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ON THE GEOMETRY AND BEHAVIOR OF n-BODY MOTIONS

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ABSTRACT. The kinematic separation of size, shape, and orientation of n-body systems is investigated together with specific issues concerning the dynamics of classical n-body motions. A central topic is the asymptotic behavior of general collisions, extending the early work of Siegel, Wintner, and more recently Saari. In particular, asymptotic formulas for the derivatives of any order of the basic kinematic quantities are included. The kinematic Riemannian metric on the congruence and shape moduli spaces are introduced via $O(3)$-equivariant geometry. For $n = 3$, a classical geometrization procedure is explicitly carried out for planary 3-body motions, reducing them to solutions of a rather simple system of geodesic equations in the 3-dimensional congruence space. The paper is largely expository and various known results on classical n-body motions are surveyed in our more geometrical setting.

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1. Introduction. The classical n-body problem studies the motion of n celestial bodies under the mutual influence of gravitational forces. In reality one studies an idealized system consisting of n point masses P_i of mass m_i in Euclidean 3-space, where the dynamical laws are given by the Newtonian potential function. In the more recent literature, one also finds studies of particle systems whose dynamics are given by various types of potential functions with similar symmetry properties, such as the inverse q force law with $q \neq 2$.

We start in Section 2 with the kinematics of many particle systems, in the general setting of classical vector algebra. Of particular importance is the decomposition of kinetic energy and the associated kinematic identities and inequalities, including the Sundman inequality which is well known from celestial mechanics. As far as dynamics is concerned, say, with the inverse q force law, $1 < q < 3$, a central topic which we will discuss is the asymptotic behavior of motions leading to a general collision (total collapse). This old topic dates back to the pioneering work of Sundman and Siegel on 3-body motions (see [17, 18, 22, 23]) and its partly generalization to $n > 3$ by Wintner [24], where the time derivatives up to second order of the basic kinematic quantities are investigated. We will extend these results and prove the *expected* asymptotic formulas for the derivatives of any order. The first part of the proofs appears in Section 2.2; here we establish the case of derivatives up to order 2, largely following Siegel's approach. The proof is completed in Section 6, where we have adapted ideas found in Wintner [24]. We mention that Saari and his collaborators have extended Wintner' ideas and techniques to study (i) collisions involving subsystems of the particles, and (ii) expanding systems and their limiting behavior as $t \to \infty$ (cf. [14]).

Unfortunately, in this treatise we have not included any discussion and corresponding results in these directions.

Section 3 is more geometric, involving equivariant geometry modulo the orthogonal group $O(3)$. Here we define the congruence and shape moduli spaces $\bar{M}$ and M^*, respectively, and their natural kinematic Riemannian geometry, which actually coincides with their $O(3)$-orbital distance metric and describes $\bar{M}$ as the Riemannian cone over the compact shape space M^*. This section is far from being exhaustive, but provides some new formalism beyond the vector algebra setting in Section 2 which will be used in later sections. We refer to Hsiang and Straume [6] for a detailed and more complete investigation of the geometry of triangles with mass distribution. For $n > 3$, we refer to [5] and its succeeding paper [7].

In Section 4, we discuss briefly another classical topic, namely the central configurations and the corresponding shape invariant n-body motions. In celestial mechanics these motions are essentially the only known exact solutions for $n \geq 3$, and they date back to Euler and Lagrange (around 1770) who investigated the case $n = 3$. The simplicity of these motions is clearly illustrated by the image curve in the shape moduli space M^*, which in these cases is a single point, namely the shape of a central configuration. We will give an explicit and uniform description of these motions.

The induced Riemannian kinematic metric on the cone $\bar{M} = CM^*$ has the *standard* form $d\bar{s}^2 = d\rho^2 + \rho^2 d\sigma^2$, where $\rho^2 = I$ is the total moment of inertia of the n-body system, representing the size of the system, and $d\sigma^2$ is the induced metric on the shape space M^*. In Section 5, we consider the case $n = 3$, where M^* is a round 2-dimensional (half-)sphere of radius $1/2$, and hence $d\sigma^2 = (1/4)(dr^2 + \sin^2 r\, d\varphi^2)$ in terms of spherical polar coordinates (r, φ). Thus, the triplet (ρ, r, φ) presents itself as a natural coordinate system in $\bar{M}$, and following the classical geometrization procedure in dynamics we work out explicitly in these coordinates the induced (or rather reduced) ODE for 3-body motions at moduli space level, for the special case of planary motions at a fixed energy level h. Its Hamiltonian version is also presented but not further investigated. The above mentioned ODE in $\bar{M}$ coincides with the geodesic equations of the conformally modified metric $d\bar{s}_h^2 = (U + h)d\bar{s}^2$ on $\bar{M}$, where $-U$ is the potential energy. Due to its conspicuous simplicity, we expect this ODE to be quite useful both in the qualitative and numerical analysis of 3-body motions. In future studies we will continue in this direction and also investigate the corresponding system for nonplanary motions.

The present work is a revised version of [21], growing out from the geometric study [6] and the more analytical methods of Siegel and Wintner. In retrospect, however, some of the problems we wanted to study are found to be more or less solved in the existing literature during the recent decades, especially by work of Saari and his collaborators (cf. [8, 14, 15, 16]). Despite this elimination, we apologize for any remaining overlappings or missing references to related works. However, in this rather expository treatise, perhaps with an untraditional approach, we discuss some theoretical but important issues in celestial mechanics which are far from being completely settled, and we hope to stimulate further studies and insight in these mathematical problems which are, after all, still rooted in the physical reality.

2. Fundamental results in the setting of classical vector algebra. In this section, we work out some fundamental identities and inequalities generally known from the classical literature, perhaps in a different setting. Many of these results are actually of a purely kinematic nature, namely independent of the nature of the forces acting on the particles. Therefore, it is natural to start out from a kinematic viewpoint.

2.1. Kinematics of many particle systems. Let $\mathbf{a}_i$, $i \le n$, be the radius vector of points P_i in a fixed Euclidean 3-space, denoted by $\mathbb{R}^3$. The position and motion of the system is recorded by the following time dependent vector in the associated Euclidean $3n$-space

$$\mathbf{X} = \mathbf{X}(t) = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) \in \mathbb{R}^{3n} = \mathbb{R}^3_{(1)} \oplus \mathbb{R}^3_{(2)} \oplus \cdots \oplus \mathbb{R}^3_{(n)} \tag{2.1}$$

with velocity vector $(d/dt)\mathbf{X} = \dot{\mathbf{X}} = (\dot{\mathbf{a}}_1, \dot{\mathbf{a}}_2, \dots, \dot{\mathbf{a}}_n)$, assumed to be continuous. We refer to the vector $\mathbf{X}$ as an *n-configuration*, and $\mathbb{R}^{3n}$ as the (unrestricted) *configuration space*. The dynamics of the system also involves the acceleration $\ddot{\mathbf{X}}$, namely when the influence of forces on the kinematic behavior is concerned. However, kinematics is the formal investigation of quantities and relationships involving $\mathbf{X}$ and $\dot{\mathbf{X}}$, where the vectors are usually regarded as independent. Thus, in the following subsection, we focus attention on constructions involving two arbitrary vectors $\mathbf{X}$ and $\mathbf{Y}$.

2.1.1. Vector algebra and geometry in $\mathbb{R}^{3n}$. Let $\mathbf{a} \cdot \mathbf{b}$ (resp., $\mathbf{a} \times \mathbf{b}$) denote the standard inner product (resp., cross product) of $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$. However, following an old idea due to Jacobi it is convenient to equip $\mathbb{R}^{3n}$ with a Euclidean metric associated with the given mass distribution $(m_1, m_2, \dots, m_n)$, which we will refer to as the *kinematic metric*, namely for general $\mathbf{X} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$ and $\mathbf{Y} = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n)$ define the inner product by

$$\mathbf{X} \cdot \mathbf{Y} = \sum m_i (\mathbf{a}_i \cdot \mathbf{b}_i). \tag{2.2}$$

Similarly, we define the cross-product $\mathbb{R}^{3n} \times \mathbb{R}^{3n} \to \mathbb{R}^3$ by

$$\mathbf{X} \times \mathbf{Y} = \sum m_i (\mathbf{a}_i \times \mathbf{b}_i). \tag{2.3}$$

and we will also need the usual exterior product construction $\mathbb{R}^{3n} \times \mathbb{R}^{3n} \to \Lambda^2 \mathbb{R}^{3n}$

$$\mathbf{X} \wedge \mathbf{Y} = \sum_{i,j} \mathbf{a}_i \wedge \mathbf{b}_j \in \sum_{i,j} \mathbb{R}^3_{(i)} \wedge \mathbb{R}^3_{(j)}, \tag{2.4}$$

where $\mathbf{a}_i \wedge \mathbf{b}_j$ is regarded as a vector in the (i,j)-summand $\mathbb{R}^3_{(i)} \wedge \mathbb{R}^3_{(j)}$ of $\Lambda^2 \mathbb{R}^{3n}$ and $\mathbb{R}^3_{(i)} \subset \mathbb{R}^{3n}$ is the ith summand with orthogonal basis $\mathbf{e}^{(i)}_r = \mathbf{e}_r$, $r = 1, 2, 3$. Here $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denotes a fixed orthonormal basis of our model 3-space $\mathbb{R}^3$ containing the vectors $\mathbf{a}_i = \sum x^{(i)}_r \mathbf{e}_r$ and $\mathbf{b}_j = \sum y^{(j)}_s \mathbf{e}_s$. However, $\Lambda^2 \mathbb{R}^{3n}$ inherits the Euclidean metric induced from (2.2), with orthonormal basis

$$\frac{1}{\sqrt{m_i m_j}} \mathbf{e}^{(i)}_r \wedge \mathbf{e}^{(j)}_s, \quad (r,i) \ne (s,j), \ r < s \text{ if } i = j, \tag{2.5}$$

in particular, $\|\mathbf{e}_r^{(i)}\| = \sqrt{m_i}$. Thus, the vector in (2.4) has norm square

$$
\begin{aligned}
\|\mathbf{X}\wedge\mathbf{Y}\|^2 &= \det\begin{vmatrix}\mathbf{X}\cdot\mathbf{X} & \mathbf{X}\cdot\mathbf{Y}\\ \mathbf{X}\cdot\mathbf{Y} & \mathbf{Y}\cdot\mathbf{Y}\end{vmatrix} = \|\mathbf{X}\|^2\|\mathbf{Y}\|^2-(\mathbf{X}\cdot\mathbf{Y})^2\\
&= \sum_{i=1}^{n} m_i^2\|\mathbf{a}_i\times\mathbf{b}_i\|^2+\sum_{i<j} m_i m_j\sum_{r,s=1}^{3}\left(x_r^{(i)}y_s^{(j)}-x_s^{(j)}y_r^{(i)}\right)^2,
\end{aligned}
\tag{2.6}
$$

where $\mathbf{a}_i = \sum x_r^{(i)}\mathbf{e}_r^{(i)}$, $\mathbf{b}_i = \sum y_s^{(j)}\mathbf{e}_s^{(j)}$, and $\mathbf{a}_i\wedge\mathbf{b}_j$ are regarded as vectors in $\mathbb{R}^{3n}$, whereas $\mathbf{a}_i\times\mathbf{b}_i$ belongs to $\mathbb{R}^3$.

In order to investigate further the connection between the three types of "vector product," (2.2), (2.3), and (2.4), we introduce the following four nonnegative quantities depending on $\mathbf{X}$ and $\mathbf{Y}$:

$$
Q_1^{\diamond} = \sum_{\mathbf{b}_i\neq\mathbf{0}} m_i\frac{(\mathbf{a}_i\diamond\mathbf{b}_i)^2}{\|\mathbf{b}_i\|^2},\qquad Q_2^{\diamond} = \sum_{\mathbf{a}_i\neq\mathbf{0}} m_i\frac{(\mathbf{a}_i\diamond\mathbf{b}_i)^2}{\|\mathbf{a}_i\|^2},
\tag{2.7}
$$

where the operator symbol $\diamond$ means either $\circ$ or $\times$, indicating the inner product or cross product in $\mathbb{R}^3$, respectively. They are evidently related by

$$
Q_1^{\circ}+Q_1^{\times} = \|\mathbf{X}\|^2,\qquad Q_2^{\circ}+Q_2^{\times} = \|\mathbf{Y}\|^2.
\tag{2.8}
$$

The notation $\mathbf{X}\diamond\mathbf{Y}$ refers to (2.2) and (2.3), and the inequalities

$$
(\mathbf{X}\diamond\mathbf{Y})^2 \le \|\mathbf{X}\|^2 Q_2^{\diamond},\qquad \text{respectively}\qquad (\mathbf{X}\diamond\mathbf{Y})^2 \le Q_1^{\diamond}\|\mathbf{Y}\|^2
\tag{2.9}
$$

follow by an application of the Cauchy-Schwarz inequality, for example

$$
\begin{aligned}
(\mathbf{X}\diamond\mathbf{Y})^2 &= \left[\sum m_i(\mathbf{a}_i\diamond\mathbf{b}_i)\right]^2 = \left[\sum\left(\sqrt{m_i}\|\mathbf{a}_i\|\right)\left(\sqrt{m_i}\frac{(\mathbf{a}_i\diamond\mathbf{b}_i)}{\|\mathbf{a}_i\|}\right)\right]^2\\
&\le \left[\sum m_i\|\mathbf{a}_i\|^2\right]\left[\sum m_i\frac{(\mathbf{a}_i\diamond\mathbf{b}_i)^2}{\|\mathbf{a}_i\|^2}\right] = \|\mathbf{X}\|^2 Q_2^{\diamond}.
\end{aligned}
\tag{2.10}
$$

Let $R_i^{\diamond}$, $i = 1,2$, be the *residues* (or *residual terms*) of the inequalities in (2.9), namely by definition

$$
(\mathbf{X}\diamond\mathbf{Y})^2+R_2^{\diamond} = \|\mathbf{X}\|^2 Q_2^{\diamond},\qquad\text{respectively}\qquad (\mathbf{X}\diamond\mathbf{Y})^2+R_1^{\diamond} = \|\mathbf{Y}\|^2 Q_1^{\diamond}.
\tag{2.11}
$$

Then by carefully checking the inequality in (2.10), we find that

$$
R_2^{\diamond} = 0 \Longleftrightarrow \frac{\mathbf{X}\diamond\mathbf{Y}}{\|\mathbf{X}\|^2} = \frac{\mathbf{a}_i\diamond\mathbf{b}_i}{\|\mathbf{a}_i\|^2},\quad\forall i,
\tag{2.12}
$$

$$
R_1^{\diamond} = 0 \Longleftrightarrow \frac{\mathbf{X}\diamond\mathbf{Y}}{\|\mathbf{Y}\|^2} = \frac{\mathbf{a}_i\diamond\mathbf{b}_i}{\|\mathbf{b}_i\|^2},\quad\forall i,
\tag{2.13}
$$

(whenever $\mathbf{a}_i$ and $\mathbf{b}_i$ are nonzero, as in (2.7)). By combining (2.8), (2.9), and (2.11), we obtain the identity

$$
\|\mathbf{X}\times\mathbf{Y}\|^2+(\mathbf{X}\cdot\mathbf{Y})^2+\mathfrak{R} = \|\mathbf{X}\|^2\|\mathbf{Y}\|^2,\quad i = 1,2,
\tag{2.14}
$$

and therefore

$$\mathfrak{R} = (R_1^{\circ} + R_1^{\times}) = (R_2^{\circ} + R_2^{\times}) \geq 0 \tag{2.15}$$

is the *residue* of the inequality

$$\|\mathbf{X}\times\mathbf{Y}\|^2 + (\mathbf{X}\cdot\mathbf{Y})^2 \leq \|\mathbf{X}\|^2\|\mathbf{Y}\|^2 \tag{2.16}$$

or equivalently, by (2.6), the inequality

$$\|\mathbf{X}\times\mathbf{Y}\|^2 \leq \|\mathbf{X}\wedge\mathbf{Y}\|^2. \tag{2.17}$$

2.1.2. The basic kinematic quantities and examples of "simple" types of motion. Now, we return to the n-configuration motion $\mathbf{X}(t)$, (2.1), and put $\mathbf{Y} = \dot{\mathbf{X}}$ in Section 2.1.1. First of all, we define the ith individual (central) moment of inertia I_i, kinetic energy T_i, and angular momentum Ω_i by

$$I_i = m_i\|\mathbf{a}_i\|^2, \qquad T_i = \frac{1}{2}m_i\|\dot{\mathbf{a}}_i\|^2, \qquad \Omega_i = m_i(\mathbf{a}_i\times\dot{\mathbf{a}}_i). \tag{2.18}$$

The corresponding total moment of inertia, kinetic energy, and angular momentum is the sum of the individual ones. They are the basic kinematic quantities, namely

$$I = \rho^2 = \|\mathbf{X}\|^2, \qquad T = \frac{1}{2}\|\dot{\mathbf{X}}\|^2, \qquad \Omega = \mathbf{X}\times\dot{\mathbf{X}}. \tag{2.19}$$

We introduce some terminology in order to characterize specific types of "well-behaved" motions. The motion is *rectilinear* (resp., *planar* or *flat*) if the particles $\mathbf{P}_i$ move along a fixed line (resp., plane) in 3-space. The motion is *collinear* (resp., *coplanar*) if at each time t the particles lie on a line (resp., plane) which may depend on t. Furthermore, we say a motion is *torque-free* if each individual angular momentum Ω_i, $i \geq 1$, is constant. Recall that in dynamics this means that $\mathbf{P}_i$ is (for some reason) subject to a central force field, see for example [12].

In the following examples we will draw some immediate kinematic consequences from the sole assumption that Ω is time independent.

- A collinear motion is torque-free, and then it is rectilinear if and only if $\Omega = \mathbf{0}$. On the other hand, if $\Omega \neq \mathbf{0}$ then a collinear motion is planar and Ω is normal to the plane of motion. Moreover, each Ω_i is a nonnegative multiple of Ω.
- A coplanar motion is necessarily planar if it is torque-free. Indeed, each $\mathbf{P}_i$ moves in a fixed plane (resp., along a fixed line) if $\Omega_i \neq \mathbf{0}$ (resp., if $\Omega_i = \mathbf{0}$), namely in a plane with normal vector Ω_i. This plane is independent of i. For $n = 3$ all motions are coplanar (or collinear), of course. In fact, for a 3-body motion (in a gravitational force field) which is torque-free and with $\Omega \neq \mathbf{0}$, the P_i are at the vertices of an equilateral triangle, and hence the motion is shape invariant.

Problem 2.1. Assume a coplanar motion $X(t)$ is perpendicular to Ω (e.g., $\Omega = 0$) at some time t_0, and assume $X(t_0)$ is not collinear. Must the motion be planar?

The answer is yes for $n = 3$; it is, for example, a direct consequence of the generalized Euler equations for 3-body motions given by [6, Theorem 4]. A weaker statement is proved in Lemma 2.2 below. In dynamics, Saari [15] answers the problem affirmatively for general n, making use of the "standard" symmetry assumptions on the potential function U. We conjecture, however, that an affirmative answer holds for purely kinematic reasons as well.

• The class of *homothetic* n-body motions is defined by the condition that $\mathbf{X}(t)$ is confined to a fixed line in $\mathbb{R}^{3n}$. This is the subclass, defined by the condition $\Omega = \mathbf{0}$, of the more general *shape invariant* motions, namely the n-configurations at times $t_1 \neq t_2$ differ by a similarity transformation of the Euclidean 3-space. Equivalently, the kinetic energy term T^{σ} representing change of shape vanishes (cf. Section 2.1.4). In Wintner [24], n-body motions of this type are referred to as *homographic*. The first solutions of the 3-body problem in the literature, dating back to Euler and Lagrange, are shape invariant. Still, with essentially no exceptions they are the only known exact solutions of the (unrestricted) n-body problem. Even so, in the classical n-body problem, the complete determination of all realizable shapes for the shape invariant solutions is still an unsolved problem for $n > 3$. We refer to Section 4 for a precise description of the shape invariant motions.

Sundman was the first to prove, for the Newtonian n-body problem, that a general collision (or total collapse) is only possible when $\Omega = 0$. Weierstrass probably knew this result, and he showed for $n = 3$ that the motion must be planar if $\Omega = 0$. We will give a simple and purely kinematic proof of this fact in the following lemma.

Lemma 2.2. *A 3-configuration motion with constant angular momentum Ω, with respect to the center of mass, is planar if and only if Ω is perpendicular to the plane of motion. In particular, the motion is planar if Ω vanishes.*

Proof. Let the origin be the center of mass. We will assume $\mathbf{a}_1 \times \mathbf{a}_2 \neq \mathbf{0}$ during the motion, say $\mathbf{a}_1 \times \mathbf{a}_2 = f(t)\mathbf{n}$ with $f(t) > 0$, where $\mathbf{n}$ is a unit normal vector of the plane of motion. Define numbers x and y by

$$x = (\mathbf{a}_1 \times \mathbf{a}_2) \cdot \dot{\mathbf{a}}_1, \qquad y = (\mathbf{a}_1 \times \mathbf{a}_2) \cdot \dot{\mathbf{a}}_2. \tag{2.20}$$

Then it is easy to see that $\dot{\mathbf{n}} = 0$, that is, the motion is planar, if and only if $x = y = 0$. On the other hand, a simple calculation gives the identity

$$-m_3 \Omega \times (\mathbf{a}_1 \times \mathbf{a}_2) = (k_{11}x + k_{12}y)\mathbf{a}_1 + (k_{21}x + k_{22}y)\mathbf{a}_2, \tag{2.21}$$

where

$$\begin{aligned} k_{11} &= m_1 m_3 + m_1^2, & k_{22} &= m_2 m_3 + m_2^2, \\ k_{12} &= k_{21} = m_1 m_2, & \det(k_{ij}) &= m_1 m_2 m_3, \end{aligned} \tag{2.22}$$

and the mass distribution has been normalized so that $\Sigma m_i = 1$. It follows that Ω and $\mathbf{n}$ are collinear if and only if $x = y = 0$. □

2.1.3. The basic kinematic identities and inequalities. Consider an n-configuration motion $\mathbf{X}(t)$, and apply the results of Section 2.1.1 with $\mathbf{Y} = \dot{\mathbf{X}}$. So far, we need not assume invariance of any quantity such as Ω. Equation (2.6) reads

$$\|\mathbf{X} \wedge \dot{\mathbf{X}}\|^2 = 2IT - \frac{1}{4}\dot{I}^2 = \sum_{i=1}^{n} \|\Omega_i\|^2 + \|\Omega_{\text{mix}}\|^2, \tag{2.23}$$

where the "mixed angular momentum" term Ω_{mix} has norm square

$$\|\Omega_{\text{mix}}\|^2 = \sum_{i<j} m_i m_j \sum_{r,s=1}^{3} \left(x_r^{(i)} \dot{x}_s^{(j)} - x_s^{(j)} \dot{x}_r^{(i)} \right)^2. \tag{2.24}$$

Moreover, (2.14) and (2.16) now reads

$$\|\Omega\|^2 + \frac{1}{4}\dot{I}^2 + \mathfrak{R} = 2IT, \tag{2.25}$$

$$\|\Omega\|^2 + \frac{1}{4}\dot{I}^2 \leq 2IT, \tag{2.26}$$

where the latter is usually referred to in the literature as the *Sundman inequality* (cf. Saari [13]), and hence it is appropriate to refer to the kinematic identity (2.25) as the *Sundman identity* and to $\mathfrak{R}$ as the *Sundman residue*.

Recall from (2.15) that $\mathfrak{R}$ decomposes in two ways. Correspondingly, it follows from (2.12) and (2.13) that equality holds in (2.26), that is, $\mathfrak{R}$ vanishes, if and only if

$$\dot{I}_i = \frac{I_i}{I}\dot{I}, \quad \Omega_i = \frac{I_i}{I}\Omega, \quad \forall i, \tag{2.27}$$

or equivalently (whenever the fractions are defined)

$$\dot{I}_i = \frac{T_i}{T}\dot{I}, \quad \Omega_i = \frac{T_i}{T}\Omega, \quad \forall i. \tag{2.28}$$

In fact, it is not difficult to see that the above conditions characterize shape invariant motions which in the case $\Omega \neq \mathbf{0}$ must be coplanar. Therefore, these motions are characterized by the *Sundman identity* with a vanishing residual term, namely

$$\|\Omega\|^2 + \frac{1}{4}\dot{I}^2 = 2IT. \tag{2.29}$$

Remark 2.3. For the classical gravitational n-body motions, or in dynamics where the potential has similar symmetry properties, shape invariant motions with nonvanishing angular momentum are, in fact, planar (i.e., coplanar, but confined to a fixed plane). We refer to Section 4.

We will further analyze the structure of the Sundman residue $\mathfrak{R}$, (2.15), starting with the following remark.

Remark 2.4. By its definition, $\mathfrak{R}$ is a nonnegative, homogeneous, and $O(3)$-invariant polynomial of degree 4 of the $6n$ variables $x_r^{(i)}, \dot{x}_s^{(j)}$. Hence, by classical invariant theory it may be written as a quadratic polynomial of inner products among the n position vectors $\mathbf{a}_i$ and their velocities.

By invoking the identity (2.23) and the obvious decomposition

$$\|\Omega\|^2 = \sum \|\Omega_i\|^2 + 2\sum_{i<j} \Omega_i \cdot \Omega_j, \tag{2.30}$$

we obtain the following explicit formula:

$$\begin{aligned} \mathfrak{R} &= \|\Omega_{\text{mix}}\|^2 - 2\sum_{i<j} \Omega_i \cdot \Omega_j \\ &= \sum_{i,j} m_i m_j \left[\|\mathbf{a}_i\|^2 \|\dot{\mathbf{a}}_j\|^2 - (\mathbf{a}_i \cdot \mathbf{a}_j)(\dot{\mathbf{a}}_i \cdot \dot{\mathbf{a}}_j) - (\mathbf{a}_i \times \mathbf{a}_j) \cdot (\dot{\mathbf{a}}_i \times \dot{\mathbf{a}}_j) \right]. \end{aligned} \tag{2.31}$$

From the last expression it is easy to see that $\mathfrak{R}$ vanishes if and only if for each $i \neq j$ the following three conditions hold (where the last two make sense only when all four vectors involved are nonzero):

- $\|\mathbf{a}_i\|\|\dot{\mathbf{a}}_j\| = \|\mathbf{a}_j\|\|\dot{\mathbf{a}}_i\|$;
- the angle $\theta_{i,j}$ between $\mathbf{a}_i$ and $\mathbf{a}_j$ equals the angle between $\dot{\mathbf{a}}_i$ and $\dot{\mathbf{a}}_j$;
- $\mathbf{a}_i$ and $\mathbf{a}_j$ span the same plane as $\dot{\mathbf{a}}_i$ and $\dot{\mathbf{a}}_j$ (if $\sin\theta_{i,j} \neq 0$).

For $\Omega \neq \mathbf{0}$, a kinematic interpretation of $\mathfrak{R}$ is still somewhat awkward. However, there is a modified version of the kinematic identity (2.25), namely the *general kinematic identity*

$$II_\omega\|\omega\|^2 + \frac{1}{4}\dot{I}^2 + \tilde{\mathfrak{R}} = 2IT, \tag{2.32}$$

where

$$I_\omega\|\omega\|^2 = \|\omega \times \mathbf{X}\|^2 \geq \frac{\|\Omega\|^2}{I}, \tag{2.33}$$

$$I_\omega = \sum I_i \sin^2\theta_i \leq I, \tag{2.34}$$

is the moment of inertia with respect to the ω-axis, where ω is the instantaneous angular velocity of the system and θ_i is the angle between $\mathbf{a}_i$ and ω. The *general residue*

$$\tilde{\mathfrak{R}} = \mathfrak{R} - \left(II_\omega\|\omega\|^2 - \|\Omega\|^2\right) \geq 0 \tag{2.35}$$

is, indeed, still nonnegative. The connection between Ω and ω is explained in the next subsection, where the three terms on the left side of (2.32) are interpreted in terms of kinetic energy. Moreover, equality holds in (2.33), and hence $\mathfrak{R} = \tilde{\mathfrak{R}}$, if and only if $\mathbf{X}$ is perpendicular to Ω, that is, either Ω vanishes or $\mathbf{X}$ is coplanar and its plane is perpendicular to Ω.

2.1.4. Decomposition of kinetic energy. We will focus attention on the kinetic energy T and its natural decomposition suggested by the geometry of Euclidean 3-space $\mathbb{R}^3$. Namely, an n-configuration is uniquely characterized by the three geometric invariants: *size, shape*, and *position*, where shape and size together define the *congruence class* of the n-configuration, and the position (relative to a fixed reference n-configuration $\mathbf{X}_0$) is measured by an element (or rather coset) of the isometry subgroup of $\mathbb{R}^3$ fixing the origin, that is, the orthogonal group $O(3)$. For a given n-configuration motion, $t \to \mathbf{X}(t)$, the rate of change of the above invariants is expressed by the corresponding components of the velocity, namely the following orthogonal decomposition

$$\dot{\mathbf{X}} = \dot{\mathbf{X}}^\rho + \dot{\mathbf{X}}^\perp = \dot{\mathbf{X}}^\rho + \left(\dot{\mathbf{X}}^\sigma + \dot{\mathbf{X}}^\omega\right). \tag{2.36}$$

The associated decomposition of kinetic energy is written as

$$\begin{gathered} T = T^\rho + T^\perp = T^\rho + T^\sigma + T^\omega, \\ T^\varphi = \frac{1}{2}\|\dot{\mathbf{X}}^\varphi\|^2, \quad \varphi \in \{\rho, \perp, \sigma, \omega\}. \end{gathered} \tag{2.37}$$

Remark 2.5. So far, the center of mass is not assumed to be the origin, so the above "congruence" notion may not be a natural one. However, after this subsection this will be assumed (cf. Section 2.2). Anyhow, (2.36) is a well-defined decomposition, with a possible translational component $\dot{\mathbf{X}}^t$ being "absorbed" into the other components.

In (2.36), $\dot{\mathbf{X}}^\rho$ is the *radial* velocity component, which by definition is parallel with $\mathbf{X}$ itself, whereas the *transversal* component $\dot{\mathbf{X}}^\perp$ is perpendicular to $\mathbf{X}$. The latter further decomposes into $\dot{\mathbf{X}}^\sigma$ and $\dot{\mathbf{X}}^\omega$, representing the change of "shape" and "position" (or orientation), respectively. It follows that $\dot{\mathbf{X}}^\rho = (\dot{\rho}/\rho)\mathbf{X}$, and consequently

$$T^\rho = \frac{\dot{I}^2}{8I} = \frac{1}{2}\dot{\rho}^2 \tag{2.38}$$

which combined with (2.23) gives

$$T^\perp = \frac{1}{2I}\|\mathbf{X}\wedge\dot{\mathbf{X}}\|^2 = \frac{1}{2I}\sum_{i=1}^{n}\|\Omega_i\|^2 + \frac{1}{2I}\|\Omega_{\text{mix}}\|^2. \tag{2.39}$$

Next, we turn to the more awkward problem of splitting off the rotational energy T^ω from $T^\perp$. The velocity $\dot{\mathbf{X}}^\omega$ is the component tangential to the $SO(3)$-orbit through $\mathbf{X}$, whose tangent space is spanned by the (Killing) vector fields

$$\mathbf{X} \longrightarrow (\mathbf{n}\times\mathbf{a}_1,\dots,\mathbf{n}\times\mathbf{a}_n) = \mathbf{n}\times\mathbf{X} \tag{2.40}$$

generated by all $\mathbf{n}\in\mathbb{R}^3 \simeq so(3)$. Therefore

$$\dot{\mathbf{X}}^\omega = \omega\times\mathbf{X} \tag{2.41}$$

for some $\omega\in\mathbb{R}^3$, and moreover, for any $\mathbf{n}$

$$\mathbf{n}\cdot(\mathbf{X}\times\dot{\mathbf{X}}^\omega) = (\mathbf{n}\times\mathbf{X})\cdot\dot{\mathbf{X}}^\omega = (\mathbf{n}\times\mathbf{X})\cdot\dot{\mathbf{X}} = n\cdot(\mathbf{X}\times\dot{\mathbf{X}}) = \mathbf{n}\cdot\Omega. \tag{2.42}$$

It follows that

$$\Omega = \mathbf{X}\times(\omega\times\mathbf{X}) \tag{2.43}$$

and for fixed $\mathbf{X}$ this identity defines a linear transformation $\omega\to\Omega$ which is easily seen to be invertible if $\mathbf{X}$ is not collinear. Anyhow, Ω vanishes if and only if $\omega\times\mathbf{X}$ vanishes. Although in the collinear case, ω is only determined up to a summand in the direction of the vectors $\mathbf{a}_i$, I_ω adjusts correspondingly in the general formula for the rotational kinetic energy

$$T^\omega = \frac{1}{2}\|\omega\times\mathbf{X}\|^2 = \frac{1}{2}I_\omega\|\omega\|^2. \tag{2.44}$$

In general, we have inequalities

$$\frac{\|\Omega\|^2}{I\cdot I_\omega} \le \|\omega\|^2 \le \frac{2T}{I_\omega}, \tag{2.45}$$

where equality on the right side means purely rotational motion, that is, $T = T^\omega$, and equality on the left side (cf. (2.33)) is equivalent to the vanishing of the Sundman residue $\mathfrak{R}$, namely (2.29) holds. Furthermore, assuming $\mathbf{X}$ is not collinear, one can show that Ω and ω are collinear vectors if and only if $\mathbf{X}$ is a coplanar configuration and Ω is perpendicular to its plane, namely

$$\omega = \frac{1}{I}\Omega, \qquad I_\omega = I, \qquad T^\omega = \frac{1}{2}\frac{\|\Omega\|^2}{I}. \tag{2.46}$$

For example, (2.46) holds for a planar motion, whereas the expression for T^ω in (2.46) is a lower bound in the general case.

It is now clear that for a planar motion $\mathbf{X}(t)$, (2.37) coincides with the Sundman identity (2.25), and for a general motion the "change of shape" kinetic energy term is given by

$$T^\sigma = \frac{1}{2I}\tilde{\mathfrak{R}} \tag{2.47}$$

so that (2.37) coincides with the general kinematic identity (2.32). The motion is said to be *shape invariant* if the term T^σ vanishes. In particular, for planar motions its expression becomes

$$T^\sigma = \frac{1}{2I}\mathfrak{R} = \frac{1}{2I}\left(\left\|\Omega_{\mathrm{mix}}\right\|^2 - 2\sum_{i<j}\Omega_i\cdot\Omega_j\right). \tag{2.48}$$

In retrospect, we recall the inequalities

$$0 \le \tilde{\mathfrak{R}} \le \mathfrak{R} \tag{2.49}$$

and the following interpretations:

- $\tilde{\mathfrak{R}} = \mathfrak{R}$ says $\mathbf{X}$ is perpendicular to Ω (and hence coplanar if $\Omega \ne \mathbf{0}$),
- $\tilde{\mathfrak{R}} = 0$ says the motion is shape invariant,
- $\mathfrak{R} = 0$ says the motion is shape invariant, and is also coplanar if $\Omega \ne \mathbf{0}$,
- $\tilde{\mathfrak{R}} = 0$ if and only if $\mathfrak{R} = 0$ holds in dynamics governed by a "typical" potential function, for example, for Newtonian n-body motions.

We have seen that the inequality $T^\omega + T^\rho \le T$, or its equivalent form

$$I I_\omega \|\omega\|^2 + \frac{1}{4}\dot{I}^2 \le 2IT \tag{2.50}$$

is generally stronger than the Sundman inequality (2.26). For any motion the inequality in (2.50) is strict, unless the motion is shape invariant, whereas equality holds in the Sundman inequality if the shape invariant motion is also homothetic or coplanar. The inequality $\tilde{\mathfrak{R}} \le \mathfrak{R}$ also expresses that, in an appropriate sense, the change of shape is (locally) maximal when the motion passes through a coplanar n-configuration.

2.2. Dynamics and basic asymptotic analysis. In this subsection, we assume that the n-configuration motion is due to a force acting on the mass points, derived from a potential function $U(\mathbf{X})$, namely

$$\ddot{\mathbf{X}} = \nabla\mathbf{U} = \left(\frac{1}{m_1}\frac{\partial U}{\partial \mathbf{a}_1}, \frac{1}{m_2}\frac{\partial U}{\partial \mathbf{a}_2}, \ldots, \frac{1}{m_n}\frac{\partial U}{\partial \mathbf{a}_n}\right) \tag{2.51}$$

is the differential equation of the motion.

2.2.1. The potential function and invariance properties. We will make the following two basic assumptions on the potential function:

- U is invariant with respect to Euclidean motions in $\mathbb{R}^3$;
- U is homogeneous of degree $-e$, that is, $U(k\mathbf{X}) = k^{-e}U(\mathbf{X})$ for all $k > 0$.

These properties are typical for a system with only mutual interaction between particles and no external forces acting on the system. As a direct consequence of the translational, respectively $O(3)$-invariance of U, we infer respectively

$$\sum m_i \ddot{\mathbf{a}}_i = \sum \frac{\partial U}{\partial \mathbf{a}_i} = \mathbf{0}, \qquad \dot{\Omega} = \sum \mathbf{a}_i \times \frac{\partial U}{\partial \mathbf{a}_i} = \mathbf{0}, \tag{2.52}$$

and moreover, U depends only on the mutual distances

$$r_{i,j} = \|\mathbf{a}_i - \mathbf{a}_j\| \tag{2.53}$$

since these numbers constitute a complete set (but not functionally independent if $n > 4$) of congruence invariants for n-configurations. In particular, the vector Ω is an invariant of the motion.

Recall that $\rho = \sqrt{I}$ is the distance from $\mathbf{X}$ to the origin in $\mathbb{R}^{3n}$, whereas the actual "size" of an n-configuration is more naturally measured by

$$J = \frac{1}{\bar{m}} \sum_{i<j} m_i m_j r_{i,j}^2, \quad \bar{m} = \Sigma m_i, \tag{2.54}$$

namely the moment of inertia with respect to the center of mass, $(1/\bar{m}) \sum m_i \mathbf{a}_i$. It follows that $I = J$ if and only if the center of mass coincides with the origin, and thanks to (2.52) we can, indeed, choose origin in this way.

Remark 2.6. In classical mechanics, it is the invariance of linear momentum, namely $\sum m_i \dot{\mathbf{a}}_i$ is constant, which enables one to choose the origin at the center of mass, without sacrificing the "inertial" property of the reference frame, needed for the validity of the force law (2.51).

Henceforth, we assume the center of mass lies at the origin, and consequently we will restrict our configuration space to the following linear subspace of $\mathbb{R}^{3n}$:

$$M \simeq \mathbb{R}^{3n-3} : m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \cdots + m_n\mathbf{a}_n = 0 \tag{2.55}$$

with the induced kinematic metric (2.2). We also use the notation

$$\begin{gathered} U(\mathbf{X}) = \rho^{-e} U(\mathbf{X}_1) = \frac{U^*}{\rho^e}, \\ \mathbf{X}_1 = \frac{1}{\rho}\mathbf{X} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n), \quad \mathbf{u}_i = \frac{1}{\rho}\mathbf{a}_i, \end{gathered} \tag{2.56}$$

where U^* denotes the restriction of U to the unit sphere $(\rho = 1) = M_1 \simeq S^{3n-4}$ of M. Clearly, U is uniquely determined by its restriction U^*.

In addition to the above two assumptions on U we will also add one more assumption:

• The singular set of U^* is the collision variety, that is, $\mathbf{a}_i = \mathbf{a}_j$ for some $i \neq j$. Moreover, assume that $U^* > 0$ and $\mathrm{U}^* \to \infty$ as $\mathbf{X}_1$ approaches the singular set.

The standard example is, of course, the classical gravitational field, where $e = 1$. The *energy integral* of the motion is written as

$$h = T - U \tag{2.57}$$

whence $-U$ is the potential energy. Recall that the classical n-body motions have the Newtonian potential function (in appropriate units)

$$U = \sum_{i<j} \frac{m_i m_j}{r_{i,j}} \tag{2.58}$$

and its gradient vector field (in M, with respect to the Jacobi metric (2.2)) has components

$$\frac{1}{m_i}\frac{\partial U}{\partial \mathbf{a}_i} = \sum_{k \neq i} m_k \frac{(\mathbf{a}_k - \mathbf{a}_i)}{||\mathbf{a}_k - \mathbf{a}_i||^3}. \tag{2.59}$$

In general, combining (2.51) with the following crucial property of a homogeneous function U of degree $-e$ with respect to the vectors $\mathbf{a}_i$,

$$\mathbf{X} \cdot \ddot{\mathbf{X}} = \sum_{i=1}^{n} \mathbf{a}_i \cdot \frac{\partial U}{\partial \mathbf{a}_i} = -eU. \tag{2.60}$$

The associated *Lagrange-Jacobi* differential equation is the result of differentiating $I = \|\mathbf{X}\|^2$ twice and making the obvious substitutions from (2.57) and (2.60), namely the following three equivalent equations:

$$\ddot{I} = 2(2T - eU) = (4 - 2e)T + 2eh, \tag{2.61}$$

$$\ddot{\rho} + \frac{\dot{\rho}^2}{\rho} - \frac{1}{\rho}[(2-e)U + 2h] = 0, \tag{2.62}$$

$$(2-e)(T^\sigma + T^\omega) = \ddot{\rho}\rho + \frac{e}{2}\dot{\rho}^2 - eh, \tag{2.63}$$

are valid for motions on a given energy level h. Thus, (2.61) is merely a differential consequence of the energy integral (2.57), and conversely, the latter integral is recovered from (2.61) by integration, with h appearing as an integration constant.

Symmetries of the equation of motion. The differential equation (2.51) is invariant under Euclidean motions as well as time translation and reversion. It is also easy to check that the equation has a 1-parameter group $\{g_s\}$ of time-space *scaling symmetries*. In effect, if $t \to \mathbf{X}(t)$ is a solution at energy level h and angular momentum Ω, then for each fixed real number s

$$t \longrightarrow \mathbf{Y}(t) = e^{\nu s}\mathbf{X}(e^{-s}t), \quad \nu = \frac{2}{2+e}, \tag{2.64}$$

is a solution with energy and angular momentum

$$h^{(s)} = e^{-e\nu s}h, \qquad \Omega^{(s)} = e^{((2-e)/2)\nu s}\Omega, \tag{2.65}$$

respectively. (The exceptional case $e = -2$ is simpler; the space homotheties $\mathbf{X} \to e^s\mathbf{X}$ are symmetries.)

2.2.2. Asymptotic estimates of general collisions. This subsection is devoted to the basic asymptotic behavior of motions leading to a total collapse (also called general collision). This event is simply characterized by the condition $I \to 0$. The main results are stated in Theorems 2.7 and 2.8, but completion of the proofs is postponed until Section 6.

Results in this direction date back to Sundman's work around 1910 on n-body motions, mostly with $n = 3$. They were substantially improved by Siegel three decades later. Among the classical results are, for example, the vanishing of the angular momentum vector Ω and asymptotic formulas of the physical quantities T and $I = \rho^2$. Sundman also found asymptotic formulas for $\dot{I}$ and $\ddot{I}$ and, moreover, he showed that U^* has a limit as $I \to 0$. These results are also contained in our Theorem 2.7.

For functions $f(t)$ and $g(t)$ recall the following notion of *asymptotic equivalence*

$$f \sim g \quad \text{if } \frac{f}{g} \longrightarrow 1 \text{ as } t \longrightarrow t_o. \tag{2.66}$$

Theorem 2.7. *Assume $U > 0$ is homogeneous of degree $-e$, $0 < e < 2$, and the motion leads to a general collision at $t = t_o$. Then*

$$\lim_{t \to t_0} U^*(t) = \mu > 0 \tag{2.67}$$

exists, and for all $i \geq 0$

$$\begin{gathered} \frac{d^i}{dt^i}(\rho) \sim \frac{d^i}{dt^i}(\kappa t^\nu), \\ \frac{d^i}{dt^i}E \sim \frac{d^i}{dt^i}\left(\frac{\mu}{\rho^e}\right) \sim \frac{d^i}{dt^i}\left(\frac{1}{2}\nu^2\kappa^2 t^{2\nu-2}\right), \end{gathered} \tag{2.68}$$

where E denotes T^ρ, T, or U, and the constants e and μ determine the constants ν and κ by

$$\nu = \frac{2}{2+e}, \qquad \mu = \frac{\nu^2}{2}\kappa^{2+e}. \tag{2.69}$$

Theorem 2.8. *Under the assumptions of Theorem 2.7, the total angular momentum Ω is identically zero. Furthermore, each individual angular momentum Ω_i as well as the "mixed angular momentum" $\tilde{\Omega}_{\text{mix}}$, (see (2.24)) must tend to zero.*

We turn to the proofs of the theorems, using ideas adapted from the work of Siegel and Wintner (cf. [18, 19, 24]).

As in the early investigations, a principal tool is the Lagrange-Jacobi equation (see (2.61)) which in the standard case $e = 1$ reads

$$\ddot{I} = 2T + 2h. \tag{2.70}$$

Another crucial property of the potential function (2.58) is $U(\mathbf{X}) \to \infty$ as $\mathbf{X} \to \mathbf{0}$, hence also $\ddot{I} \to \infty$ by the above equation. Consequently, $I \to 0$ in finite time, and $\dot{I} > 0$ holds near collision time.

We first clarify our conditions on U. The condition $e > 0$ is needed since there is no a priori reason for $\mathbf{X}_1 \in M_1$ to approach the singular set (where $U^* \to \infty$) as $I \to 0$. However, the additional condition $e < 2$ will be needed later, see (2.96).

From (2.61) it follows that $\ddot{I} \to \infty$, so again the collision must occur at finite time. By translating and (eventually) reversing time we will assume collision occurs at $t = 0^+$, and henceforth the motion is studied during a small time interval $(0, t_0)$, where we may also assume $\dot{I} > 0$.

We start by showing that the constant vector $\Omega = \sum \Omega_i$ must be zero. To this end, put

$$K = \|\mathbf{X} \wedge \dot{\mathbf{X}}\| = \|\tilde{\Omega}_{\text{mix}}\|^2 + \sum_{i=1}^{n} \|\Omega_i\|^2 \tag{2.71}$$

and deduce the inequality

$$\dot{I}\ddot{I} \geq (2-e)K\dot{I}I^{-1} + 2h\dot{I} \tag{2.72}$$

by combining (2.23) and (2.61). Define $K_0 = \inf_{(t,t_0)} K$ and integrate the inequality from t to t_0 to obtain

$$\begin{aligned} \frac{1}{2}\dot{I}_0^2 &\geq \frac{1}{2}(\dot{I}_0^2 - \dot{I}^2) \geq (2-e)K_o \ln\left(\frac{I_0}{I}\right) + 2h(I_0 - I), \\ K_0 &\leq \frac{2hI - 2hI_0 + 2^{-1}\dot{I}_0^2}{(2-e)\ln(I_0/I)} \longrightarrow 0 \quad \text{as } t \longrightarrow 0. \end{aligned} \tag{2.73}$$

This proves that $\Omega = 0$.

Recall the splitting (2.37) of kinetic energy; here $T^{\omega} = 0$ since $\Omega = 0$. Thus, the identity $T = T^{\rho} + T^{\sigma} = U + h$ gives the asymptotic formula

$$U^* \sim A(t) + B(t) \quad \text{as } t \longrightarrow 0^+, \tag{2.74}$$

where

$$A(t) = \rho^e T^{\rho}, \qquad B(t) = \rho^e T^{\sigma}. \tag{2.75}$$

Lemma 2.9. *The term $A(t)$ has a limit, namely* $\lim_{t\to 0^+} A(t) = \mu > 0$ *exists.*

Proof. By combining formulas (2.38) and (2.62),

$$\dot{A} = \dot{\rho}\rho^{-1+e}((2-e)T^{\sigma} + eh). \tag{2.76}$$

Hence, for $t \leq t_0 \leq \varepsilon$

$$A(t_0) - A(t) = \int_t^{t_0} (2-e)\rho^{-1+e}\dot{\rho}T^{\sigma}\,dt + h(\rho(t_0)^e - \rho(t)^e) \tag{2.77}$$

and since both A and the integrand are nonnegative, $\lim A(t) = \mu$ must exist. It remains to show that $\mu > 0$.

Suppose we have $\mu = 0$. Since $\min(U^*) > 0$, (2.62) implies

$$\ddot{\rho}\rho^{1+e} = (2-e)U^* + 2h\rho^e - 2A = (2-e)U^* + o(1) \geq C > 0 \quad (\text{as } t \longrightarrow 0^+) \tag{2.78}$$

and therefore

$$\dot\rho\ddot\rho = \frac{\dot\rho}{\rho^{1+e}}\left((2-e)U^* + o(1)\right) \geq C\frac{\dot\rho}{\rho^{1+e}}. \tag{2.79}$$

Consequently, by integration

$$\begin{aligned}\dot\rho(t_0)^2 - \dot\rho(t)^2 &= \int_t^{t_0}\frac{d}{dt}(\dot\rho^2)dt = 2\int_t^{t_0}(\dot\rho\ddot\rho)dt \geq 2C\int_t^{t_0}\frac{\dot\rho}{\rho^{1+e}}dt \\ &= \frac{2C}{e}\left(\frac{1}{\rho(t)^e} - \frac{1}{\rho(t_0)^e}\right) \longrightarrow \infty \quad \text{as } t \longrightarrow 0^+\end{aligned} \tag{2.80}$$

and this is clearly contradictory. □

From Lemma 2.9 we have now $A(t) = \mu + o(1)$, hence by definition of A, $\dot\rho = \sqrt{2\mu}\rho^{-e/2} + o(\rho^{-e/2})$ and consequently

$$\nu\rho^{1/\nu} = \int_0^\rho \rho^{e/2}\,d\rho = \int_0^t \dot\rho\rho^{e/2}\,dt = \int_0^t\left(\sqrt{2\mu}+o(1)\right)dt = \sqrt{2\mu}t + o(t) \tag{2.81}$$

which gives $\rho \sim \kappa t^\nu$ and $\kappa = (\nu^{-1}\sqrt{2\mu})^\nu$. Moreover, $\dot\rho \sim \sqrt{2\mu}\rho^{-e/2} \sim \nu\kappa t^{\nu-1}$, and then the asymptotic expression we seek for T^ρ follows from $T^\rho \sim \mu\rho^{-e}$ or $(1/2)\dot\rho^2$.

Next, we turn to the asymptotic formula for $\ddot\rho$. Implicit in the work of Siegel [17, 18] is a special property of the potential function U which, in our interpretation, leads to an upper bound estimate of $\dot U$ on time intervals where U^* and its gradient (or the gradient of U evaluated on M_1) have a given bound. The next lemma explains this.

Lemma 2.10. *Let $0 < t_2 < t_1 < \varepsilon$ and assume that $U(\mathbf{X}_1) \leq C_1$ and $\|\nabla U(\mathbf{X}_1)\| < C_1$ for $t \in [t_2, t_1] = \mathfrak{I}$. Then there is a constant $C > 0$, depending on ε, such that $|\dot U| \leq Ct^{2\nu-3}$ for all $t \in \mathfrak{I}$.*

Proof. Since $\dot U = \nabla U \cdot \dot{\mathbf{X}}$,

$$|\dot U| \leq \|\nabla U(\mathbf{X})\|\|\dot{\mathbf{X}}\| = \frac{1}{\rho^{e+1}}\|\nabla U(\mathbf{X}_1)\|\sqrt{2T}. \tag{2.82}$$

On the other hand, for small t

$$T = U + h = \frac{1}{\rho^e}(U^* + h\rho^e) \leq C_2\rho^{-e}, \tag{2.83}$$

and consequently

$$|\dot U| \leq \frac{C_1\sqrt{2C_2}}{\rho^{e+1+e/2}} \leq \frac{C}{t^{(e+1+e/2)\nu}} = \frac{C}{t^{3-2\nu}}. \tag{2.84}$$

□

Lemma 2.11. *The term $B(t)$ tends to zero, that is, $B(t) \to 0$ as $t \to 0^+$.*

Proof. Since B is nonnegative, the claim is $\limsup B(t) = 0$. We first establish

$$\liminf B(t) = 0 \quad \text{as } t \longrightarrow 0^+. \tag{2.85}$$

Using the asymptotic formulas for ρ and $\dot\rho$, consider (2.77), from which it follows that the integral

$$\int_{0^+}^{t_0}\rho^{-1+e}\dot\rho T^\sigma\,dt \tag{2.86}$$

is convergent. However, the integrand is

$$\rho^{-1+e}\dot{\rho}T^{\sigma} = \left(\frac{\nu}{t} + o(t^{-1})\right)B(t) \tag{2.87}$$

and the integral of $1/t$ is divergent, so this is not possible unless $\liminf B(t) = 0$.

It remains to show that $\limsup B(t) = 0$. Assume to the contrary, that $\limsup B(t) > 0$. Consequently, for given $\varepsilon > 0$, there is a sequence $\varepsilon > t_1 > t_2 > \cdots > t_k > \cdots$ with $\lim t_k = 0$, such that for some $\delta > 0$

$$\delta \le B(t) \le 3\delta, \quad \text{for } t \in [t_{2i}, t_{2i-1}] = J_i; \qquad B(t_{2i-1}) = 3\delta, \qquad B(t_{2i}) = \delta. \tag{2.88}$$

Therefore, for $t \in J_i$ and each i, by Lemma 2.9 there is a constant C_1 such that

$$U^* = A + B - \rho^e h \le \mu + 3\delta + o(1) \le C_1, \tag{2.89}$$

where $o(1) \to 0$ as $t_1 \to 0$. In particular, the curve $\mathbf{X}_1(t)$ will stay in a compact region on the sphere M_1 and disjoint from the singular set of U^*. In this region the norm of the gradient is also bounded, say, by C_1. Hence, we may apply Lemma 2.10, and for some constant C

$$|\dot{T}| = |\dot{U}| \le Ct^{2\nu-3} \quad \text{for } t \in J_i. \tag{2.90}$$

Consequently, by (2.83), (2.90), and the asymptotic formulas for ρ and $\dot{\rho}$ already found, there is a constant C_2 such that

$$\left|\frac{d}{dt}(\rho^e T)\right| = |\rho^e \dot{T} + e\rho^{e-1}\dot{\rho}T| \le C_2\frac{1}{t} \quad \text{for } t \in J_i. \tag{2.91}$$

Since $A \sim \mu$, we may also assume t_1 is so small that

$$|A(t_{2i-1}) - A(t_{2i})| \le \delta. \tag{2.92}$$

Then, by (2.75), (2.89), and (2.91)

$$\begin{aligned} 2\delta &= B(t_{2i-1}) - B(t_{2i}) \\ &= (\rho^e T - A)_{t=t_{2i-1}} - (\rho^e T - A)_{t=t_{2i}} \\ &\le C_2 \int_{t_{2i}}^{t_{2i-1}} \frac{dt}{t} + \delta \end{aligned} \tag{2.93}$$

from which we infer

$$\int_{t_{2i}}^{t_{2i-1}} \frac{B(t)}{t}\,dt \ge \delta \int_{t_{2i}}^{t_{2i-1}} \frac{dt}{t} \ge \frac{\delta^2}{C_2} \quad \text{for each } i. \tag{2.94}$$

On the other hand, the integral (2.86) is convergent and consequently also

$$\int_0^{t_1} \frac{B(t)}{t}\,dt < \infty. \tag{2.95}$$

Clearly, (2.94) and (2.95) are not compatible, and this completes the proof that $\limsup B(t) = 0$. □

From Lemma 2.11 we infer $T^{\rho} \sim T$, and evidently $T \sim U$. Then the asymptotic expression for $\ddot{\rho}$ follows from (2.62). Returning to K defined in (2.23) or (2.71),

$$K = 2IT - \frac{\dot{I}^2}{4} = 2I(T - T^{\rho}) = 2IT^{\sigma} = 2\rho^{2-e}B \longrightarrow 0 \quad \text{as } t \longrightarrow 0^+ \tag{2.96}$$

and this completes the proof of Theorem 2.8.

Finally, the asymptotic formulas for the time derivatives of $E = T^{\rho}, T$, or U in Theorem 2.7 rest upon the asymptotic formulas for all

$$\frac{d^3}{dt^3}\rho, \frac{d^4}{dt^4}\rho, \dots, \frac{d^i}{dt^i}\rho, \quad i > 2, \tag{2.97}$$

whose proof will be delayed until Section 6. Assuming these formulas for the moment, we use (2.62) to find successively $U, \dot{U}, \ddot{U}, \dots, (d^i/dt^i)U$, as a polynomial in $\rho, \dot{\rho}, \dots, (d^{i+2}/dt^{i+2})\rho$. Similar polynomial expressions are easily derived for the derivatives of $T^{\rho} = (1/2)\dot{\rho}^2$, of course. Now, the asymptotic expressions for the derivatives of E follows by inserting the asymptotic expressions of $\rho, \dot{\rho}, \dots$ into the corresponding polynomial. However, for a fixed $i > 0$, the three cases of E will have the same monomial as its asymptotic expression, say $(d^i/dt^i)(E) \sim p_i t^{q_i}$, since the three cases agree for $i = 0$, and the common monomial is

$$p_0 t^{q_0} = \frac{1}{2}\nu^2\kappa^2 t^{2\nu - 2}. \tag{2.98}$$

We leave this topic here by stating the following problem concerning the asymptotic behavior in the remaining "directions," namely with regard to shape and position (or orientation). In Section 6.2 we will return to this problem.

Problem 2.12. In the case of a general collision, what is the asymptotic behavior of the curve $X_1(t) = \rho^{-1}X(t)$ on the unit sphere M_1? Must $X_1(t)$ converge, that is, is there a limiting shape and orientation?

3. The moduli space of size and shape. In this section, we will focus attention on the kinematics of motions in the *congruence* moduli space $\bar{M}$, whose points represent the size and shape of n-configurations. As a mathematical object, $\bar{M}$ is the quotient space

$$M \xrightarrow{\pi} \bar{M} = M/O(3) \tag{3.1}$$

consisting of the congruence classes $\pi(\mathbf{X}) = \bar{\mathbf{X}}$. Namely, $\bar{M}$ is the orbit space of the natural orthogonal $O(3)$-representation on $M \simeq \mathbb{R}^{3(n-1)}$ by $n-1$ copies of the standard representation on $\mathbb{R}^3$. The above quotient construction is well understood in the framework of equivariant Riemannian geometry, by which $\bar{M}$ becomes a (stratified) Riemannian space and π a Riemannian submersion. As will be seen below, this description of the metric is, in fact, consistent with the decomposition of kinetic energy in Section 2.1.4. We refer to [2] for basic results on equivariant differential geometry.

3.1. The kinematic metric on $\bar{M}$. Recall that one can define *kinetic energy* on M and $\bar{M}$ as functions T and $\bar{T}$ on the respective tangent bundles of M and $\bar{M}$

$$\begin{array}{ccc} TM & \xrightarrow{T} & R \\ \Big\downarrow{\scriptstyle d\pi} & & \\ T\bar{M} & \xrightarrow{\bar{T}} & R \end{array} \tag{3.2}$$

such that their restriction to each tangent space (fibre) is a positive definite quadratic form. The associated Riemannian metrics are customarily written as

$$ds^2 = 2T dt^2, \qquad d\bar{s}^2 = 2\bar{T} dt^2. \tag{3.3}$$

On M we want this, of course, to be the kinematic metric defined by the Jacobi inner product (2.2), that is, $ds^2 = \sum m_i ds_i^2$ where ds_i^2 is the standard metric on the ith summand of $\mathbb{R}^{3n}$.

On the other hand, the natural choice of $\bar{T}$ is suggested by the decomposition (2.37) of T, namely

$$T = \bar{T} + T^{\omega}, \qquad \bar{T} = T^{\rho} + T^{\sigma}. \tag{3.4}$$

We claim that $\bar{T}$ is, indeed, a function defined on $T\bar{M}$ with the appropriate properties. This will be seen in the next subsection, where $d\bar{s}^2 = 2\bar{T}dt^2$ is recognized as the orbital distance metric on $\bar{M}$. Consequently, we will also refer to $d\bar{s}^2$ as the *kinematic metric* on $\bar{M}$.

By inserting the expression for T^{ω} found in Section 2 into (3.4), we have by (3.3)

$$ds^2 = d\bar{s}^2 + 2T^{\omega} dt^2 = d\bar{s}^2 + I_{\omega} \|\omega\|^2 dt^2 \tag{3.5}$$

which expresses the rotational kinetic energy as the "lost" term in the passage from M to $\bar{M}$, cf. (3.2). In the special case of planary motion, or if Ω vanishes, the above formula reads

$$ds^2 = d\bar{s}^2 + \frac{\|\Omega\|^2}{\rho^2} dt^2. \tag{3.6}$$

3.2. The orbital geometry of $\bar{M}$. We first describe the Euclidean space M as the Riemannian cone

$$M = C(M_1) : ds^2 = d\rho^2 + \rho^2 d\phi^2 \tag{3.7}$$

over M_1, where $(M_1, d\phi^2)$ is the unit sphere $(\rho = 1)$ of M with its spherical metric $d\phi^2$. As usual, ρ is the norm function

$$\rho(\mathbf{X}) = \sqrt{I} = \|\mathbf{X}\| \tag{3.8}$$

which together with coordinates on M_1 constitute polar coordinates on M. Clearly ρ is also a function on the $O(3)$-orbit space, which can be described as the Riemannian cone

$$\bar{M} = C(M^*) : d\bar{s}^2 = d\rho^2 + \rho^2 d\sigma^2 \tag{3.9}$$

over its "unit sphere" ($\rho = 1$)

$$(M^*, d\sigma^2) = (M_1, d\phi^2)/O(3). \tag{3.10}$$

Here $d\bar{s}^2$ (and its restriction $d\sigma^2$ to M^*) denotes the orbital distance metric, and as will be seen below, the notation in (3.9) is consistent with (3.3).

As a cone, $\bar{M}$ (resp., M) is a union of rays emanating from its vertex $\bar{\mathbf{0}}$ (resp., $\mathbf{0}$). These curves are also the geodesics reaching the vertex, and ρ measures the distance from $\bar{\mathbf{X}}$ (resp., $\mathbf{X}$) to the vertex. Note that scalar multiplication in M induces a "multiplication" by positive scalars k in $\bar{M}$, namely $k\bar{\mathbf{X}} = \pi(k\mathbf{X})$, and a ray consists of all positive multiples of a unique point on M^*. The unit vector field in the (outward) ray direction (in $\bar{M}$ or M) is denoted by $\partial/\partial\rho$.

We will also refer to the subspace M^* of $\bar{M}$ as the moduli space of *similarity classes* (or *shapes*), or briefly the *shape space*. Clearly, the crucial data of the geometry of $\bar{M}$ is encoded into $(M^*, d\sigma^2)$, see Section 3.3. The simplest but important case $n = 3$ has been thoroughly investigated in [6].

Finally, we show that the kinetic energy as defined by the orbital distance metric is, after all, the function $\bar{T}$ defined by (3.4). To this end, let $\mathbf{X} \neq \mathbf{0}$ be a given point in M and $\bar{\mathbf{X}}$ its image in $\bar{M}$. Consider the following tangent spaces and their orthogonal decompositions:

$$T_{\mathbf{X}}M \simeq M = M^\rho \oplus M^\sigma \oplus M^\omega, \tag{3.11}$$

$$T_{\bar{\mathbf{X}}}\bar{M} = \bar{M}^\rho \oplus \bar{M}^\sigma. \tag{3.12}$$

Here M^ρ is the radial line (in the direction of $\partial/\partial\rho$) through $\mathbf{X}$, mapped by $d\pi$ to the tangent line $\bar{M}^\rho$ through $\bar{\mathbf{X}}$ generated by $\partial/\partial\rho$, and $M^\omega = \ker d\pi$ is the tangent space of the SO(3)-orbit (cf. Section 2.1.4). Moreover, M^σ is mapped isomorphically to the other summand $\bar{M}^\sigma$. Now, the inner product on $T_{\bar{\mathbf{X}}}\bar{M}$, by definition of Riemannian submersion, is determined by the condition that $d\pi : M^\rho \oplus M^\sigma \to T_{\bar{\mathbf{X}}}\bar{M}$ is an isometry. In particular, we have

$$\begin{gathered} d\pi(\mathbf{X}) = \|\mathbf{X}\| \frac{\partial}{\partial\rho} = \rho\frac{\partial}{\partial\rho}, \\ \frac{1}{\rho} d\pi : M^\sigma \xrightarrow{\simeq} \bar{M}^\sigma \xrightarrow{\simeq} T_{\mathbf{X}^*}M^*, \end{gathered} \tag{3.13}$$

where $\mathbf{X}^* = (1/\rho)\bar{\mathbf{X}}$ is the image of $\mathbf{X}$ in M^*, the map $M^\sigma \xrightarrow{\simeq} \bar{M}^\sigma$ is an isometry, and $\bar{M}^\sigma = T_{\mathbf{X}^*}M^*$ if $\rho = 1$.

Let $\mathbf{X}(t)$ be a motion in M and $\bar{\mathbf{X}}(t)$ the induced motion in $\bar{M}$. Let $\dot{\mathbf{X}} = \dot{\mathbf{X}}^\rho + \dot{\mathbf{X}}^\sigma + \dot{\mathbf{X}}^\omega$ be the splitting of the velocity in accordance with (3.11). By definition, $(d/dt)\bar{\mathbf{X}} = d\pi(\dot{\mathbf{X}}^\rho + \dot{\mathbf{X}}^\sigma)$, and consequently the kinetic energy of $\bar{\mathbf{X}}(t)$ will be (as promised)

$$\frac{1}{2}\left\|\frac{d}{dt}\bar{\mathbf{X}}\right\|^2 = \frac{1}{2}\left(\|\dot{\mathbf{X}}^\rho\|^2 + \|\dot{\mathbf{X}}^\sigma\|^2\right) = T^\rho + T^\sigma = \bar{T}. \tag{3.14}$$

Remark 3.1. Geodesics in $\bar{M}$ and M^* are the locus $\bar{\Gamma}$ and Γ^* of moduli curves $\bar{\mathbf{X}}(t)$ and $\mathbf{X}^*(t)$ of "*linear*" *motions* of n-configurations with $\Omega = \mathbf{0}$, where by "linear" motion we mean a motion in 3-space with constant velocity (i.e., the potential function U is constant). For example, in the case $n = 3$, Γ^* is an arc of a great circle on the 2-sphere M^*.

3.3. A brief description of M^*. Observe that $(M^*, d\sigma^2)$ is a compact $(3n-7)$-dimensional stratified Riemannian manifold

$$M^* \supset \Pi^* \supset E^*, \tag{3.15}$$

where the three strata $(M^* - \Pi^*)$, $(\Pi^* - E^*)$, and E^* correspond to the $O(3)$-orbit types $1 \subset O(1) \subset O(2)$, respectively. Here Π^* and E^* represent the shapes of coplanar and collinear n-configurations, respectively.

The orbital stratification of M^* involves projective spaces as explained by the following diagram where the horizontal maps are inclusions and the vertical ones are orbit space projections:

$$\begin{tikzcd}
S^{n-2} \arrow[r] \arrow[d] & S^{2n-3} \arrow[r] \arrow[d] & S^{3n-4} = M_1 \arrow[dd] \\
\mathbb{R}P^{n-2} = S^{n-2}/O(1) \arrow[r] \arrow[d, equal] & \mathbb{C}P^{n-2} = S^{2n-3}/\mathrm{SO}(2) \arrow[d] & \\
E^* \arrow[r] & \mathbb{C}P^{n-2}/Z_2 \arrow[r] \arrow[d, equal] & M_1/\mathrm{SO}(3) \arrow[d] \\
 & S^{2n-3}/O(2) = \Pi^* \arrow[r] & M^* = M_1/O(3).
\end{tikzcd} \tag{3.16}$$

The group $Z_2 \simeq O(1)$ acts on $\mathbb{C}P^{n-2}$ by complex conjugation, with $E^* = \mathbb{R}P^{n-2}$ as fixed point set and Π^* as orbit space. Similarly, M^* is the quotient of $M_1/\mathrm{SO}(3)$ by the induced action of $O(3)/\mathrm{SO}(3) \simeq Z_2$, with Π^* as fixed point set.

Note that the two sets $E^* \subset \Pi^*$ will not change if we work with congruence modulo $\mathrm{SO}(3)$ rather than $O(3)$. However,

$$M_1/\mathrm{SO}(3) - \Pi^* \longrightarrow M_1/O(3) - \Pi^* \tag{3.17}$$

is a 2-fold covering if $n > 3$. On the other hand, in $M_1/\mathrm{SO}(3)$ there are only two orbital strata, namely E^* and its complement. This is due to the fact that the action of $\mathrm{SO}(3)$ has only two isotropy types, namely $1 \subset \mathrm{SO}(2)$. In particular, $M_1/\mathrm{SO}(3) - E^*$ is a smooth manifold.

We also mention that $\bar{M} = M/O(3)$ (resp., M^*) can be naturally identified with the set of real, positive semidefinite symmetric $(n-1) \times (n-1)$-matrices of rank ≤ 3 (and with Euclidean norm 1, resp.). We refer to [5, 7] for further analysis of the above equivariant geometry.

3.3.1. The special case $n = 3$. Any 3-body system (called mass triangle) is, of course, a coplanar 3-configuration. Therefore, in (3.16) we have $\Pi^* = M^* = S^3/O(2)$; this is a 2-disk D^2 which geometrically is a closed hemisphere of the 2-sphere $\mathbb{C}P^1 = S^2(1/2)$ of radius $1/2$. The boundary of the hemisphere is the (equator) circle $E^* = \mathbb{R}P^1 = S^1(1/2)$ which represents the degenerate triangles, namely the collinear configurations.

On the other hand, the non-degenerate triangle $\mathbf{X} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ in 3-space can be oriented, say, by the frame $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_1 \times \mathbf{a}_2\}$. Hence, it is natural to consider oriented congruence classes of mass triangles, in which case M^* will be the base space of the usual Hopf fibration $S^3 \to S^2$, namely

$$\begin{aligned} M^* &= S^3/\mathrm{SO}(2) = \mathbb{C}P^1 = S^2\left(\frac{1}{2}\right) = M_+^* \cup M_-^*, \\ M_\pm^* &= S^5/\mathrm{SO}(3) = S^5/O(3), \qquad M_+^* \cap M_-^* = E^*, \end{aligned} \tag{3.18}$$

where the two hemispheres $M_\pm^* \simeq D^2$ represent triangles of opposite orientation. This modified definition of M^* is consistent with the "observation" that the normal vector of a triangle in motion should change continuously; the triangle degenerates when the motion crosses E^*, and as the motion enters the other hemisphere the triangle has, indeed, the opposite orientation. We refer to [6].

3.4. The gradient fields of U and U^*. Let U be the potential function on M, as in Section 2.2. By $O(3)$-invariance, U may also be regarded as a function on $\bar{M}$, and then its restriction to M^* is given by the restriction of U to the unit sphere M_1, namely

$$U^* = U^*(\mathbf{X}^*) = U(\mathbf{X}_1) = \rho^e U(\bar{\mathbf{X}}), \tag{3.19}$$

where $\mathbf{X}^* = (1/\rho)\bar{\mathbf{X}} = \pi(\mathbf{X}_1)$ and $\mathbf{X}_1 = (1/\rho)\mathbf{X}$ lies on the unit sphere M_1.

The gradient vector field of U in M is homogeneous of degree $-(e+1)$, namely

$$\nabla U(\mathbf{X}) = \frac{1}{\rho^{e+1}} \nabla U(\mathbf{X}_1) = \frac{1}{\rho^{e+1}} \mathbf{W}, \tag{3.20}$$

where $\mathbf{W} = \mathbf{W}(\mathbf{X}_1)$ is the restriction of ∇U to M_1 and

$$\mathbf{W} = \mathbf{W}^\rho + \mathbf{W}^\perp = \mathbf{W}^\rho + \mathbf{W}^\sigma + \mathbf{W}^\omega \tag{3.21}$$

is the orthogonal decomposition according to (2.36). However, the $O(3)$-invariance of U implies that $\mathbf{W}^\omega$ vanishes. Moreover, $\mathbf{W}^\rho = k\mathbf{X}_1 (= k(\partial/\partial\rho))$ for some k, and using the formula

$$\nabla U \cdot \mathbf{X} = -eU, \quad \text{(cf. (2.60))} \tag{3.22}$$

we find that $k = -eU^*$.

The bundle map $d\pi : TM \to T\bar{M}$ of (3.2) identifies $\nabla U(\mathbf{X})$ with $\nabla U(\bar{\mathbf{X}})$, and furthermore, it identifies $\mathbf{W}^\sigma$ at $\mathbf{X}_1$ with the gradient of U^* at $\mathbf{X}^*$. Consequently, we may write

$$\mathbf{W} = -(eU^*)\frac{\partial}{\partial\rho} + \nabla U^* \tag{3.23}$$

and the gradient of U in $\bar{M}$ is given by

$$\nabla U(\bar{\mathbf{X}}) = -\frac{eU^*(X^*)}{\rho^{e+1}}\frac{\partial}{\partial\rho} + \frac{1}{\rho^{e+1}}\nabla U^*(\mathbf{X}^*). \tag{3.24}$$

Remark 3.2. Note that $\nabla U + e\rho U(\partial/\partial\rho)$ is a global vector field on M, away from origin, whose restriction to the sphere M_1 is tangential to M_1 and may be identified with ∇U^*. In fact, the restriction is just $\mathbf{W}^\sigma$.

3.5. Motions in the moduli space $\bar{M}$. Consider a smooth curve $\bar{\Gamma}$ in $\bar{M}$, away from the collision subvariety, and let Γ^* be its image in M^*. Then Γ^* is smooth whenever $\bar{\Gamma}$ is transversal to the vector field $\partial/\partial\rho$; we assume this is the case, with the possible exception of some isolated points. Let θ be the arc-length parameter along Γ^* measured from a chosen starting point.

By a construction due to Hsiang [4], define the *cone surface* $C(\Gamma^*)$ to be the surface sweeped out by the rays passing through a point moving along Γ^*. It is a flat surface immersed in $\bar{M}$, with induced Riemannian metric

$$d\bar{s}^2|_{C(\Gamma^*)} = d\rho^2 + \rho^2 d\theta^2 \tag{3.25}$$

which describes (locally, for θ ranging over an interval of length $< 2\pi$) a Euclidean sector with (ρ,θ) as polar coordinates centered at the cone vertex $\bar{\mathbf{0}}$. The surface contains the curves $\bar{\Gamma}$ and Γ^*, or more precisely, a "stretched out" version of them, and here Γ^* is the circular (or equidistant) curve of distance $\rho = 1$ from $\bar{\mathbf{0}}$.

Let the surface $C(\Gamma^*)$ be positively oriented by the orthonormal frame $\{\partial/\partial\rho, (1/\rho)\partial/\partial\theta\}$. Define the parameter α (along $\bar{\Gamma}$) to be the angle between the ray direction $\partial/\partial\rho$ and the tangent vector direction. More precisely, for a given orientation of $\bar{\Gamma}$, the unit tangent vector $\mathbf{t}$ and the angle α are related as follows:

$$\mathbf{t} = \frac{d\bar{\Gamma}}{d\bar{s}} = \cos\alpha\frac{\partial}{\partial\rho} + \sin\alpha\frac{1}{\rho}\frac{\partial}{\partial\theta}, \quad \cos\alpha = \frac{d\rho}{d\bar{s}}, \ \sin\alpha = \rho\frac{d\theta}{d\bar{s}}. \tag{3.26}$$

Consider a smooth n-configuration motion $\mathbf{X}(t)$ and the induced motions $\bar{\mathbf{X}}(t)$ and $\mathbf{X}^*(t)$ in $\bar{M}$ and M^* along the curves $\bar{\Gamma}$ and Γ^* (as above) with arc-length functions $\bar{s}$ and θ, respectively. The kinetic energy of $\bar{\mathbf{X}}(t)$ is, by definition,

$$\bar{T} = T^\rho + T^\sigma, \quad \text{where } T^\rho = \frac{1}{2}\dot{\rho}^2, \ T^\sigma = \frac{1}{2}\rho^2\dot{\theta}^2. \tag{3.27}$$

Recall that T^σ is the ray-transversal component explaining the change of shape; in particular, $\dot{\theta}$ is the speed of the shape curve $\mathbf{X}^*(t)$. The dependence of $\dot{\theta}$ on $\bar{T}$ can also be expressed by

$$\dot{\theta}^2 = \frac{2\bar{T}}{\rho^2 + (d\rho/d\theta)^2}, \quad \text{where } \dot{\rho} = \dot{\theta}\frac{d\rho}{d\theta} \tag{3.28}$$

with special attention to the event $\dot{\theta} = 0$ (or $\lim\dot{\theta} = 0$), which is conceivable in both cases $\bar{T} = 0$ and $\bar{T} \neq 0$.

On the other hand, on a time interval where $\bar{T}$ does not vanish, the angle α between the curve and the ray direction is well defined and we have by (3.25) and (3.26)

$$\sin^2\alpha = \frac{T^\sigma}{\bar{T}} = \frac{B}{A+B} \quad \text{(cf. (2.75))}. \tag{3.29}$$

The event $\sin\alpha = 0$ means that $\bar{\mathbf{X}}(t)$ is tangent to a ray, provided $\bar{T} \neq 0$, and α may possibly be defined by a limit procedure whenever $\bar{T}$ vanishes. In any case, assuming $\dot{\theta}$ and α are defined (or have limits)

$$\sin\alpha = 0 \iff \dot{\theta} = 0 \iff \frac{d\rho}{d\theta} = \pm\infty. \tag{3.30}$$

3.5.1. Dynamics in $\bar{M}$. Consider a motion $\bar{\mathbf{X}}(t)$ in $\bar{M}$ induced from a solution $\mathbf{X}(t)$ of the equation of motion $\ddot{\mathbf{X}} = \nabla U(\mathbf{X})$ in M (cf. (2.51)) with potential function U (as in Section 2.2, with $0 < e < 2$). We will also assume $\Omega = \mathbf{0}$, consequently $T = \bar{T}$ and the Lagrange-Jacobi equation (2.63) reads

$$T^{\sigma} = \frac{1}{2-e}\left[\ddot{\rho}\rho + \frac{e}{2}\dot{\rho}^2 - eh\right]. \tag{3.31}$$

It follows that the kinetic energy T^{σ} due to change of shape is, in fact, a second-order differential consequence of the radial motion, for a fixed total energy level h. As functions of $t > 0$, ρ and T are differentiable (in fact, analytic), and so is the nonnegative function $T^{\sigma} = T - (1/2)\dot{\rho}^2$.

Of particular interest is the case of a general collision, say, at $t = 0^+$. Then, we consider such a motion for small $t \in (0, t_1]$; in particular, $\rho \sim \kappa t^{\nu}$ and $\dot{\rho} \sim \kappa\nu t^{\nu-1}$ as $t \to 0^+$, by Theorem 2.7. Moreover, we know (by Theorem 2.7) there is a limit, $U^*(t) = U^*(\mathbf{X}^*(t)) \to \mu$, although it is unclear whether the shape curve $\mathbf{X}^*(t)$ itself must (necessarily) converge. This is certainly true if the level set $U^* = \mu$ is known to be discrete. But this open problem is subsumed by the following and more general challenge.

Problem 3.3. If T^{σ} is not identically zero, what can be said about its asymptotic behavior as $\rho \to 0$? For example, can we have $T^{\sigma} \sim \tau t^r$ for some constants τ and r? Must $T^{\sigma} > 0$ hold for t sufficiently small?

Here is some preliminary information relevant for the above problem. The case where T^{σ} vanishes identically is just the shape invariant case (see Section 4), namely the motion in $\bar{M}$ is confined to a single ray; in particular $U^* = \mu$ is constant. In general, however, we note that $T^{\sigma} = 0$ implies $\dot{T}^{\sigma} = 0$, and $T^{\sigma} = 0$ for arbitrarily small t would imply, for each $k > 0$, the existence of a decreasing sequence $t_i \to 0$ such that $(d^k/dt^k)T^{\sigma} = 0$ at $t = t_i$. At any time t where T^{σ} vanishes, (3.31) reduces to the identity

$$\ddot{\rho} + \frac{eU^*}{\rho^{e+1}} = 0 \tag{3.32}$$

which for $t \to 0$ "approaches" the ODE

$$\ddot{\rho} + \frac{e\mu}{\rho^{e+1}} = 0 \tag{3.33}$$

of the shape invariant case. In the latter case a series expansion of $\rho(t)$ is easily derived (see Section 4.3). We make some additional observations:

- The shape curve $\mathbf{X}^*(t)$ converges to a point $p \in M^*$ if it is rectifiable (as $t \to 0$).

In fact, the curve must have limit points in the compact space M^*, but two distinct limit points would certainly lead to an unbounded arc-length function

$$\begin{aligned} \theta(t) &= \int_t^{t_1} ||\dot{\mathbf{X}}^*(t)||dt \\ &= -\int_t^{t_1} \dot{\theta}(t)dt, \quad 0 < t < t_1, \end{aligned} \tag{3.34}$$

where length is measured from some initial time $t_1 > 0$ and

$$\dot\theta = -\frac{\sqrt{2T^\sigma}}{\rho} \le 0 \quad (\text{by } (3.27)),$$
$$\dot\theta \sim \frac{o(1)}{t} \quad \text{as } t \longrightarrow 0_+ \quad (\text{by } (3.31)). \tag{3.35}$$

- There is the following convergent integral:

$$\int_{0+}^{t_1} t\cdot\dot\theta(t)^2\,dt = \text{const}\int_{0+}^{t_1}\frac{1}{t}\rho^e T^\sigma\,dt < \infty \quad (\text{cf. } (2.95)). \tag{3.36}$$

However, this does not imply convergence of the integral in (3.34); a case like $\dot\theta(t) \sim 1/t\ln t$ provides a counterexample.

- If $T^\sigma \sim ct^r$ as $t \to 0_+$, for some exponent r, then $\dot\theta(t) \sim c't^s$, with $s = r/2-\nu > -1$, and consequently the shape curve has finite length and hence converges.

Proposition 3.4. *Consider an n-body motion $\mathbf{X}(t)$ leading to a general collision at $t = 0+$. Then the following hold:*

(i) $\alpha \sim (2+e)/2t\dot\theta = o(1)$ *as* $t \to 0$.

(ii) $d\rho/d\bar{s} \to 1$ *as* $t \to 0$. *In particular, the moduli curve $\bar\Gamma$ in $\bar M$ has finite length $\bar L(t)$ measured from the vertex $\bar 0$, and moreover, $\bar L(t) \sim \rho(t)$ as $t \to 0$.*

(iii) *If there is a limiting shape*

$$p = \lim \mathbf{X}^*(t) \in M^* \quad \textit{as } t \longrightarrow 0_+, \tag{3.37}$$

then $\bar\Gamma$ is tangent to the ray through p at the cone vertex.

For the proof of the above statements, we first combine Lemmas 2.9, 2.11, and (3.29) to obtain $\sin\alpha \to 0$ as $t \to 0$. Moreover, by (3.26),

$$\frac{d\rho}{d\theta} = \frac{\dot\rho}{\dot\theta} = \rho\cot\alpha \sim \frac{\rho}{\alpha}, \qquad \text{or} \qquad \alpha \sim \frac{\rho}{\dot\rho}\dot\theta \sim \frac{1}{\nu}t\dot\theta = \frac{2+e}{2}t\dot\theta \tag{3.38}$$

and this proves (i). (ii) is an immediate consequence of (i) and (3.26), and (iii) is due to the fact $\alpha \to 0$.

4. The shape invariant motions. If the induced shape curve $\mathbf{X}^*(t)$ in M^* is a single point p, or equivalently the moduli curve $\bar{\mathbf{X}}(t)$ stays on a fixed ray in $\bar M$, the n-configuration motion $\mathbf{X}(t)$ is called shape invariant. In terms of kinetic energy the condition is that the term T^σ vanishes. Then we will write explicitly

$$\mathbf{X}(t) = (\mathbf{a}_1(t), \mathbf{a}_2(t), \dots, \mathbf{a}_n(t)) \in M,$$
$$\mathbf{a}_i(t) = \rho(t)A(t)\mathbf{u}_i, \tag{4.1}$$

where $\rho(t) > 0$, $A(t) \in \mathrm{SO}(3)$, and the constant vectors $\mathbf{u}_i \in \mathbb{R}^3$ are distinct. We may also normalize so that $\sum m_i\|\mathbf{u}_i\|^2 = 1$, and moreover, $A(t_0) = Id$ for some initial time t_0.

Our aim is to describe all shape invariant solutions of the equation of motion $\ddot{\mathbf{X}} = \nabla U(\mathbf{X})$, where U is the given potential function, as in Section 2.2. For a fixed shape

$p \in M^*$ we will write

$$\mu = U^*(p) = U(\mathbf{u}_1, \dots, \mathbf{u}_n) \tag{4.2}$$

and then the total energy integral reads

$$h = \frac{1}{2}\dot{\rho}^2 + T^\omega - \frac{\mu}{\rho^e}. \tag{4.3}$$

The procedure leading to all shape invariant solutions consists of two steps:

(i) Determination of the *central* shapes $p \in M^*$, namely the critical points of the function U^*. We remark that n-configurations $\mathbf{X} \in M$ having central shape are often referred to as *central configurations* in the literature (see Definition 4.3 in Section 4.2). Planar solutions are also referred to as *relative equilibria* (cf. Smale [20]).

(ii) Integration of a central force problem, involving the inverse $(1+e)$-force law. We refer to this as the *Kepler problem*, see Section 4.3.

Remark 4.1. Clearly, all motions are necessarily shape invariant when $n = 2$; in fact, M^* is a single point. On the other hand, step (ii) is independent of $n \geq 2$. Namely, for a given central configuration, the integration problem is just the "2-body problem."

The two steps are discussed separately in separate subsections, combined to the final description in Section 4.4. But firstly, in Section 4.1 we will focus attention on the special case of vanishing angular momentum.

4.1. Homothetic motions. The subclass of shape invariant motions with constant $A(t)$ in (4.1), that is,

$$\mathbf{X}(t) = \rho(t)(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n), \tag{4.4}$$

will be referred to as *homothetic motions*. There are various equivalent characterizations, in purely kinematic terms.

Proposition 4.2. *Homothetic motions* $\mathbf{X}(t)$ *are characterized by the following equivalent conditions:*

(i) $\dot{\mathbf{X}}(t)$ *is proportional to* $\mathbf{X}(t)$*;*

(ii) *the motion is shape invariant with vanishing angular momentum* ($\Omega = \mathbf{0}$)*;*

(iii) *the motion is shape invariant with vanishing individual angular momenta* $(\Omega_i = 0)$.

The proof is easy. For example, we see why condition (ii) leads to an expression like (4.4). For the motion $\mathbf{X}(t)$ in (4.1)

$$\begin{aligned} \Omega &= \sum m_i \mathbf{a}_i \times \dot{\mathbf{a}}_i = \rho^2 A\Big(\sum m_i \mathbf{u}_i \times S(\mathbf{u}_i)\Big) \\ &= \rho^2 A\Big(\sum m_i \mathbf{u}_i \times (\mathbf{s} \times \mathbf{u}_i)\Big) = \rho^2 A\Big(\mathbf{s} - \sum m_i (\mathbf{u}_i \cdot \mathbf{s})\mathbf{u}_i\Big), \end{aligned} \tag{4.5}$$

where $\mathbf{s} = \mathbf{s}(t)$ is the vector in $\mathbb{R}^3$ representing the skew-symmetric matrix $S = A^{-1}\dot{A}$ in the sense that $S\mathbf{v} = \mathbf{s} \times \mathbf{v}$ for $\mathbf{v} \in \mathbb{R}^3$. Hence, Ω vanishes if and only if

$$\mathbf{s} = \sum m_i (\mathbf{u}_i \cdot \mathbf{s})\mathbf{u}_i. \tag{4.6}$$

Unless the vectors $\mathbf{u}_i$ are collinear, this identity can only hold for $\mathbf{s} = \mathbf{0}$, and hence A is constant. On the other hand, if all $\mathbf{u}_i$ are collinear, then the identity implies $S\mathbf{u}_i = \mathbf{0}$ and again $A(t)\mathbf{u}_i$ is independent of t.

4.2. The central configurations. Recall from Section 3.4, the gradient field of U on M can be written as

$$\nabla U(\mathbf{X}) = \frac{1}{\rho^{e+1}}\mathbf{W} = \frac{1}{\rho^{e+1}}\left[-eU^*(p)\frac{\partial}{\partial \rho} + \nabla U^*(p)\right]. \tag{4.7}$$

Here $p \in M^*$ is the shape of $\mathbf{X}$, $\mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n)$, with

$$\mathbf{w}_i = \frac{1}{m_i}\frac{\partial U}{\partial \mathbf{a}_i}(\mathbf{X}_1), \tag{4.8}$$

is the value of ∇U at the point $\mathbf{X}_1 = \rho^{-1}\mathbf{X}$, and ∇U^* is the component of $\mathbf{W}$ tangential to the unit sphere M_1. Moreover, as the notation indicates, the latter component has been identified with the gradient of U^* in M^* at the point $p = \mathbf{X}^*$. Thus, the condition of vanishing $\nabla U^*(p)$ in (4.7) is equivalent to the following definition.

Definition 4.3. For a given potential U, $\mathbf{X} = (a_1, \dots, \mathbf{a}_n)$ is a central n-configuration (and $\mathbf{X}^*$ is a *central shape*) if $\nabla U(\mathbf{X}) = \lambda \mathbf{X}$ for some constant λ, that is,

$$\lambda \mathbf{a}_i = \frac{1}{m_i}\frac{\partial U}{\partial \mathbf{a}_i}(\mathbf{X}), \quad \forall i. \tag{4.9}$$

Note that λ is determined by (4.7), namely $\lambda = -e\rho^{-2}U(\mathbf{X})$. In particular, since U is homogeneous it suffices to determine configurations of size $\rho = 1$, namely $\mathbf{X} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ is a unit vector and hence by (4.2) and (4.7), condition (4.9) is equivalent to

$$-(e\mu)\mathbf{u}_i = \frac{1}{m_i}\frac{\partial U}{\partial \mathbf{a}_i}(\mathbf{u}_1, \dots, \mathbf{u}_n), \quad \forall i. \tag{4.10}$$

In the special case of the Newtonian potential function U, (4.10) reads

$$-\mu \mathbf{u}_i = \sum_{j \neq i} \frac{m_j(\mathbf{u}_j - \mathbf{u}_i)}{||\mathbf{u}_j - \mathbf{u}_i||^3}, \quad \forall i, \tag{4.11}$$

where

$$\mu = U^*(p) = \sum_{i<j} \frac{m_i m_i}{||\mathbf{u}_i - \mathbf{u}_j||}. \tag{4.12}$$

Thus the determination of central configurations appears as a nontrivial problem in vector algebra. For a given n and mass distribution, the cardinality of the set of critical points of U* is the *number of central shapes*. We state the following basic and still unsolved problem for the Newtonian potential.

Problem 4.4 (see Wintner [24]). Is the number of central shapes finite for all mass distributions?

Remark 4.5. The less precise and more common saying “the number of central configurations” actually means “the number of central shapes.” Anyhow, one counts the number of solutions $\mathbf{X}$ up to scaling and congruence (modulo $O(3)$, or rather

SO(3) which doubles the number of non-coplanar solutions). By definition of mass distribution, the masses are positive numbers. Indeed, there are examples showing that the number of solutions can be infinite if negative masses are allowed.

4.2.1. Lagrange's multiplier method. In his study of 3-body motions along a fixed line, Euler [3] discovered the collinear central configurations and found the three different shapes by solving a specific algebraic equation of degree 5 (cf. also Siegel [18]). Five years later, in 1772, Lagrange found the remaining central shape of three bodies, namely the shape of the regular triangle. In doing so Lagrange used his socalled multiplier method, well known in elementary calculus today. His proof is very simple and works equally well for $n = 3$ and 4 (and more generally for n-configurations in $(n-1)$-space).

Crucial to the proof is his use of the "correct" coordinates for the purpose, namely the mutual distances $r_{ij} = \|\mathbf{a}_i - \mathbf{a}_j\|$, $i < j$. These constitute a complete set of invariants for n-configurations $\mathbf{X} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ in $\mathbb{R}^{n-1}$ up to congruence, that is, modulo $O(n-1)$. For the regular part of the congruence moduli space

$$\bar{M} = \mathbb{R}^{(n-1)^2}/O(n-1), \tag{4.13}$$

namely those points such that $\mathbf{a}_1, \dots, \mathbf{a}_n$ span $\mathbb{R}^{n-1}$, the functions r_{ij} can be used as coordinates since they are functionally independent and restricted only by inequalities. Clearly, the Newtonian potential U achieves an optimally simple form

$$U = \sum_{i<j} \frac{m_i m_j}{r_{ij}} \tag{4.14}$$

and moreover, Lagrange's quadric formula for the moment of inertia

$$I = \sum_{i<j} m_i m_j r_{ij}^2 \quad \Big(\text{here we assume} \sum m_i = 1\Big) \tag{4.15}$$

makes the constraint $I = 1$ which restricts $\bar{\mathbf{X}}$ to the shape space M^* almost trivial. The resulting equation

$$\nabla U = \lambda \cdot \nabla I \tag{4.16}$$

yields immediately $2\lambda = -r_{ij}^{-3}$, and consequently all r_{ij} are equal (for any mass distribution). In particular, for $n = 4$ the only central shape which is not in a plane is the regular tetrahedron.

Indeed, the above standard definition of central configurations may itself be viewed as an application of Lagrange's method, since condition (4.9) is precisely (4.16) when we replace the multiplier λ by $(1/2)\lambda$. However, whereas (4.9) takes place in the high dimensional space M, Lagrange utilized the fact that U and I are actually functions on the quotient space $\bar{M}$, and here they have simple algebraic expressions.

4.2.2. Old and recent results on central configurations (for the gravitational potential). Around 1900, Moulton [9] generalized Euler's result on collinear central configurations for all n by showing that the total number is always $(1/2)n!$. Simpler and more geometric proofs have appeared since then. For example, in 1970 Smale

[20] proved this result in terms of elementary Morse theory, and during the following years Smale and his student Palmore, among others, continued the study of coplanar central configurations along the same lines using topological methods such as Morse theory. However, so far (anno 1998) Problem 4.4 is still unsettled for each $n > 3$.

The problem with the Morse theoretic approach is that critical points of U^* may be degenerate (cf. Palmore [10]), so the critical set may possibly be infinite. In fact, it is either finite or contains a continuum (cf. Palmore [11]). There are examples showing that for $n > 3$ the cardinality depends on the mass distribution. The cardinality is, in fact, known (by Palmore, Moeckel) to be finite for almost all masses, but the problem still remains to tell more constructively, for a given mass distribution, what is the cardinality.

Finally, we describe what is known about the case $n = 4$. First of all, there are $(1/2)4! = 12$ noncongruent collinear configurations and one nondegenerate tetrahedron, namely the regular one. However, the planar solutions are not known for general mass distributions, but we mention that Albouy [1] has recently completed the special case of four equal masses. Indeed, there are only three (proper) planar shapes modulo a permutation of masses; namely, the square, the regular triangle with the fourth mass at the center, and a special isosceles triangle with the fourth mass at the axis of symmetry. To describe the latter solution, consider an isosceles triangle with base edge of length 2 and height ≈ 1.82. The fourth mass point (resp., center of mass) has approx. height 0.65 (resp., 0.62) above the base. In summary, in the equal mass case there are $(12+19+1) = 32$ solutions in $M^* = SM/O(3)$, or 33 solutions modulo SO(3) since the regular tetrahedron configuration amounts to two classes modulo SO(3).

4.3. The Kepler problem. By definition, the *Kepler problem* (for fixed $e > 0, \mu > 0$) is given by the following (integrable) differential equation:

$$\ddot{\mathbf{x}} + \frac{e\mu}{\|\mathbf{x}\|^{2+e}}\mathbf{x} = \mathbf{0} \tag{4.17}$$

in 3-space. Clearly, the angular momentum vector $\boldsymbol{\omega} = \mathbf{x} \times \dot{\mathbf{x}}$ is an integral of the motion, and either

(i) $\boldsymbol{\omega} = 0$ and the motion is rectilinear (i.e., along a fixed line), or

(ii) the motion takes place in the plane perpendicular to $\boldsymbol{\omega} \neq 0$.

In the second case, write $\boldsymbol{\omega} = \omega\mathbf{k}$ and let (ρ, θ) be polar coordinates in the plane of motion. Then (4.17) translates to the following ODE

$$\rho^{1+e}\ddot{\rho} - \rho^{2+e}\dot{\theta}^2 + e\mu = 0, \qquad 2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} = 0, \tag{4.18}$$

where the second equation expresses the invariance of $\omega = \rho^2\dot{\theta}$.

By introducing the integration constant ω we eliminate $\dot{\theta}$ in the first equation of (4.18) and obtain

$$\ddot{\rho} + \frac{e\mu}{\rho^{1+e}} - \frac{\omega^2}{\rho^3} = 0 \tag{4.19}$$

which has the energy expression

$$h = \frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\frac{\omega^2}{\rho^2} - \frac{\mu}{\rho^e} \tag{4.20}$$

as a first integral. This implies solvability by quadrature of the above Kepler problem. The motion is rectilinear (or 1-dimensional) if and only if $\omega = 0$.

Recall the classical case $(e = 1)$ where the solutions with $\omega \neq 0$ are conic sections (ellipses, parabolas, or hyperbolas)

$$\rho = \rho(\theta) = \frac{\omega^2\mu^{-1}}{1+\epsilon\cos\theta}, \qquad h = \frac{\mu^2}{2\omega^2}(\epsilon^2 - 1), \tag{4.21}$$

with eccentricity ϵ. Thus, explicit and elementary expressions for the solution curves exist if we eliminate time t and regard ρ as a function of θ, but for this purpose the above approach is not the simplest one. However, starting from (4.20) and using the identity

$$\dot{\rho} = \frac{d\rho}{d\theta}\dot{\theta}, \tag{4.22}$$

it is easy to verify that the above expression for $\rho(\theta)$ is, indeed, a solution.

4.3.1. The 1-dimensional Kepler problem revisited. Here the Kepler problem is the ODE (4.19) with $\omega = 0$, or its integrated form (4.20). Namely, one can start from any of the three equivalent equations:

$$\rho^{1+e}\ddot{\rho} + e\mu = 0, \tag{4.23a}$$

$$\rho\ddot{\rho} + \frac{e}{2}\dot{\rho}^2 - eh = 0, \tag{4.23b}$$

$$\rho^e\dot{\rho}^2 - 2h\rho^e - 2\mu = 0, \tag{4.23c}$$

where $\rho = \rho(t)$ may be regarded as the coordinate of a real line. On each side of the “collapse” singularity $\rho = 0$, $\rho(t)$ is found by quadrature, implicitly defined by an equation $H(\rho) = t$ which involves the integration constants h and $\rho_0 = \rho(0)$. Then one is left with the inversion procedure, which is rather an algebraic problem. The latter can be solved, for example, by assuming a series expansion of $\rho(t)$ and define its coefficients recursively by substituting the series into a series expansion of $H(\rho)$. However, the coefficients are derived more efficiently from the ODE itself.

As an illustration, we determine the solution $\rho(t) \geq 0$ for $t \geq 0$, with the singular initial condition $\rho_0 = 0$. To simplify notation, let $\xi = \mu^{-1/(2+e)}\rho$ and let

$$\ddot{\xi} + \frac{e}{\xi^{1+e}} = 0 \tag{4.24}$$

be the corresponding “normalized” version of (4.23a), whose energy integral is

$$\hbar = \mu^{-\upsilon}h = \frac{1}{2}\dot{\xi}^2 - \xi^{-e}, \quad \text{where } \upsilon = \frac{2}{2+e}, \quad \text{(cf. Section 2.2)}. \tag{4.25}$$

It follows that $H_\hbar(\xi) = t$, where

$$H_\hbar(\xi) = \frac{1}{\sqrt{2}}\int_0^\xi \frac{x^{e/2}}{\sqrt{1+\hbar x^e}}\,dx \tag{4.26}$$

and we seek the solution as a series $\xi(t) = t^\upsilon(c_0 + c_1t^\upsilon + c_2t^{2\upsilon} + c_3t^{3\upsilon} + \cdots)$.

The function in (4.26) can be expanded as

$$H_\hbar(\xi)=\frac{\xi^{1+e/2}}{\sqrt{2}}\left[\frac{1}{(1+(1/2)e)}-\frac{\hbar}{2(1+(3/2)e)}\xi^{e}+\frac{3\hbar^2}{8(1+(5/2)e)}\xi^{2e}\right.$$
$$\left.-\frac{5\hbar^3}{16(1+(7/2)e)}\xi^{3e}+\cdots\right]. \tag{4.27}$$

In the classical case $e=1$, $\upsilon=2/3$, the explicit expression for the function is

$$H_\hbar(\xi)=\begin{cases}\dfrac{\sqrt{2}}{3}\xi^{3/2}, & \hbar=0,\\ \dfrac{1}{\sqrt{2}}\hbar^{-3/2}\left[\sqrt{\hbar\xi}\sqrt{1+\hbar\xi}-\ln\left(\sqrt{\hbar\xi}+\sqrt{1+\hbar\xi}\right)\right], & \hbar>0,\\ \dfrac{1}{\sqrt{2}}(|\hbar|^{-3/2})\left[\arcsin\sqrt{-\hbar\xi}-\sqrt{-\hbar\xi}\sqrt{1+\hbar\xi}\right], & \hbar<0.\end{cases} \tag{4.28}$$

Note that, in addition to being invertible as a function of ξ, $H_\hbar$ is also continuous with respect to $\hbar$, namely $\lim_{\hbar\to 0}H_\hbar(\xi)=H_0(\xi)$. For $\hbar<0$, it follows from (4.28) that ξ reaches its maximal value $\xi_1=(-\hbar)^{-1}$ at time $t_1=\pi(-2\hbar)^{-3/2}$.

We obtain the series solution

$$\rho(t)=\sqrt[3]{\mu}\,\xi(t)=t^{2/3}\left(k_0+k_1t^{2/3}+k_2t^{4/3}+k_3t^2+\cdots\right), \tag{4.29}$$

where

$$k_0=\kappa=\left(\frac{9}{2}\mu\right)^{1/3},\qquad k_2=-\frac{3}{7}\frac{k_1^2}{k_0},\qquad k_3=\frac{23}{63}\frac{k_1^3}{k_0^2},\ldots. \tag{4.30}$$

Here k_1 is arbitrary and is a factor of each k_i for $i>1$. But this series can also be derived directly, and perhaps more easily, from (4.23a). From (4.23b) or (4.23c) we deduce that the energy constant h is related to k_1 by $k_1=(9/10)(h/k_0)$. The value of k_0 is, of course, in agreement with the general asymptotic formula for ρ in Section 2.2.

Finally, note that replacement of the series $(k_0+k_1t^{2/3}+k_2t^{4/3}+\cdots)$ in (4.29) by a power series $(k_0+k_1t+k_2t^2+\cdots)$ leads only to the solution with $k_i=0$ for all $i\geq 1$, namely the solution $\rho(t)=\kappa t^{2/3}$ corresponding to $h=0$.

4.4. Description of the shape invariant solutions. Let $\mathbf{X}(t)$ denote a solution of the differential equation $\ddot{\mathbf{X}}=\nabla U(\mathbf{X})$. Then, by the identity (4.7), the following two conditions on the motion (for a given time interval) are equivalent:

(i) $\nabla U(\mathbf{X}(t))$ is a multiple of $\mathbf{X}(t)$;

(ii) the motion has central shape, that is, each $\mathbf{X}^*(t)\in M^*$ is a critical point of the function U^*.

Remark 4.6. We do not claim that central shape implies shape invariance (i.e., constant shape), as would be the case, for example, if the critical set of U^* is finite. However, a constant shape must necessarily be central, according to the following lemma.

Lemma 4.7. *If the motion is shape invariant, then its shape must be central.*

Proof. Consider the orthogonal decomposition of the acceleration vector

$$\ddot{\mathbf{X}}=\nabla U=k\mathbf{X}+\ddot{\mathbf{X}}^{\sigma} \tag{4.31}$$

of $\mathbf{X}(t)$ (cf. (3.20), (3.21)). In general, the shape curve $\mathbf{X}^*(t)$ has a "horizontal" lifting $\mathbf{H}(t)$ in M_1, namely $\pi : M_1 \to M^*$ takes $\mathbf{H}(t)$ to $\mathbf{X}^*(t)$ and $d\pi$ maps the velocity $\dot{\mathbf{H}} = \dot{\mathbf{H}}^\sigma$ isometrically to $(d/dt)\mathbf{X}^*(t)$.

Now, if $\ddot{\mathbf{X}}^\sigma \neq \mathbf{0}$ at $t = t_0$, then also $\ddot{\mathbf{H}}^\sigma \neq \mathbf{0}$ at $t = t_0$ and consequently $\dot{\mathbf{H}}$ cannot vanish identically near $t = t_0$. This argument shows that $\ddot{\mathbf{X}}^\sigma$ must vanish identically if $\mathbf{X}(t)$ is a curve of constant shape $\mathbf{X}^*(t) = \mathbf{X}^*(t_0)$, namely ∇U is a multiple of $\mathbf{X}$. □

We turn to the uniform description of all shape invariant solutions, for $n \geq 2$.

Theorem 4.8. *A shape invariant motion is a solution of the differential equation $\ddot{\mathbf{X}} = \nabla U(\mathbf{X})$ if and only if the following two conditions hold (cf. notation in (4.1)):*

(i) *the constant vectors $\mathbf{u}_i$ constitute a central n-configuration, namely a solution of equation system (4.10);*

(ii) *ρ and A satisfy the differential equation*

$$\rho^{1+e} A^{-1} \frac{d^2}{dt^2}(\rho A) = -(e\mu) \cdot Id \tag{4.32}$$

when both sides are regarded as linear operators restricted to the subspace $V \subset \mathbb{R}^3$ spanned by the vectors $\mathbf{u}_i$.

Proof. The motion $\mathbf{X}(t)$ is a solution of the differential equation if and only if

$$\frac{d^2}{dt^2}(\rho A) = \frac{1}{m_i}\frac{\partial U}{\partial \mathbf{a}_i}(\mathbf{X}) = \rho^{-(1+e)} A\mathbf{w}_i, \quad \forall i, \tag{4.33}$$

where

$$\mathbf{w}_i = \frac{1}{m_i}\frac{\partial U}{\partial \mathbf{a}_i}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n). \tag{4.34}$$

By Lemma 4.7, the shape of the motion is central, and hence $\mathbf{w}_i = -(e\mu)\mathbf{u}_i$ by (4.10). Therefore (4.33) can be written as in (4.32). □

Corollary 4.9. *Each solution (ρ, A) of the differential equation (4.32) must satisfy $A(t) \in \mathrm{SO}(2)$ for some fixed subgroup $\mathrm{SO}(2) \subset \mathrm{SO}(3)$. Furthermore, if*

$$A(t) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \theta = \theta(t), \tag{4.35}$$

then (ρ, A) is a solution of (4.32) if and only if (ρ, θ) is a solution of the Kepler problem (4.18).

Proof. The last statement is easily checked once we know that $A(t)$ belongs to the rotation group of a fixed plane.

Suppose to the contrary, that there is a solution where the rotations $A(t)$ do not belong to a single subgroup $\mathrm{SO}(2)$. We consider the "2-body problem" with potential function

$$U = \frac{n_1 n_2}{\|\mathbf{b}_1 - \mathbf{b}_2\|^e} \tag{4.36}$$

and choose the numbers $n_i > 0$ together with two vectors $\mathbf{v}_i$ satisfying the conditions

$$\sum n_i \mathbf{v}_i = 0, \qquad \sum n_i \|\mathbf{v}_i\|^2 = 1, \qquad U(\mathbf{v}_1, \mathbf{v}_2) = \mu. \tag{4.37}$$

Then $\mathbf{Y} = (\mathbf{b}_1, \mathbf{b}_2)$, with $\mathbf{b}_i(t) = \rho(t)A(t)\mathbf{v}_i$, is a non-planar solution of the differential equation $\ddot{\mathbf{Y}} = \nabla U(\mathbf{Y})$. However, this equation is really a Kepler problem (4.17) for $\mathbf{x} = \mathbf{b}_1$, say. This is a contradiction since a solution of a Kepler problem is always planar. $\square$

Corollary 4.10. *A shape invariant solution of the equation $\ddot{\mathbf{X}} = \nabla U(\mathbf{X})$ with angular momentum $\Omega \neq \mathbf{0}$ must be planar.*

Proof. Consider a solution $\mathbf{X}(t)$, as in (4.1). By Corollary 4.9, $A(t)$ belongs to a group $SO(2)$, say, the rotation group of the xy-plane as in Corollary 4.9.

We claim that $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ in (4.1) is an n-configuration in the xy-plane. This is a consequence of Theorem 4.8, according to which the space $V \subset \mathbb{R}^3$ spanned by the vectors $\mathbf{u}_i$ lies in the kernel of the linear operator

$$\rho^{1+e} A^{-1} \frac{d^2}{dt^2}(\rho A) + (e\mu) \cdot Id = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix}, \tag{4.38}$$

where $a = c - \rho^{2+e}\dot{\theta}^2$, $b = \rho^{1+e}(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})$, $c = \rho^{1+e}\ddot{\rho} + e\mu$.

Suppose some $\mathbf{u}_i$ is outside the xy-plane, hence $c = 0$. However, $a^2 + b^2 > 0$ since $\dot{\theta} \neq 0$, and this will force V to be the z-axis and hence also $\Omega = \mathbf{0}$. $\square$

A simple calculation shows that Ω in the above corollary is the normal vector $\omega\mathbf{k}$ where $\omega = \rho^2\dot{\theta}$.

Corollary 4.11. *A homothetic motion $\mathbf{X}(t)$, (4.4), is a solution of the equation $\ddot{\mathbf{X}} = \nabla U(\mathbf{X})$ if and only if $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ is a central configuration and $\rho(t)$ is a solution of the 1-dimensional Kepler problem (4.23).*

Remark 4.12. The first (and exact) solutions of the 3-body problem, dating back to Euler and Lagrange, are the shape invariant motions where the bodies rotate rigidly about the center of mass with constant angular velocity. They calculated the possible shapes for such motions, namely the collinear central configurations and the shape of a regular triangle. From the terminology due to Smale [20], central configurations in a plane are also referred to as relative equilibria since those shape invariant solutions of the n-body problem where the bodies rotate rigidly and uniformly around origin become fixed points in a rotating coordinate system.

For example, let $\mathbf{X} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$ be a fixed central configuration in the plane and consider the solution $\mathbf{X}(t)$ of the classical n-body problem obtained by uniformly rotating the mass points around the origin (= center of mass) with a specific angular speed ω. What must be the value of $\pm\omega$? This problem is easily solved by regarding the vectors as complex numbers and writing $\mathbf{a}_i(t) = e^{i\omega t}\mathbf{a}_i$. Then from the Newtonian equation and the definition of a central configuration

$$\ddot{\mathbf{X}}(t) = \nabla(\mathbf{X}(t)) = \lambda\mathbf{X}(t), \tag{4.39}$$

it follows

$$\omega^2 = -\lambda = \frac{U(\mathbf{X})}{\rho^2} = \frac{\sum_{i<j} m_i m_j \|\mathbf{a}_i - \mathbf{a}_j\|^{-1}}{\sum m_i \|\mathbf{a}_i\|^2}. \tag{4.40}$$

5. The induced equation of motion at moduli space level. We will study the equations of motion in the *congruence* moduli space $\bar{M}$, induced from the equation $\ddot{\mathbf{X}} = \nabla U(\mathbf{X})$ in M. However, we will only consider planar motions or motions with vanishing angular momentum. As will be seen, the resulting differential equations in $\bar{M}$ depend on $h = T - U$ and $\omega = \|\Omega\|$ as parameters with fixed values. For $n = 3$ the equations are worked out explicitly in terms of (global) coordinates of $\bar{M}$ as a cone over the sphere M^*, and the associated Hamiltonian formulation is also given for the sake of completeness.

5.1. Variational principles and the dynamical metric. To achieve the reduced equations on $\bar{M}$ we shall first replace the above Newtonian type equation by any of its equivalent systems derived from an action principle involving either the Lagrange function $L = T + U$ or kinetic energy T. Namely, recall from classical mechanics the two action integrals

$$J_1 = \int L\,dt, \qquad J_2 = \sqrt{2}\int T\,dt, \tag{5.1}$$

corresponding to Hamilton's principle and the least action principle, respectively. In the second case, Jacobi's "geometrization trick" amounts to the reformulation of J_2 as the integral

$$J_2 = \int \sqrt{T}\,da = \int \sqrt{U+h}\,da = \int da_h, \tag{5.2}$$

where the Euclidean metric

$$da^2 = 2T\,dt^2 = \|\dot{\mathbf{X}}\|^2 dt^2 \tag{5.3}$$

is the *kinematic metric* on M, and its conformal modification by the function $U + h$

$$da_h^2 = (U+h)\,da^2 \tag{5.4}$$

may be called the *dynamical metric* on M at energy level h. Thus, by (5.2) the solution curves in M, for a given value of h, are geodesics of the modified metric (5.4).

Now, we turn to the induced equation of motion in the moduli space $\bar{M}$, for planar motions $t \to \mathbf{X}(t)$ or motions with vanishing angular momentum ($\omega = 0$). The action principles in (5.2) can be pushed down to $\bar{M}$, expressed in terms of the reduced versions of kinetic energy, potential energy and Lagrangian function, namely

$$\begin{gathered} \bar{T} = T - \frac{1}{2}\frac{\omega^2}{\rho^2}, \qquad \bar{U} = U - \frac{1}{2}\frac{\omega^2}{\rho^2}, \qquad \bar{L} = \bar{T} + \bar{U}, \\ \bar{J}_1 = \int \bar{L}\,dt, \qquad \bar{J}_2 = \sqrt{2}\int \bar{T}\,dt = \int d\bar{s}_{h,\omega}, \end{gathered} \tag{5.5}$$

where

$$d\bar{s}_{h,\omega}^2 = (\bar{U}+h)\,d\bar{s}^2 \tag{5.6}$$

is the induced *dynamical metric* on $\bar{M}$ at level (h,ω), namely the conformal modification of the kinematic metric $d\bar{s}^2$ by the function $(\bar{U}+h)$.

It follows that the moduli curves $\bar{\mathbf{X}}(t)$ of the solutions $\mathbf{X}(t)$ in M, for given values of h and ω, are the solutions of each of the following two equivalent systems:

(i) the Euler-Lagrange equations of the Lagrangian $\bar{L} = \bar{T} + \bar{U}$, regarded as a function on the tangent bundle of $\bar{M}$;

(ii) the geodesic equations of the dynamical metric $d\bar{s}^2_{h,\omega}$ on $\bar{M}$ (cf. (5.6)).

Note that the solutions in case (i) are curves parameterized by time t, whereas in case (ii) the connection between t and the arc-length parameter $\bar{s}_{h,\omega}$ is given by the relation

$$\left(\frac{d\bar{s}_{h,\omega}}{dt}\right)^2 = \bar{T}\left(\frac{d\bar{s}}{dt}\right)^2 = 2\bar{T}^2. \tag{5.7}$$

5.2. Geodesic curvature and critical rays in $\bar{M}$. By Theorem 4.8, we already know that the critical rays in $\bar{M}$, that is, rays passing through critical points of U^* on M^*, are the only rays which are also geodesics of the dynamical metric. Another proof of this fact will be demonstrated here, as follows. Being a conformal modification of the kinematic metric $d\bar{s}^2$, the geodesics of the metric in (5.6) are characterized by

$$\bar{K}(\mathbf{n}) = \frac{1}{2}\frac{d}{d\mathbf{n}}\ln(\bar{U}+h), \tag{5.8}$$

where $\bar{K}(\mathbf{n})$ is the geodesic curvature in the normal direction $\mathbf{n}$, with respect to $d\bar{s}^2$. However, all rays are geodesics of $d\bar{s}^2$, so the ray through $p \in M^*$ is a geodesic of the modified metric if and only if

$$\frac{d}{d\mathbf{n}}\ln(\bar{U}+h) = 0 \tag{5.9}$$

holds along the ray, for any normal vector $\mathbf{n}$. This condition is, in fact, independent of the coordinate ρ, so we choose $\rho = 1$ and hence the vectors $\mathbf{n}$ span the tangent space of M^* at p. It follows that (5.9) holds if and only if p is a critical point of U^*.

5.3. The induced ODE in $\bar{M}$ for the case $n = 3$. In the simplest case $n = 3$, global coordinates for $\bar{M}$ are readily available and we shall do explicit calculations to derive the equation of motion in $\bar{M}$, either as the Euler-Lagrange equations for the Lagrangian $\bar{L}$, or equivalently as the geodesic equations of the metric $d\bar{s}^2_{h,\omega}$, expressed in terms of t as the independent variable.

Recall from Section 3, the subspace $(M^*, d\sigma^2) \subset (\bar{M}, d\bar{s}^2)$ is the 2-sphere of radius $1/2$ in the kinematic metric. Here we shall, however, find it convenient to represent M^* by the standard sphere $S^2(1)$ of radius 1, and consequently we must use the modified expression

$$d\bar{s}^2 = d\rho^2 + \frac{\rho^2}{4}ds^2 = d\rho^2 + \frac{\rho^2}{4}(dr^2 + \sin^2 r\, d\varphi^2) \tag{5.10}$$

for the kinematic metric in $\bar{M}$, where $ds^2 = 4d\sigma^2$ is the metric of $S^2(1)$ and (r,φ) are spherical polar coordinates centered at any chosen point p_0 on the sphere, $0 \le r \le \pi$, $0 \le \varphi \le 2\pi$. In particular, $r = 0$ at the point p_0. Note that $\bar{M}$, as a cone over S^2, is homeomorphic to $\mathbb{R}^3$ and away from the cone vertex $\bar{M}$ is, in fact, diffeomorphic to $\mathbb{R}^3 - \{\mathbf{0}\}$.

Theorem 5.1. *For planar 3-body motions of total energy h, the corresponding congruence moduli curves $\bar{\mathbf{X}}(t)$ are the solutions of the following ODE:*

$$0 = \ddot{\rho} + \frac{\dot{\rho}^2}{\rho} - \frac{1}{\rho}\left(\frac{2-e}{\rho^e}U^*(r,\varphi) + 2h\right), \tag{5.11a}$$

$$0 = \ddot{r} + 2\frac{\dot{\rho}}{\rho}\dot{r} - \frac{1}{2}\sin(2r)\dot{\varphi}^2 - \frac{4}{\rho^{2+e}}\frac{\partial U^*}{\partial r}, \tag{5.11b}$$

$$0 = \ddot{\varphi} + 2\frac{\dot{\rho}}{\rho}\dot{\varphi} + 2\cot(r)\dot{r}\dot{\varphi} - \frac{4}{\rho^{2+e}}\frac{1}{\sin^2 r}\frac{\partial U^*}{\partial \varphi}. \tag{5.11c}$$

We make the following remarks concerning the above theorem:

• As usual, the potential function U is homogeneous of degree $-e$, $0 < e < 2$, namely $U = U^*(r,\varphi)/\rho^e$ where U^* is the restriction of U to the 2-sphere $M^* = (\rho = 1)$.

• By combining the energy conservation law $h = \bar{T} - \bar{U}$ with the expression for $2\bar{T}$ given by the kinematic metric, we obtain the 1-order equation

$$\frac{2}{\rho^e}U^* + 2h - \frac{\omega^2}{\rho^2} = \dot{\rho}^2 + \frac{\rho^2}{4}\left(\dot{r}^2 + \sin^2(r)\dot{\varphi}^2\right) \tag{5.12}$$

as a first step of the integration of the system (5.11). Conversely, solving (5.12) for ω^2 leads to an expression which is easily seen to be a first integral of the system (5.11). This procedure actually introduces $\omega^2 = \|\Omega\|^2$ as a nonnegative constant of integration, and in the remaining integration problem we may replace (5.11b) or (5.11c) by (5.12).

• Equation (5.11a) is just the Lagrange-Jacobi equation, see (2.62).

• The system of equations (5.11a), (5.11b), (5.11c), and (5.12) is symmetric with respect to the choice of spherical coordinates (centered at any point) on $M^* = S^2$. For example, the center may lie on the eclipse circle E^* which represents the collinear 3-configurations.

• For $n > 3$, (5.11b) and (5.11c) will be replaced by $3n-7$ "similar" equations of order 2, defined by the dynamical metric of $\bar{M}$ and in a 1-1 correspondence with the chosen coordinates of M^*.

• The above ODE system in $\bar{M}$ is of order 4 when (5.12) is included and time t is eliminated.

We will derive the ODE of the theorem via the Riemannian viewpoint, starting with the calculation of the Christoffel symbols $\{\Gamma_{i,j}^k\}$ in the coordinates $(\rho,r,\varphi) = (x_1,x_2,x_3)$. The following three vector fields:

$$\mathbf{v}_1 = \frac{1}{f}\frac{\partial}{\partial \rho}, \qquad \mathbf{v}_2 = \frac{1}{f}\frac{2}{\rho}\frac{\partial}{\partial r}, \qquad \mathbf{v}_3 = \frac{1}{f}\frac{2}{\rho\sin r}\frac{\partial}{\partial \varphi}, \tag{5.13}$$

where $f^2 = \bar{U} + h$, constitute an orthonormal frame in $(\bar{M}, d\bar{s}_{h,c}^2)$ whenever defined. Define

$$\begin{gathered} g_{1,1} = f^2, \qquad g_{2,2} = \frac{1}{4}f^2\rho^2, \qquad g_{3,3} = \frac{1}{4}f^2\rho^2\sin^2 r, \\ g^{i,i} = g_{i,i}^{-1}, \quad g_{i,j} = g^{i,j} = 0 \quad \text{for } i \neq j, \end{gathered} \tag{5.14}$$

and hence by a standard procedure of Riemannian geometry

$$\Gamma^k_{i,j} = \frac{1}{2}\sum_m \left\{\frac{\partial g_{j,m}}{\partial x_i} + \frac{\partial g_{m,i}}{\partial x_j} - \frac{\partial g_{i,j}}{\partial x_m}\right\} g^{m,k}, \tag{5.15}$$

$$\frac{d^2}{du^2}x_k + \sum_{i,j}\Gamma^k_{i,j}\frac{d}{du}x_i\frac{d}{du}x_j = 0, \quad k = 1,2,3, \tag{5.16}$$

where $u = \bar{s}_{h,\omega}$ is the arc-length parameter. As an example, choose $k = 1$ in (5.16) and obtain the first geodesic equation

$$\begin{aligned} -\rho'' = {} & \frac{1}{2f^2}\left[-\frac{eU^*}{\rho^{1+e}} + \frac{\omega^2}{\rho^3}\right](\rho')^2 - \left[\rho + \frac{\rho^2}{2f^2}\left(-\frac{eU^*}{\rho^{1+e}} + \frac{\omega^2}{\rho^3}\right)\right] \\ & \cdot \frac{1}{4}\left[(r')^2 + \sin^2 r(\varphi')^2\right] + \frac{1}{\rho^e f^2}\left[U^*_r\rho' r' + U^*_\varphi \rho'\varphi'\right] \end{aligned} \tag{5.17}$$

using the notation $x' = (d/du)x$, $U^*_r = \partial U^*/\partial r$ and so forth. By (5.7) and (5.10), we have the identities

$$\begin{aligned} &\dot{u}^2 = 2\bar{T}^2 = 2f^4, \qquad 2f^2 = \dot{\rho}^2 + \frac{\rho^2}{4}\left(\dot{r}^2 + \sin^2 r\dot{\varphi}^2\right), \\ &x' = \frac{\dot{x}}{\dot{u}}, \quad x'' = \frac{\dot{u}\ddot{x} - \ddot{u}\dot{x}}{\dot{u}^3}, \quad \text{where } x = \rho, r \text{ or } \varphi, \end{aligned} \tag{5.18}$$

by means of which the above equation for ρ'' transforms to

$$\begin{aligned} 0 = {} & \ddot{\rho} + \frac{1}{2f^2}\left[-\frac{eU^*}{\rho^{1+e}} + \frac{\omega^2}{\rho^3}\right]\dot{\rho}^2 \\ & - \frac{2f^2 - \dot{\rho}^2}{\rho^2}\left[\rho + \frac{\rho^2}{2f^2}\left(-\frac{eU^*}{\rho^{1+e}} + \frac{\omega^2}{\rho^3}\right)\right] + \left(\frac{\dot{U}^*}{f^2\rho^e} - \frac{(d/dt)\bar{T}}{\bar{T}}\right)\dot{\rho}. \end{aligned} \tag{5.19}$$

Now it is straightforward to verify that the last equation further simplifies to the Lagrange-Jacobi equation, namely (5.11a). Similarly, the cases $k = 2,3$ yield (5.11b) and (5.11c), respectively.

5.3.1. Shape invariant motions as solutions of the reduced ODE. At the level of $\bar{M}$, the shape invariant solutions of the dynamical equation $\ddot{\mathbf{X}} = \nabla U(\mathbf{X})$ are the solutions of the system (5.11) with $\dot{r} = \dot{\varphi} = 0$. Let $p_0 = (r_0, \varphi_0) \in M^*$ be the shape of a solution. By equation (5.11b) and (5.11c) it follows that

$$\frac{\partial U^*}{\partial r}(p_0) = \frac{\partial U^*}{\partial \varphi}(p_0) = 0, \tag{5.20}$$

namely p_0 is a critical point of U^*, and when the constant $\mu = U^*(p_0)$ is substituted into equation (5.11a), $\rho(t)$ will be a solution of the resulting Lagrange-Jacobi equation

$$\ddot{\rho} + \frac{\dot{\rho}^2}{\rho} - \frac{1}{\rho}\left(\frac{2-e}{\rho^e}\mu + 2h\right) = 0. \tag{5.21}$$

Furthermore, one checks that the expression

$$\omega^2 = \rho^2\left(2h + \frac{2\mu}{\rho^e} - \dot{\rho}^2\right) \tag{5.22}$$

is a first integral of (5.21), which in turn simplifies to the Kepler equation (4.19)

$$\ddot{\rho} + \frac{e\mu}{\rho^{1+e}} - \frac{\omega^2}{\rho^3} = 0 \tag{5.23}$$

when $\dot{\rho}^2$ is eliminated in (5.21) by means of ω^2.

5.3.2. Collinear 3-body motions. By definition, a collinear motion has its shape curve $X^*(t)$ confined to the equator circle $E^* \subset M^* = S^2$. The equations of motion for this special case are obtained from the system (5.11), as follows. For example, choose polar coordinates (r,φ) centered at a point $p_0 \in E^*$, say E^* is given by $\varphi = 0$ and r is the arc length along E^*. Then ODE (5.11) is reduced to the system

$$\ddot{\rho} + \frac{\dot{\rho}^2}{\rho} - \frac{1}{\rho}\left(\frac{2-e}{\rho^e}U^* + 2h\right) = 0, \qquad \ddot{r} + 2\frac{\dot{\rho}}{\rho}\dot{r} - \frac{4}{\rho^{2+e}}\frac{\partial U^*}{\partial r} = 0, \tag{5.24}$$

since equation (5.11c) vanishes identically. In fact, $U^*_\varphi(r,0)$ vanishes since it is the normal derivative of U^* along the equator circle E^* and moreover, $U^*(r,\varphi) = U^*(r,-\varphi)$ holds. Finally, the energy conservation equation (5.12) reduces to

$$\frac{2}{\rho^e}U^* + 2h - \frac{\omega^2}{\rho^2} = \dot{\rho}^2 + \frac{\rho^2}{4}\dot{r}^2. \tag{5.25}$$

Alternatively, by choosing the (north) pole as the center of polar coordinates, E^* is given by $r = \pi/2$ and φ is the arc length along E^*. The resulting system of equations is again given by (5.24), with r replaced by φ.

5.3.3. The Hamiltonian formulation. We will give a Hamiltonian version of the ODE of Theorem 5.1, in terms of the variables (ρ,r,φ) and their conjugates (P,R,Φ). Starting from the Lagrangian and Hamiltonian functions on $\bar{M}$

$$\bar{L} = \bar{T} + \bar{U} = \frac{1}{2}\dot{\rho}^2 + \frac{\rho^2}{8}(\dot{r}^2 + \sin^2 r\,\dot{\varphi}^2) + \frac{U^*(r,\varphi)}{\rho^e} - \frac{\omega^2}{\rho^2}, \tag{5.26}$$

$$\bar{H} = \bar{T} - \bar{U} = \frac{1}{2}\dot{\rho}^2 + \frac{\rho^2}{8}(\dot{r}^2 + \sin^2 r\,\dot{\varphi}^2) - \frac{U^*(r,\varphi)}{\rho^e}, \tag{5.27}$$

we define P, R, and Φ by

$$P = \frac{\partial \bar{L}}{\partial \dot{\rho}} = \dot{\rho}, \qquad R = \frac{\partial \bar{L}}{\partial \dot{r}} = \frac{1}{4}\rho^2\dot{r}, \qquad \Phi = \frac{\partial \bar{L}}{\partial \dot{\varphi}} = \frac{1}{4}\rho^2\sin^2 r\,\dot{\varphi}. \tag{5.28}$$

It follows that

$$\bar{H} = \frac{1}{2}P^2 + \frac{2}{\rho^2}R^2 + \frac{2}{\rho^2\sin^2 r}\Phi^2 - \frac{U^*(r,\varphi)}{\rho^e} \tag{5.29}$$

and the equations of motion are

$$\begin{aligned}
&\dot{\rho} = P, \qquad \dot{r} = \frac{4}{\rho^2}R, \qquad \dot{\varphi} = \frac{4}{\rho^2\sin^2 r}\Phi,\\
&\dot{P} = \frac{4}{\rho^3}R^2 + \frac{4}{\rho^3\sin^2 r}\Phi^2 - \frac{e}{\rho^{1+e}}U^*(r,\varphi),\\
&\dot{R} = \frac{4\cot r}{\rho^2\sin^2 r}\Phi^2 + \frac{1}{\rho^e}\frac{\partial U^*}{\partial r}, \qquad \dot{\Phi} = \frac{1}{\rho^e}\frac{\partial U^*}{\partial \varphi}.
\end{aligned} \tag{5.30}$$

5.4. The scaling symmetry. Recall from Section 2.2.1, for any n there is the 1-parameter group of space-time scaling symmetries of the equation of motion $\ddot{\mathbf{X}} = \nabla U(\mathbf{X})$ in M. This group survives as an induced 1-parameter group $\{\bar{g}_s\}$ defined on the space $R \times \bar{M}$, and it is a symmetry group of the induced equation of motion on $\bar{M}$. The action is trivial on the coordinates of M^* and is otherwise given by

$$\bar{g}_s : (t, \rho) \longrightarrow (e^s t, e^{\nu s} \rho), \qquad \nu = \frac{2}{2+e}. \tag{5.31}$$

In particular, if $t \to (\rho(t), r(t), \varphi(t))$ is a solution of the ODE described in Theorem 5.1, then the curve

$$t \longrightarrow (e^{\nu s} \rho(e^{-s} t), r(e^{-s} t), \varphi(e^{-s} t)) \tag{5.32}$$

is also a solution.

The infinitesimal generator Y of the group $\{\bar{g}_s = e^{sY}\}$, prolonged up to order 1, is

$$Y = t\frac{\partial}{\partial t} + \nu\rho\frac{\partial}{\partial \rho} - \frac{\nu}{2} P \frac{\partial}{\partial P} + \frac{\nu}{2} R \frac{\partial}{\partial R} + \frac{\nu}{2} \Phi \frac{\partial}{\partial \Phi}. \tag{5.33}$$

6. Asymptotic behavior at a general collision. In the first subsection we continue the asymptotic analysis of the derivatives of the size function $\rho(t)$, needed to complete the proof of Theorem 2.7. In the second subsection we recall the history and discuss the present state of Problem 2.12 stated at the end of Section 2.

The results in this section should be valid for potential functions U as in the asymptotic analysis in Section 2.2. However, for simplicity and explicit calculations we will choose the "standard model"

$$U = \sum_{i<j} \frac{m_i m_j}{||\mathbf{a}_i - \mathbf{a}_j||^e}, \quad 0 < e < 2, \tag{6.1}$$

where in the proofs below we may as well choose exponent $e = 1$. But the letter e also appears in e^x with the usual meaning.

6.1. Asymptotic estimates and completion of the proof of Theorem 2.7. Let $\mathbf{X}(t) = (\mathbf{a}_1(t), \ldots, \mathbf{a}_n(t))$ be an n-body motion leading to a general collision at $t = 0$. The normalized motion $\mathbf{X}_1(t) = \rho(t)^{-1}\mathbf{X}(t)$ is a curve on the unit sphere M_1. However, since we know $\rho(t) \sim \kappa t^\nu$ as $t \to 0$, it is natural to study the following approximate normalized motion (where the scaling factor κ is omitted for simplicity)

$$\tilde{\mathbf{X}}(t) = t^{-\nu}\mathbf{X}(t) = (\tilde{\mathbf{a}}_1(t), \ldots, \tilde{\mathbf{a}}_n(t)), \quad \text{where } \tilde{\mathbf{a}}_i = \frac{1}{t^\nu}\mathbf{a}_i. \tag{6.2}$$

By expressing the equation of motion and its energy conservation, namely

$$\nabla U = \ddot{\mathbf{X}}, \qquad T - U = h, \tag{6.3}$$

in terms of the motion $\tilde{\mathbf{X}}(t)$, we will extend the results from Section 2.2 as rather simple consequences. To this end, we modify appropriately the approach of Wintner (cf. [24, Sections 363–364]), where related "normalized" functions are used. First we need some new notation.

It is convenient to replace time t by the variable

$$u = -\log t \,(\text{or } t = e^{-u}) \tag{6.4}$$

hence $t \to 0$ means $u \to \infty$. Then there is the following identity for differential operators:

$$t^k \frac{d^k}{dt^k} = (-1)^k \left(n_{k,1} \frac{d}{du} + \cdots + n_{k,k} \frac{d^k}{du^k} \right), \quad n_{k,k} = 1,\ n_{k,i} \in \mathbb{Z}^+. \tag{6.5}$$

In particular, for a function $f(t)$ and its "transform"

$$\bar{f}(u) = f(t) = f(e^{-u}) \tag{6.6}$$

it is easy to verify the following equivalence

$$t^k \frac{d^k f}{dt^k} \approx 0, \quad 1 \le k \le m \Longleftrightarrow \frac{d^k}{du^k} \bar{f} \approx 0, \quad 1 \le k \le m, \tag{6.7}$$

where we have used the notation

$$f_1 \approx f_2 \quad \text{iff } f_1 - f_2 = o(1) \text{ as } t \longrightarrow 0 \text{ (or } u \longrightarrow \infty). \tag{6.8}$$

However, following the standard convention we simply write f instead of $\bar{f}$ when the choice of independent variable is clear, confer (6.6).

For any function f of $\mathbf{X}$, write $\tilde{f}$ for the same function applied to $\tilde{\mathbf{X}}$. Both functions will also be regarded as functions of t, namely

$$\tilde{f}(\tilde{\mathbf{X}}) = f(\tilde{\mathbf{X}}), \qquad f(t) = f(\mathbf{X}(t)), \qquad \tilde{f}(t) = f\left(\frac{1}{t^\nu} \mathbf{X}(t) \right). \tag{6.9}$$

For example, we have $\tilde{\rho}(t) = t^{-\nu} \rho(t)$ and

$$\tilde{U}(\tilde{\mathbf{X}}) = \sum_{i<j} \frac{m_i m_j}{\|\tilde{\mathbf{a}}_i - \tilde{\mathbf{a}}_j\|^e}, \qquad \tilde{U}(t) = t^{\nu e} U(t). \tag{6.10}$$

Suppose $f(t)$ is a function with the property that

$$t^{-q} f(t) \longrightarrow f_0 \neq 0 \quad \text{as } t \longrightarrow 0, \text{ for some constants } q \text{ and } f_0. \tag{6.11}$$

Such a constant q is unique if it exists; we will refer to q as the *order* of f (near $t = 0$) and write

$$\tilde{f}(t) = t^{-q} f(t) \approx f_0. \tag{6.12}$$

It is easy to check that (6.12) is consistent with (6.9).

Lemma 6.1. *Suppose $\tilde{f}(t) = t^{-q} f(t) \approx f_0$ is a function of order q near $t = 0$, and q is not an integer ≥ 0. Then there is the following equivalence:*

$$t^k \frac{d^k}{dt^k} \tilde{f}(t) \approx 0, \quad 1 \le k \le m, \tag{6.13a}$$

$$\Updownarrow$$

$$\frac{d^k}{dt^k} f(t) \sim \frac{d^k}{dt^k} (f_0 t^q), \quad 1 \le k \le m. \tag{6.13b}$$

Proof. We will make use of the Leibniz formula

$$\frac{d^m}{dt^m}\tilde{f}(t) = \frac{d^m}{dt^m}\left(t^{-q}f(t)\right) = \sum_{i=0}^{m}\binom{m}{i}\frac{d^{m-i}}{dt^{m-i}}\left(t^{-q}\right)\frac{d^i}{dt^i}f(t) \tag{6.14}$$

and use induction on m to prove the equivalence. We start by dividing each term of the identity

$$\frac{df}{dt} = \tilde{f}(t)\frac{d}{dt}\left(t^q\right) + t^q\frac{d}{dt}\tilde{f}(t) \tag{6.15}$$

by $(d/dt)(f_0t^q)$. Then the right-hand side is ≈ 1, that is, its limit is 1 as $t \to 0$, if and only if $t(d/dt)\tilde{f} \approx 0$. This proves (6.13) for $m = 1$. Next, by induction, assume the equivalence holds for all $k \le m-1$, and we will establish the equivalence for $k = m$ as a consequence.

(i) Assume (6.13a) holds (for all $k \le m$). By the induction assumption, (6.13b) holds for all $k \le m-1$, namely

$$\frac{d^k}{dt^k}f = f_0\frac{d^k}{dt^k}\left(t^q\right) + o\left(t^{q-k}\right), \quad 1 \le k \le m-1 \tag{6.16}$$

and we need to establish this identity for $k = m$ as well. Now, by (6.14) and the assumptions,

$$\begin{aligned}
0 \approx t^m\frac{d^m}{dt^m}\tilde{f}(t) &= t^m\sum_{i=0}^{m}\binom{m}{i}\frac{d^{m-i}}{dt^{m-i}}\left(t^{-q}\right)\left[\frac{d^i}{dt^i}\left(f_0t^q\right) + o\left(t^{q-i}\right)\right] \\
&\quad + t^{m-q}\left(\frac{d^m}{dt^m}f - \frac{d^m}{dt^m}\left(f_0t^q\right) - o\left(t^{q-m}\right)\right) \\
&= t^m\sum_{i=0}^{m}\binom{m}{i}\frac{d^{m-i}}{dt^{m-i}}\left(t^{-q}\right)\frac{d^i}{dt^i}\left(f_0t^q\right) + t^m\sum_{i=0}^{m}\binom{m}{i}\frac{d^{m-i}}{dt^{m-i}}\left(t^{-q}\right)o\left(t^{q-i}\right) \\
&\quad + t^{m-q}\left(\frac{d^m}{dt^m}f - \frac{d^m}{dt^m}\left(f_0t^q\right)\right) + o(1) \\
&= t^m\frac{d^m}{dt^m}f_0 + t^m\sum_{i=0}^{m}c_{m,q,i}t^{-q-m+i}o\left(t^{q-i}\right) + t^{m-q}\left(\frac{d^m}{dt^m}f - \frac{d^m}{dt^m}\left(f_0t^q\right)\right) + o(1) \\
&\approx t^{m-q}\left(\frac{d^m}{dt^m}f - \frac{d^m}{dt^m}\left(f_0t^q\right)\right) \quad \text{(all } c_{m,q,i} \text{ are constants!)}.
\end{aligned} \tag{6.17}$$

Therefore, since the last expression is ≈ 0 we conclude

$$\frac{d^m}{dt^m}f \sim \frac{d^m}{dt^m}\left(f_0t^q\right). \tag{6.18}$$

(ii) Conversely, assume (6.13b) holds, that is, (6.16) holds in the range $1 \le k \le m$. Then by (6.14)

$$t^m\frac{d^m}{dt^m}\tilde{f} = t^m\sum_{i=0}^{m}\binom{m}{i}\frac{d^{m-i}}{dt^{m-i}}\left(t^{-q}\right)\left[\frac{d^i}{dt^i}\left(f_0t^q\right) + o\left(t^{q-i}\right)\right] = t^m\frac{d^m}{dt^m}f_0 + o(1) \approx 0 \tag{6.19}$$

and thus (6.13a) also holds for $k = m$. $\square$

Now, we reformulate the two basic equations of (6.3) in terms of the motion $\tilde{\mathbf{X}}(t)$. Straightforward calculations give

$$\frac{d^2}{dt^2}\mathbf{X} = \frac{d^2}{dt^2}(t^\nu\tilde{\mathbf{X}}) = t^{\nu-2}\left[\nu(\nu-1)\tilde{\mathbf{X}} + (1-2\nu)\frac{d}{du}\tilde{\mathbf{X}} + \frac{d^2}{du^2}\tilde{\mathbf{X}}\right], \tag{6.20}$$

$$\nabla U = t^{\nu-2}\left(\frac{1}{m_1}\frac{\partial\tilde{U}}{\partial\tilde{\mathbf{a}}_1},\dots,\frac{1}{m_n}\frac{\partial\tilde{U}}{\partial\tilde{\mathbf{a}}_n}\right), \qquad T = t^{2\nu-2}\left[\tilde{T}_u + \frac{\nu^2}{2}\tilde{\rho}^2 - \frac{\nu}{2}\tilde{\rho}\frac{d}{du}\tilde{\rho}\right], \tag{6.21}$$

where

$$\frac{1}{m_i}\frac{\partial\tilde{U}}{\partial\tilde{\mathbf{a}}_i} = e\sum_{k\neq i}\frac{m_k(\tilde{\mathbf{a}}_k - \tilde{\mathbf{a}}_i)}{\|\tilde{\mathbf{a}}_i - \tilde{\mathbf{a}}_k\|^{e+2}}, \qquad \tilde{T}_u = \frac{1}{2}\sum m_i\left\|\frac{d}{du}\tilde{\mathbf{a}}_i\right\|^2. \tag{6.22}$$

By substituting the above equations into (6.3) we obtain equation (6.23a) and (6.23b) below, and (6.23c) is merely a reformulation of asymptotic estimates already obtained in Section 2.2.2:

$$t^{2-\nu}\cdot\nabla U = \tilde{\nabla}\tilde{U} = \nu(\nu-1)\tilde{\mathbf{X}} + (1-2\nu)\frac{d}{du}\tilde{\mathbf{X}} + \frac{d^2}{du^2}\tilde{\mathbf{X}}, \tag{6.23a}$$

$$\tilde{T}_u = -\frac{\nu^2}{2}\tilde{\rho}^2 + \frac{\nu}{2}\tilde{\rho}\frac{d}{du}\tilde{\rho} + \tilde{U} + he^{-(2-2\nu)u}, \tag{6.23b}$$

$$\tilde{\rho} \approx \kappa, \qquad \tilde{U} \approx \frac{\mu}{\kappa^e}, \qquad \frac{d}{du}\tilde{\rho} \approx \frac{d^2}{du^2}\tilde{\rho} \approx 0. \tag{6.23c}$$

The term involving h in (6.23b) vanishes when $u \to \infty$, since $2 - 2\nu > 0$. So, by inserting the expressions of (6.23c) into (6.23b) it follows that the left side of (6.23b) is ≈ 0, or equivalently each $(d/du)\tilde{\mathbf{a}}_i \approx 0$. On the other hand, $\|\tilde{\mathbf{X}}\| \approx \kappa$ and hence each $\tilde{\mathbf{a}}_i$ is bounded, and moreover, by (6.23c) it follows that $\tilde{U}$ is bounded and therefore all $\|\tilde{\mathbf{a}}_i - \tilde{\mathbf{a}}_k\|$ have a lower bound. Then we deduce from (6.21) that all partial derivatives of $\tilde{U}$, with respect to the components of $\tilde{\mathbf{a}}_i$ and of any order, are also bounded, regarded as functions of u. In particular, in (6.23a) $\tilde{\nabla}\tilde{U}$ is bounded, hence also $(d^2/du^2)\tilde{\mathbf{X}}$ is bounded. Now, the idea is to apply the operator d/du successively to (6.23a), and then it follows that $(d^k/du^k)\tilde{\nabla}\tilde{U}$ and hence also $(d^{k+1}/du^{k+1})\tilde{\mathbf{X}}$ is bounded, for all $k \geq 1$.

In fact, we can deduce from the boundedness that $(d^k/du^k)\tilde{\mathbf{X}} \approx 0$ for each $k \geq 1$, thanks to the following lemma of "Tauberian type."

Lemma 6.2 (see [24, Section 363]). *Let $f(u)$ be a function defined for $u > 0$, and assume $f(u)$ has a limit and $(d^2/du^2)f(u)$ is bounded, as $u \to \infty$. Then $(d/du)f(u) \to 0$ as $u \to \infty$.*

Finally, by applying the operator d/du successively to (6.23b) we deduce that the highest derivative of $\tilde{\rho}$ is always bounded. Again from the above lemma we conclude that $(d^k/du^k)\tilde{\rho} \approx 0$ for all $k \geq 1$.

Corollary 6.3. *If the n-body motion $\mathbf{X}(t)$ leads to a general collision at $t = 0$, then the following asymptotic formulas hold:*

$$\frac{d^k}{dt^k}\rho(t) \sim \frac{d^k}{dt^k}(\kappa t^\nu), \quad k \geq 0, \tag{6.24}$$

$$t^k\frac{d^k}{dt^k}\mathbf{X}_1 = o(1), \quad k \geq 1. \tag{6.25}$$

Proof. Property (6.24) follows by combining $(d^k/du^k)\tilde{\rho} \approx 0$ with the statements in (6.7) and (6.13). This also completes the proof of Theorem 2.7 in Section 2.2.

To establish (6.25), first observe that

$$\mathbf{X}_1 = \frac{1}{\rho}\mathbf{X} = \frac{1}{\tilde{\rho}}\tilde{\mathbf{X}}, \tag{6.26}$$

and then apply the operator d/du successively to the rightmost expression. Since all derivatives of $\tilde{\rho}$ and $\tilde{\mathbf{X}}$ are ≈ 0 and $\tilde{\rho} \approx \kappa \neq 0$, it is easily seen that $(d^k/du^k)\mathbf{X}_1 \approx 0$ for all $k \geq 1$. Now property (6.25) of the corollary follows from (6.7). □

6.2. Convergence of shape and the Painlevé-Wintner spin problem. In the 3-body problem, Euler and Lagrange found that if the bodies started initially at rest, with the shape of a central configuration, then the motion would lead to a triple collision. These solutions are the archetypical examples of homothetic motions (cf. Section 4.1), that is, shape invariant motions with vanishing angular momentum. They also motivated Sundman to determine the possible limiting shapes of triple collision motions in the general 3-body problem. He proved that there is always a limiting shape and, moreover, it coincides with the shape of one of those simple solutions discovered by Euler and Lagrange. Thus, Euler, Lagrange and Sundman partially answered the initial case $n = 3$ of Problem 2.12. Namely, in our terminology, the shape curve $\mathbf{X}^*(t)$ in M^* has a limit as $\mathbf{X}(t) \to \mathbf{0}$. The remaining problem is concerned with the actual convergence of the position (or orientation) of the limiting configuration, or equivalently whether the unit vector $\mathbf{X}_1(t)$ has a limit. This problem, which (at least) dates back to Painlevé in the 19th century and was revived and extended to all $n \geq 3$ by Wintner [24], is described as the *Painlevé-Wintner spin problem* in [14, 16]. Loosely speaking, can the collision orbit "spin" about its limiting collision point?

We first recall the case $n = 3$, where Siegel's analytical work on triple collisions around 1940 is tantamount to proving that $\mathbf{X}_1(t)$ has a limit, cf. [18, 19]. Crucial to Siegel's proof are two characteristics of the motion, namely (i) the shape has a limit, and (ii) there is no rotational contribution to the motion. For the same reasons, Siegel's conclusion should hold for all $n > 3$ as well, in view of the statements in [14, 16]. What we can say more precisely is expressed by the following lemma.

Lemma 6.4. *If the n-body motion $\mathbf{X}(t)$ leads to a general collision at $t = 0$, then its unit vector curve $\mathbf{X}_1(t)$ converges as $t \to 0$ if and only if the shape curve $\mathbf{X}^*(t)$ converges.*

For the proof we will invoke differential geometric ideas. Recall from Section 2.1.4 and Theorem 2.8 in Section 2.2.2 that $\mathbf{X}(t)$ has zero angular momentum and hence a vanishing rotational velocity component $\dot{\mathbf{X}}^\omega$, namely

$$\dot{\mathbf{X}} = \dot{\mathbf{X}}^\rho + \dot{\mathbf{X}}^\sigma = \dot{\mathbf{X}}^\rho + \rho\dot{\mathbf{X}}_1. \tag{6.27}$$

We consider the (stratified) Riemannian submersion

$$M_1 = S^{3n-4} \longrightarrow M^* \tag{6.28}$$

and the connection on M_1 whose "horizontal" tangent vectors are those perpendicular to the SO(3)-orbits, cf. Section 3.2. Note that the velocity $\dot{\mathbf{X}}_1 = (1/\rho)\dot{\mathbf{X}}^\sigma$ is "horizontal" and is identified with the velocity $\dot{\mathbf{X}}^*$, so $\mathbf{X}_1(t)$ is a "horizontal" lifting of the shape curve. It is a standard fact that a lifting is uniquely determined by its initial position $\mathbf{X}_1(t_1)$. By continuity of the lifting construction, it also follows that $\mathbf{X}_1(t)$ has a limit (as $t \to 0$) if and only if $X^*(t)$ has a limit.

Unfortunately, up to now we cannot claim that the shape curve always has a limit; it is only proved that the curve approaches the critical set of the function U^*. However, although it is an open problem whether there are mass distributions for which the "number" of central configurations is infinite, it is difficult to imagine a total collapse motion for which the shape does not converge. The following argument may, perhaps, be useful for the complete proof of the convergence. It follows from property (6.25) of the last corollary that the motion $\mathbf{X}(t)$ and its velocity, acceleration etc. all tend to align (that is, approaching collinearity as vectors in the configuration space M) towards a general collision. For example, if ψ is the angle between $\mathbf{X}$ and $\dot{\mathbf{X}}$, then from the asymptotic estimates of $\dot{\rho}$ and kinetic energy T we deduce

$$\cos\psi = \frac{\mathbf{X}\cdot\dot{\mathbf{X}}}{\|\mathbf{X}\|\,\|\dot{\mathbf{X}}\|} = \frac{\dot{\rho}}{\|\dot{\mathbf{X}}\|} \longrightarrow 1. \tag{6.29}$$

Therefore, the motion is "resembling" a homothetic motion in the limit, namely the type of motions characterized by the collinearity of $\mathbf{X}(t)$ and its velocity, see Section 4.1. This suggests that $\mathbf{X}(t)$ is approaching a specific line in M whatsoever.

Finally, we mention the following property of total collapse motions. In their book, Siegel and Moser [19] showed that for 3-body motions leading to a triple collision, the motion must be rectilinear if the limiting shape is of collinear type. This result has been generalized to n-body motions by Hulkower and Saari by considering the dimensions of stable and unstable manifolds associated with the dynamical equations.

Theorem 6.5 (see [16]). *If the limiting shape of a total collapse n-body motion is a collinear (resp., coplanar) n-configuration, then the motion itself is confined to a straight line (resp., a plane).*

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