is said to be an $(n-1)$-dimensional polyhedron if there exist an integer N and $c_i \in \mathbb{R}^n$, $\delta_i \in \mathbb{R}$, $i = 0,1,\dots,N$ such that $\hat{\Gamma} = \{x \in \mathbb{R}^n : c_0'x = \delta_0,\ c_i'x \le \delta_i,\ i = 1,\dots,N\}$. The set $\{x \in \mathbb{R}^n : c_0'x = \delta_0,\ c_i'x < \delta_i,\ i = 1,\dots,N\}$ is said to be the interior of $\hat{\Gamma}$.

Lemma 5.2 (see [3]). *Let a hypersurface $\Gamma \subset \mathbb{R}^n$ be bounded and such that there exists a set $\hat{\Gamma} \subset \mathbb{R}^n$ and a continuous bijection $\mathcal{M} : \mathbb{R}^n \to \mathbb{R}^n$ which satisfy the following assumptions:*

(i) $\Gamma = \mathcal{M}(\hat{\Gamma})$;

(ii) $\hat{\Gamma} = \cup_{i=1}^{\mathcal{N}} \hat{\Gamma}_i$, *where $\mathcal{N}$ is an integer, $\hat{\Gamma}_i$ is $(n-1)$-dimensional polyhedron;*

(iii) $\mathcal{M} : \hat{\Gamma}_i \to \mathbb{R}^n$ *are C^1-smooth bijections, $i = 1,\dots,\mathcal{N}$;*

(iv) $|\mathbf{n}(x) - \mathbf{n}_i| \le \delta_0$, *if $\mathcal{M}^{-1}(x)$ belongs to the interior of $\hat{\Gamma}_i$, $i = 1,\dots,\mathcal{N}$, where $x \in \Gamma$, $\delta_0 \le n^{-2}/2$ is a constant, $\mathbf{n}(x)$ is the normal to Γ in x, and $\mathbf{n}_i$ is the normal to $\hat{\Gamma}_i$; it is assumed that the orientations of these normals are fixed and $|\mathbf{n}(x)| = 1$, $|\mathbf{n}_i| = 1$;*

(v) $\mathcal{M}(x) = x$, *if x is a top point of some $\hat{\Gamma}_i$.*

Then $N(j,x,\Gamma) < +\infty$ for a.e. x. Moreover, $g \in \mathcal{W}^* \cap H^{-1}$ and $g_\varepsilon(\cdot) \to g$ in $\mathcal{W}^*$.

6. Existence of the occupation time density and an analog of Meyer-Tanaka formula.

Set

$$
g_\varepsilon(x,t) \triangleq \frac{1}{\varepsilon} \operatorname{Ind}\{x \in \Gamma(\varepsilon,t)\}, \qquad l_\varepsilon(t) \triangleq \int_0^t g_\varepsilon(y(s),s). \tag{6.1}
$$

It is natural to interpret the limit of $l_\varepsilon(T)$ as the occupation time of $y(t)$ on $\Gamma(t)$.

Condition 6.1. (i) The hypersurface $\Gamma(t)$ is such that the assumptions of Lemma 5.2 hold for $\Gamma = \Gamma(t)$ for a.e. $t \in [0,T]$ and $g = g(t) \in X^{-1}$, where $g(t) \triangleq \lim_{\varepsilon\to 0} g_\varepsilon(t)$ (by Lemma 5.2, the limit exists in H^{-1} for a.e. $t \in [0,T]$).

(ii) The initial vector $a = y(0)$ has probability density function $\rho \in L_\infty(\mathbb{R}^n)$.

(iii) The function $\beta(x,t)$ in (2.1) is continuous.

Note that the assumptions of Lemma 5.2 hold for disks, spheres, and many other piecewise C^1-smooth $(n-1)$-dimensional surfaces. Moreover, it can be easy to find examples when the surface $\Gamma(t)$ changes in time, approaching a fractal, but $g \in X^{-1}$.

Example 6.2. Let $n = 2$, $T = 2$, $\Gamma(t) = \{(x_1, x_2) : x_2 = \sin(x_1(1-t)^{-1/3}),\ x_1 \in [-1,1]\}$. Then $N((2,x_2),2,\Gamma(t)) \equiv 1$ and

$$\begin{aligned}
\|g\|^2_{X^{-1}} &= \int_0^T \|g(t)\|^2_{H^{-1}}\,dt \\
&\le \text{const} \int_0^T \Big[1 + \sup_{x_2} N\big((2,x_2),1,\Gamma(t)\big)^2\Big]dt \\
&\le \text{const}\left(2 + \int_0^2 (1-t)^{-2/3}dt\right) < +\infty.
\end{aligned} \tag{6.2}$$

Hence $g = g(t) \in X^{-1}$.

The following example presents a fractal Γ which is constant in time.

Example 6.3. Let $n = 2$, $\Gamma(t) \equiv \Gamma = \{(x_1,x_2) : x_2 = x_1 \sin(x_1^{-1/3}), x_1 \in [-1,1]\}$. Then $N((2,x_2),2,\Gamma(t)) \equiv 1$ and

$$\|g\|^2_{H^{-1}} \le \left(1 + \int_{-1}^1 dx_2 N\big((2,x_2),1,\Gamma(t)\big)^2\right) \le \text{const}\left(1 + \int_{-1}^1 x_1^{-2/3}dx_1\right) < +\infty. \tag{6.3}$$

Hence $g \in H^{-1}$.

Denote by β_j the columns of the matrix β, $j = 1,\dots,n$. Let $\mathcal{F}_t$ be the filtration of complete σ-algebras of events, generated by $\{a, w(s), \nu(B,s),\ s \le t,\ B \in \bar{\mathcal{B}}_n\}$.

Introduce the set $\widetilde{\mathcal{Y}}$ of all bounded functions $\xi(t) = \xi(t,\omega) : [0,T]\times\Omega \to \mathbb{R}^n$ which are progressively measurable with respect to $\mathcal{F}_t$, and introduce the set $\widetilde{\mathcal{X}}$ of all bounded functions $\psi(u,t) = \psi(u,t,\omega) : \mathbb{R}^n \times [0,T]\times\Omega \to \mathbb{R}^n$ which are progressively measurable with respect to $\mathcal{F}_t$ for all u.

Introduce the Hilbert space $\mathcal{Y}_2$ as the completion of $\widetilde{\mathcal{Y}}$ with respect to the norm $\|\xi\|_{\mathcal{Y}_2} \triangleq \mathbf{E}\int_0^T |\xi(t)|^2 dt$, and introduce the Hilbert space $\mathcal{X}_2$ as the completion of $\widetilde{\mathcal{X}}$ with respect to the norm $\|\psi\|_{\mathcal{X}_2} \triangleq \mathbf{E}\int_0^T dt \int_{\mathbb{R}^n} |\psi(u,t)|^2 \Pi(du)$.

We present now an analog of the Meyer-Tanaka formula (cf. [9] or [11, page 169]).

Theorem 6.4. *Assume that Conditions 3.1, 4.3, 4.6, and 6.1 are satisfied. Let $V \triangleq L(0)g$ (by definition, this V belongs Y^1). Let V and $\partial V/\partial x$ be Borel measurable representatives V and $\partial V/\partial x$ of corresponding equivalence classes in $L_2(Q)$. Then*

$$\frac{\partial V}{\partial x}(y(t),t)\beta_j(y(t),t) \in \mathcal{Y}_2, \quad V(y(t)+\theta(y(t),u,t),t) - V(y(t),t) \in \mathcal{X}_2, \tag{6.4}$$

and $\mathbf{E}|l_\varepsilon(T) - \hat{\mathbf{t}}(T)|^2 \to 0$ as $\varepsilon \to 0$, where

$$\begin{aligned}
\hat{\mathbf{t}}(T) \triangleq V(a,0) &+ \sum_{j=1}^n \int_0^T \frac{\partial V}{\partial x}(y(t),t)\beta_j(y(t),t)\,dw_j(t) \\
&+ \int_0^T dt \int_{\mathbb{R}^m} \big(V(y(t)+\theta(y(t),u,t),t) - V(y(t),t)\big)\tilde{\nu}(du,dt).
\end{aligned} \tag{6.5}$$

Corollary 6.5. *Let $p \triangleq \mathbf{L}(0)^*\rho$. In the assumptions and notation of Theorem 6.4,*

$$\begin{aligned}\mathbf{E}\hat{\mathfrak{t}}(T)^2 = \int_{\mathbb{R}^n} |V(x,0)|^2\rho(x)\,dx \\ + \int_Q \Bigg(\sum_{j=1}^n \left|\frac{\partial V}{\partial x}(x,t)\beta_j(x,t)\right|^2 + \int_{\mathbb{R}^m} |V(x+\theta(x,u,t),t) \\ - V(x,t)|^2\Pi(du)\Bigg)p(x,t)\,dx\,dt.\end{aligned} \tag{6.6}$$

Note that if Condition 6.1(ii) is satisfied, then $\|\rho\|_{H^0}^2 = \int_{\mathbb{R}^n}\rho(x)^2dx \le \|\rho\|_{L_\infty(\mathbb{R}^n)}$, and $\rho \in H^0$. By definition, p in Corollary 6.5 is the solution of the boundary value problem (4.2) with $\varphi = 0$, $g = 0$. Moreover, by Theorem 4.1, it follows that $p(x,t)$ is the probability density function of the process $y(t)$, and, by Corollary 4.7, $p \in L_\infty(Q)$.

Proof of Theorem 6.4. Set

$$\begin{aligned}\xi_j(t) &\triangleq \frac{\partial V}{\partial x}(y(t),t)\beta_j(y(t),t),\\ \psi(u,t) &\triangleq V(y(t)+\theta(y(t),u,t),t) - V(y(t),t).\end{aligned} \tag{6.7}$$

Let $h_\varepsilon(x,t) \in X^0 \cap C([0,T];C^2(\mathbb{R}^n))$ be such that $\|h_\varepsilon - g_\varepsilon\|_{X^0} \le \varepsilon$. Set

$$\begin{aligned}V_\varepsilon \triangleq L(0)h_\varepsilon, \qquad \lambda_\varepsilon(t) \triangleq \int_0^t h_\varepsilon(y(s),s)\,ds,\\ \xi_{j,\varepsilon}(t) \triangleq \frac{\partial V_\varepsilon}{\partial x}(y(t),t)\beta_j(y(t),t),\\ \psi_\varepsilon(u,t) \triangleq V_\varepsilon(y(t)+\theta(y(t),u,t),t) - V_\varepsilon(y(t),t).\end{aligned} \tag{6.8}$$

By definition, we have that $h_\varepsilon = -\partial V_\varepsilon/\partial t - \mathcal{A}V_\varepsilon$ and $V_\varepsilon(x,T) = 0$. By the generalized Itô formula (cf. [6, page 272]), it follows that

$$\begin{aligned}-V_\varepsilon(a,0) &= V_\varepsilon(y(T),T) - V_\varepsilon(a,0)\\ &= -\int_0^T h_\varepsilon(y(t),t)\,dt + \sum_{j=1}^n\int_0^T \xi_j(y(t),t)\,dw_j(t)\\ &\quad + \int_0^T dt\int_{\mathbb{R}^m}(V_\varepsilon(y(t)+\theta(y(t),u,t),t) - V_\varepsilon(y(t),t))\tilde{\nu}(du,dt).\end{aligned} \tag{6.9}$$

Hence

$$\begin{aligned}\lambda_\varepsilon(T) = V_\varepsilon(a,0) + \sum_{j=1}^n\int_0^T \frac{\partial V_\varepsilon}{\partial x}(y(t),t)\beta_j(y(t),t)\,dw_j(t)\\ + \int_0^T dt\int_{\mathbb{R}^m}(V_\varepsilon(y(t)+\theta(y(t),u,t),t) - V_\varepsilon(y(t),t))\tilde{\nu}(du,dt).\end{aligned} \tag{6.10}$$

By [6, Lemmas 2, 3 and Theorem 4, pages 293–296], it follows that the functions V_ε and $\partial V_\varepsilon(x,t)/\partial x$ are bounded and continuous, then $\xi_{\varepsilon,j}(t) \in \mathcal{Y}^0$ and $\psi_\varepsilon(u,t) \in \mathcal{Z}^0$.

By Theorem 4.1, $p = p(x,t) \triangleq \hat{\mathbf{L}}(0)^*\rho$ is the probability density function of the process $y(t)$. Let $W_\varepsilon \triangleq V_\varepsilon - V$. By Corollary 3.9, $\|W_\varepsilon\|_{Y^1} \to 0$ as $\varepsilon \to 0$. Then

$$\begin{aligned}
\mathbf{E}\left|V_\varepsilon(a,0)-V(a,0)\right|^2 &= \int_{\mathbb{R}^n} \|W_\varepsilon(x,0)\|^2 \rho(x)\,dx \\
&\le \|\rho\|_{L_\infty(\mathbb{R}^n)} \|W_\varepsilon\|_{X^0} \longrightarrow 0, \\
\mathbf{E}\int_0^T |\xi_{j,\varepsilon}(t)-\xi_j(t)|^2 dt &= \sum_{j=1}^n \mathbf{E}\int_0^T \left|\frac{\partial W_\varepsilon}{\partial x}(y(t),t)\beta_j(y(t),t)\right|^2 dt \\
&= \int_Q \left|\frac{\partial W_\varepsilon}{\partial x}(x,t)\right|^2 p(x,t)\,dx\,dt \\
&\le \operatorname{const}\|p\|_{L_\infty(Q)} \|W_\varepsilon\|_{Y^1} \longrightarrow 0, \\
\mathbf{E}\int_0^T dt \int_{\mathbb{R}^n} \left|\psi_\varepsilon(u,t)-\psi(u,t)\right|^2 \Pi(du) &= \mathbf{E}\int_0^T \left|(\mathcal{J}(t)W_\varepsilon)(y(t),t)\right|^2 dt \\
&= \int_Q \left|(\mathcal{J}(t)W_\varepsilon)(x,t)\right|^2 p(x,t)\,dx\,dt \\
&\le \operatorname{const}\|p\|_{L_\infty(Q)} \|\mathcal{J}W_\varepsilon\|_{X^0} \longrightarrow 0.
\end{aligned} \tag{6.11}$$

Then

$$\begin{aligned}
\mathbf{E}\left|\hat{\mathbf{t}}(T)-\lambda_\varepsilon(T)\right|^2 &= \mathbf{E}\left|V_\varepsilon(a,0)-V(a,0)\right|^2 \\
&\quad + \sum_{j=1}^n \mathbf{E}\int_0^T \left|\xi_{j,\varepsilon}(t)-\xi_j(t)\right|^2 dt \\
&\quad + \mathbf{E}\int_0^T dt \int_{\mathbb{R}^n} \left|\psi_\varepsilon(u,t)-\psi(u,t)\right|^2 \Pi(du) \longrightarrow 0.
\end{aligned} \tag{6.12}$$

Furthermore,

$$\begin{aligned}
\mathbf{E}\left|l_\varepsilon(T)-\lambda_\varepsilon(T)\right|^2 &= \int_0^T \int_{\mathbb{R}^n} \left|g_\varepsilon(x,t)-h_\varepsilon(x,t)\right|^2 p(x,t)\,dt \\
&\le \|p\|_{L_\infty(Q)} \|g_\varepsilon - h_\varepsilon\|_{X^0} \longrightarrow 0.
\end{aligned} \tag{6.13}$$

This completes the proof of Theorem 6.4. □

Proof of Corollary 6.5. The proof can be easily obtained similar to (6.12). □

7. Equations for the characteristic function of the occupation time

Theorem 7.1. *Assume that Conditions 3.1, 4.3, 4.6, and 6.1 are satisfied, and that $g \in \mathcal{X}$. Let $\nu \in \mathbb{R}$ be given, and let $z \triangleq i\nu$, where $i = \sqrt{-1}$. Let $V \triangleq zL(zg)g$. Then $V \in Y^1$, and*

$$1 + (V(\cdot,0),\rho)_{H^0} = \mathbf{E}\exp\{z\hat{\mathbf{t}}(T)\}. \tag{7.1}$$

Proof. Let h_ε be such that $h_\varepsilon(x,t) \in X^0 \cap C([0,T];C^2(\mathbb{R}^n))$ and $\|h_\varepsilon - g_\varepsilon\|_{\mathscr{X}} \le \varepsilon$. Set $\lambda_\varepsilon(t) \triangleq \int_0^t h_\varepsilon(y(s),s)ds$. Let

$$V_\varepsilon(x,s) \triangleq \mathbf{E}\left\{z\int_s^T h_\varepsilon(y(t),t)\exp\left(z\int_s^t h_\varepsilon(y(r),r)dr\right)dt \mid y(s)=x\right\}. \tag{7.2}$$

It is easy to see that

$$V_\varepsilon(x,s) = \mathbf{E}\left\{\exp\left(z\int_s^T h_\varepsilon(y(t),t)dt\right) \mid y(s)=x\right\} - 1. \tag{7.3}$$

By [6, Theorem 1, page 301], applied after a small modification for a non-homogeneous integro-differential equation, it follows that $V_\varepsilon = zL(zh_\varepsilon)h_\varepsilon$, that is, V_ε is the solution of the problem

$$\frac{\partial V_\varepsilon}{\partial t} + \mathscr{A}V_\varepsilon + zh_\varepsilon V_\varepsilon = -zh_\varepsilon, \qquad V_\varepsilon(x,T) = 0. \tag{7.4}$$

By Lemma 5.2 it follows that $\|g - g_\varepsilon\|_{\mathscr{X}} \to 0$ as $\varepsilon \to 0+$. Hence $\|g - h_\varepsilon\|_{\mathscr{X}} \to 0$ as $\varepsilon \to 0+$. By Corollary 3.9, it follows that $\|V - V_\varepsilon\|_{Y^1} \to 0$. Hence $(V_\varepsilon(\cdot,0),\rho)_{H^0} \to (V(\cdot,0),\rho)_{H^0}$. It was shown in the proof of Theorem 6.4 that $\mathbf{E}|\lambda_\varepsilon(T) - \hat{\mathbf{t}}(T)|^2 \to 0$. Then $\lambda_\varepsilon(T)$ converges to $\hat{\mathbf{t}}(T)$ in distribution, and $\mathbf{E}e^{z\lambda_\varepsilon(T)} \to \mathbf{E}e^{z\hat{\mathbf{t}}(T)}$ for each $z = i\nu$, $\nu \in \mathbb{R}$. Then the proof follows. $\square$

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