IJMMS 28:4 (2001) 247–249
PII. S0161171201006652
http://ijmms.hindawi.com

ON ALMOST PERIODIC SOLUTIONS OF THE DIFFERENTIAL EQUATION $x''(t) = Ax(t)$ IN HILBERT SPACES

GASTON M. N'GUEREKATA

(Received 31 January 2001)

Abstract. We prove almost periodicity of solutions of the equation $x''(t) = Ax(t)$ when the linear operator A satisfies an inequality of the form $\operatorname{Re}(Ax,x) \geq 0$.

2000 Mathematics Subject Classification. 34G10, 34K14.

1. Introduction. Let H be a Hilbert space equipped with norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)$. Almost periodic functions (in Bochner's sense) are continuous functions $f:\mathbb{R}\to H$ such that for every $\epsilon > 0$, there exists a positive real number l such that every interval $[a,a+l]$ contains at least a point τ such that

$$\sup_{t\in\mathbb{R}} \|f(t+\tau)-f(t)\| < \epsilon. \tag{1.1}$$

The Bochner's criterion (cf. [1, 3, 4]) states that a function $f:\mathbb{R}\to H$ is almost periodic if and only if for every sequence of real numbers $(\sigma_n)_{n=1}^{\infty}$ there exists a subsequence $(s_n)_{n=1}^{\infty}$ such that $(f(t+s_n))_{n=1}^{\infty}$ is uniformly convergent in $t\in\mathbb{R}$.

We proved in [2] that if $A = A_+ + A_-$, where A_+ is a symmetric linear operator and A_- is a skew-symmetric linear operator such that $\operatorname{Re}(A_+x,A_-x) \geq -c\|A_+x\|^2$ for every $x\in H$, then every solution of $x'(t) = Ax(t)$, $t\in\mathbb{R}$, with a relatively compact range in H is almost periodic.

In this note, we use the technique described in [2] to prove similar results for some classes of linear differential equation of second order $x''(t) = Ax(t)$.

2. Main results

Theorem 2.1. *Assume that the linear operator A satisfies the inequality of the form $\operatorname{Re}(Ax,x)\geq 0$, for any $x\in H$. Then solutions of the differential equation*

$$x''(t) = Ax(t), \quad t\in\mathbb{R}, \tag{2.1}$$

(that are functions $x(t)\in C^2(\mathbb{R},H)$) with relatively compact ranges in H, are almost periodic.

Proof. Consider $x(t)$ a solution of (2.1) with a relatively compact range in H and let $\phi:\mathbb{R}\to\mathbb{R}$ be defined by $\phi(t) = \|x(t)\|^2$. Then ϕ is a bounded function over $\mathbb{R}$.

Moreover, for every $t \in \mathbb{R}$, we have

$$\begin{aligned}\phi'(t) &= (x'(t), x(t)) + (x(t), x'(t)),\\ \phi''(t) &= 2\left[\|x'(t)\|^2 + \operatorname{Re}(Ax(t), x(t))\right]\\ &\geq 0,\end{aligned} \tag{2.2}$$

which shows that ϕ is a convex function over $\mathbb{R}$, therefore it is constant

$$\phi(t) = \phi(0), \quad \forall t \in \mathbb{R}, \tag{2.3}$$

or

$$\|x(t)\| = \|x(0)\|, \quad \forall t \in \mathbb{R}. \tag{2.4}$$

We fix $s \in \mathbb{R}$ and consider the function $y_s(\cdot) : \mathbb{R} \to H$ defined by

$$y_s(t) = x(t+s). \tag{2.5}$$

Then $y_s(t)$ obviously satisfies (2.1). Now fix s_1 and s_2 in $\mathbb{R}$. Then $y_{s_1}(t) - y_{s_2}(t)$ also satisfies (2.1); therefore we have

$$\|y_{s_1}(t) - y_{s_2}(t)\| = \|y_{s_1}(0) - y_{s_2}(0)\|, \quad \forall t \in \mathbb{R}, \tag{2.6}$$

which gives

$$\|x(t+s_1) - x(t+s_2)\| = \|x(s_1) - x(s_2)\|, \quad \forall t \in \mathbb{R}. \tag{2.7}$$

Let $(\sigma_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then by relative compactness of $x(t)$, there exists a subsequence $(s_n)_{n=1}^{\infty} \subset (\sigma_n)_{n=1}^{\infty}$ such that $(x(s_n))_{n=1}^{\infty}$ is Cauchy. Hence for any given $\epsilon > 0$, there exists N such that if $n, m > N$, then

$$\|x(s_n) - x(s_m)\| < \epsilon. \tag{2.8}$$

Consequently,

$$\sup_{t \in \mathbb{R}} \|x(t+s_n) - x(t+s_m)\| < \epsilon. \tag{2.9}$$

We conclude that $x(t)$ is almost periodic by the Bochner's criterion. □

Remark 2.2. Examples of such problem occur when A is a positive or monotone linear operator.

References

[1] C. Corduneanu, *Almost Periodic Functions*, 2nd ed., Chelsea Publishing, New York, 1989. Zbl 0672.42008.

[2] G. M. N'Guerekata, *Remarques sur les solutions presque-périodiques de l'équation* $[(d/dt) - A]x = 0$ [*Remarks on the almost-periodic solutions of the equation* $[(d/dt) - A]x = 0$], Canad. Math. Bull. **25** (1982), no. 1, 121–123 (French). MR 84b:34087. Zbl 484.34030.

[3] ______, *Almost-periodicity in linear topological spaces and applications to abstract differential equations*, Int. J. Math. Math. Sci. **7** (1984), no. 3, 529–540. MR 86c:34125. Zbl 561.34045.

[4] ______, *Almost automorphy, almost periodicity and stability of motions in Banach spaces*, Forum Math. **13** (2001), no. 4, 581–588. CMP 1 830 248.

Gaston M. N'Guerekata: Department of Mathematics, Morgan State University, Baltimore, MD 21251, USA

E-mail address: gnguerek@morgan.edu